Homework 8

Math 25a

Due November 8, 2017

Topics covered: Linear maps, eigenvectors, eigenvalues, polynomials of linear operators, page-rank algorithm

Instructions:

- The homework is divided into one part for each CA. You will submit each part to the corresponding CA's mailbox on the second floor of the science center.
- If your submission to any one CA takes multiple pages, then staple them together. A stapler is available in the Cabot library in the science center.
- If you collaborate with other students, please mention this near the corresponding problems.
- Some problems from this assignment come from the 3rd edition of Axler's book. I've indicated this next to the problems. For example, Axler 1.B.4 means problem 4 from the exercises to Section B of Chapter 1. Sometimes the problem in Axler is slightly different, so make sure you do the problem as listed in the assignment.

1 For Ellen

Problem 1 (Axler 6.A.1)**.** *Show that* $\langle (x_1, x_2), (y_1, y_2) \rangle = |x_1 y_1| + |x_2 y_2|$ *is not an inner product on* $\mathbb{R}^2$.

Solution. □

Problem 2 (Axler 6.A.4)**.** *Let* V *be a real inner product space.*

(a) Show that $\langle u + v, u - v \rangle = ||u||^2 - ||v||^2$ *for every* $u, v \in V$.

(b) Use this to argue that the diagonals of a rhombus are perpendicular to each other.

Solution. □

Problem 3 (Axler 6.A)**.** *Suppose* $T \in L(V)$ *is such that* $||Tv|| \leq ||v||$ *for every* $v \in V$. *Prove that* $T - \sqrt{2}I$ *is invertible.*

Solution. □

2 For Michele

Problem 4. *Use Cauchy–Schwarz to prove that* $(ac+bd)^2 \leq (a^2+b^2)(c^2+d^2)$ *for any real numbers* a, b, c, d. *(You could also solve this by expanding both sides, but use Cauchy–Schwarz to get familiar with how it works.)*

Solution. □

Problem 5. *Fix* $n \geq 1$. *Prove that* $(x_1 + \cdots + x_n)^2 \leq n(x_1^2 + \cdots + x_n^2)$ *for all* $x_1, \ldots, x_n \in \mathbb{R}$. *Hint: whatever you do, for the love of algebra, do not expand both sides. Instead use Cauchy–Schwarz.*

Solution. □

Problem 6 (Axler 6.B.1)**.** *Fix* $t \in \mathbb{R}$. *Let* $u_t = (\cos t, \sin t)$ *and* $v_t = (-\sin t, \cos t)$ *and* $w_t = (\sin t, -\cos t)$. *Show that* (u, v) *and* (u, w) *are each orthonormal bases of* $\mathbb{R}^2$. *Show that every orthonormal basis of* $\mathbb{R}^2$ *has this form (i.e. for any orthonormal basis* (e_1, e_2), *there is* $t \in \mathbb{R}$ *so that* (e_1, e_2) *is either equal to* (u_t, v_t) *or* (u_t, w_t)*).*

Solution. □

3 For Charlie

Problem 7. *On Poly*$_2(\mathbb{R})$*, consider the inner product* $\langle p, q\rangle = \int_0^1 p(x)q(x)dx$*. Apply Gram-Schmidt to the basis* $1, x, x^2$ *to produce an orthonormal basis for Poly*$_2(\mathbb{R})$*. (Please do this without a computer and show your steps. If you like, check that you get the same answer as Mathematica.)*

Solution. □

Problem 8 (Axler 6.B.10)**.** *Suppose* V *is a real inner product space and* $v_1, \ldots, v_m$ *is a linearly independent list of vectors in* V*. Prove that there are exactly* 2^m *orthogonal lists* $e_1, \ldots, e_m$ *of vectors such that in* V *such that*

$$\operatorname{span}(v_1, \ldots, v_j) = \operatorname{span}(e_1, \ldots, e_j)$$

for each $j = 1, \ldots, m$*.*

Solution. □

Problem 9 (Axler 6.B.9)**.** *What happens when Gram–Schmidt is applied to a list of vectors that's not linearly independent?*

Solution. □

4 For Natalia

Problem 10 (Axler 6.B.13). *Let $v_1, \dots, v_k$ be linearly independent vectors in V. Show that there is a vector $w \in V$ so that $\langle v_i, w \rangle > 0$ for each $i = 1, \dots, k$.*

Solution. □

Problem 11 (Axler 6.C.11). *In $\mathbb{R}^4$, let U be the subspace spanned by $(1, 1, 0, 0)$ and $(1, 1, 1, 2)$. Find the point on U that is closest to $(1, 2, 3, 4)$ (i.e. find $u \in U$ such that $||u - (1, 2, 3, 4)||$ is as small as possible).*

Solution. □

Problem 12 (Axler 6.C.7). *Suppose V is finite dimensional and $P \in L(V)$ is such that $P^2 = P$ and every vector of $\ker P$ is orthogonal to every vector of $\operatorname{Im} P$. Prove that there exists a subspace $U \subset V$ such that $P = P_U$, where P_U is the orthogonal projection with respect to U (we called it $\pi : V \to U$ in class).*

Solution. □