Homework 9

Math 25a

Due November 15, 2017

Topics covered: inner product spaces, adjoints, spectral theorem

Instructions:

- The homework is divided into one part for each CA. You will submit each part to the corresponding CA's mailbox on the second floor of the science center.
- If your submission to any one CA takes multiple pages, then staple them together. A stapler is available in the Cabot library in the science center.
- If you collaborate with other students, please mention this near the corresponding problems.
- Some problems from this assignment come from the 3rd edition of Axler's book. I've indicated this next to the problems. For example, Axler 1.B.4 means problem 4 from the exercises to Section B of Chapter 1. Sometimes the problem in Axler is slightly different, so make sure you do the problem as listed in the assignment.

1 For Charlie

In all of the problems below, V always denotes a finite dimensional real inner product space, unless otherwise stated.

Problem 1 (Axler 7.A.1)**.** *Define* $T \in L(\mathbb{R}^n)$ *by* $T(x_1, \ldots, x_n) = (0, x_1, \ldots, x_{n-1})$*. Find a formula for the adjoint* T^**. (It might help to look at Axler §7.A, Example 7.3.)*

Solution. □

Problem 2 (Axler 7.A.3)**.** *Suppose* $T \in L(V)$ *and* $U \subset V$ *is a subspace. Prove that* U *is invariant under* T *if and only if* $U^\perp$ *is invariant under* T^**.*

Solution. □

Problem 3 (Axler 7.A.6)**.** *Make* $Poly_2(\mathbb{R})$ *into an inner product space by defining*

$$\langle p, q \rangle = \int_0^1 p(x)q(x)dx$$

Define $T \in L(Poly_2(\mathbb{R}))$ *by* $T(a_2x^2 + a_1x + a_0) = a_1x$*.*

(a) Show that T *is not self-adjoint.*

(b) Observe that the matrix of T *with respect to the basis* $(1, x, x^2)$ *is symmetric:*

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

How is it possible that this matrix is symmetric, but T *is not self-adjoint?!*

Solution. □

2 For Michele

Problem 4 (Axler 7.A.8)**.** *Show that the set of self-adjoint operators on V is a subspace of $L(V)$. Is the set of orthogonal operators a subspace of $L(V)$?*

Solution. □

Problem 5 (Axler 7.B.1)**.** *True or false: There exists $T \in L(\mathbb{R}^3)$ such that T is not self-adjoint (with respect to the usual inner product) and such that there exists a basis of $\mathbb{R}^3$ consisting of eigenvectors of T.*

Solution. □

Problem 6 (Axler 6.A.18)**.** *Suppose $p > 0$. Prove that there is an inner product on $\mathbb{R}^2$ such that the associated norm is given by*

$$||(x, y)|| = (x^p + y^p)^{1/p}$$

for all $(x, y) \in \mathbb{R}^2$ if and only if $p = 2$.

Solution. □

3 For Natalia

Problem 7 (Axler 7.B.11)**.** *Prove or give a counterexample: every self-adjoint operator* $T \in L(V)$ *has a cube root, i.e. there exists* $S \in L(V)$ *so that* $S^3 = T$*.*

Solution. □

Problem 8 (Axler 6.A.19)**.** *Prove that*

$$\langle u, v\rangle = \frac{||u+v||^2 - ||u-v||^2}{4}$$

for all $u, v \in V$*. Explain why this implies that an isometry of* V *preserves angles.*

Solution. □

Problem 9 (Axler 7.B.10)**.** *Give an example of a real inner product space* V *and* $T \in L(V)$ *and real numbers* b, c *with* $b^2 < 4c$ *such that* $T^2 + bT + cI$ *is not invertible.*

Solution. □

4 For Ellen

Problem 10 (Axler 7.A.4). *Prove that $T \in L(V)$ is injective if and only if T^* is surjective, and prove that T is surjective if and only if T^* is injective.*

Solution. □

Problem 11 (Axler 7.A.7). *Suppose $S, T \in L(V)$ are self-adjoint. Prove that ST is self-adjoint if and only if S and T commute, i.e. $ST = TS$.*

Solution. □

Problem 12 (Axler 7.A.11). *Suppose $P \in L(V)$ and $P^2 = P$. Prove there exists a subspace $U \subset V$ such that $P = P_U$ if and only if P is self-adjoint. (Recall that P_U denotes the orthogonal projection onto U.)*

Solution. □