Homework 10

Math 25a

Due December 5, 2018

Topics covered (lectures 20-21): orthogonal complements/projections, adjoints, spectral theorem

Instructions:

- The homework is divided into one part for each CA. You will submit each part to the corresponding CA's mailbox on the second floor of the science center.
- If your submission to any one CA takes multiple pages, then staple them together. A stapler is available in the Cabot library in the science center.
- If you collaborate with other students, please mention this near the corresponding problems.
- Some problems from this assignment come from the 3rd edition of Axler's book. I've indicated this next to the problems. For example, Axler 1.B.4 means problem 4 from the exercises to Section B of Chapter 1. Sometimes the problem in Axler is slightly different, so make sure you do the problem as listed in the assignment.

1 For Joey

Problem 1 (Axler 6.B.13). *Let $v_1, \ldots, v_k$ be linearly independent vectors in V. Show that there is a vector $w \in V$ so that $\langle v_i, w \rangle > 0$ for each $i = 1, \ldots, k$. Hint: use the representation theorem.*

Solution. □

Problem 2 (Axler 7.A.1). *Define $T \in L(\mathbb{R}^n)$ by $T(x_1, \ldots, x_n) = (0, x_1, \ldots, x_{n-1})$. Find a formula for the adjoint*

$$T^*(x_1, \ldots, x_n) =$$

Solution. □

Problem 3 (Axler 7.B.14). *Let U be a finite dimensional real vector space and fix $T \in L(U)$. Prove that the following are equivalent.*

(a) There exists a basis $u_1, \ldots, u_n \in U$ of eigenvectors of T.

(b) There exists an inner product on U such that T is self-adjoint with respect to this inner product.

Solution. □

2 For Laura

Problem 4 (Axler 7.C.7)**.** *Let V be a finite dimensional real inner product space. Assume $T \in L(V)$ is positive. Show that T is invertible if and only if $\langle Tv, v\rangle > 0$ for every $v \in V \setminus \{0\}$.*

Solution. □

Problem 5 (Axler 6.C.11)**.** *In $\mathbb{R}^4$ with the standard inner product, let U be the subspace spanned by $(1,1,0,0)$ and $(1,1,1,2)$. Find the point on U that is closest to $(1,2,3,4)$, i.e. find $u \in U$ such that $|u-(1,2,3,4)|$ is as small as possible.*

Solution. □

Problem 6. *For $v, w \in \mathbb{R}^3$, defined the cross product $v \times w \in \mathbb{R}^3$ as the unique vector so that*

$$\langle u, v \times w\rangle = \det(u, v, w)$$

for all $u \in \mathbb{R}^3$. Writing $v = (v_1, v_2, v_3)$ and $w = (w_1, w_2, w_3)$, prove that the cross product is given by the formula you already know.[1]

Solution. □

[1] Or if you don't already know the formula, derive it.

3 For Beckham

Problem 7 (Axler 7.A.11). *Suppose $P \in L(V)$ and $P^2 = P$. Prove there exists a subspace $U \subset V$ such that $P = P_U$ if and only if P is self-adjoint. (Here P_U denotes the orthogonal projection onto U.)*

Solution. □

Problem 8 (Axler 7.B.1). *True or false: There exists $T \in L(\mathbb{R}^3)$ such that T is not self-adjoint (with respect to the usual inner product) and such that there exists a basis of $\mathbb{R}^3$ consisting of eigenvectors of T.*

Solution. □

Problem 9 (Axler 7.A.6). *Make $Poly_2(\mathbb{R})$ into an inner product space by defining*

$$\langle p, q\rangle = \int_0^1 p(x)q(x)dx$$

Define $T \in L(Poly_2(\mathbb{R}))$ by $T(a_2x^2 + a_1x + a_0) = a_1x$. Is it self-adjoint? Hint: it's matrix with respect to the standard basis is symmetric, but the answer is no. Explain this.

Solution. □

4 For Davis

Problem 10 (Axler 7.B.11). *Prove or give a counterexample: every self-adjoint operator* $T \in L(V)$ *has a cube root, i.e. there exists* $S \in L(V)$ *so that* $S^3 = T$.

Solution. □

Problem 11. *Consider* $V = \mathbb{R}^3$ *with the standard inner product. Show that if* $S \in L(V)$ *is orthogonal, then there exists* $x \in V$ *such that* $S^2 x = x$. *(It might first help to show that* S *has an eigenvector.)*

Solution. □

Problem 12. *True or false: the identity* $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in M_2(\mathbb{R})$ *has infinitely many square roots.*

Solution. □