Homework 7

Math 25a

Due October 25, 2017

Topics covered: Linear maps, eigenvectors, eigenvalues, polynomials of linear operators, page-rank algorithm

Instructions:

- The homework is divided into one part for each CA. You will submit each part to the corresponding CA's mailbox on the second floor of the science center.
- If your submission to any one CA takes multiple pages, then staple them together. A stapler is available in the Cabot library in the science center.
- If you collaborate with other students, please mention this near the corresponding problems.
- Some problems from this assignment come from the 3rd edition of Axler's book. I've indicated this next to the problems. For example, Axler 1.B.4 means problem 4 from the exercises to Section B of Chapter 1. Sometimes the problem in Axler is slightly different, so make sure you do the problem as listed in the assignment.

1 For Michele

Problem 1. *Suppose $S : U \to V$ and $T : V \to W$ are two linear maps with composition $TS = 0$ in $L(U, W)$.*

(a) *Show that the image of S is a subspace of the kernel of T. Terminology: If $TS = 0$ and $\ker T = \operatorname{Im} S$, then we say that $U \to V \to W$ is exact.*

(b) *Given an example of linear maps $S : \mathbb{R} \to \mathbb{R}^2$ and $T : \mathbb{R}^2 \to \mathbb{R}$ such that $TS = 0$ but the composition $\mathbb{R} \to \mathbb{R}^2 \to \mathbb{R}$ is not exact.*

Solution. □

Problem 2. (a) *Let $V = \mathrm{Poly}_7(F)$, and $U = \mathrm{Poly}_5(F)$, and $W = F^2$. Consider the linear maps $S(p(x)) = (x^2 - x)p(x)$ and $T(q(x)) = (q(0), q(1))$. Show that $TS = 0$ and that the sequence is exact at V.*

(b) *Show further that S is injective and T is surjective. More terminology: if $U \xrightarrow{S} V \xrightarrow{T} W$ is exact and S is injective and T is surjective, then we say $0 \to U \xrightarrow{S} V \xrightarrow{T} W \to 0$ is a short exact sequence.*

Solution. □

Problem 3 (Axler 5.A.21)**.** *Suppose $T \in L(V)$ is invertible, and let $\lambda \in F$ be nonzero. Show λ is an eigenvalue of T if and only if λ^{-1} is an eigenvalue of T^{-1}. (Does this match your intuition?) Show that T and T^{-1} have the same eigenvectors.*

Solution. □

2 For Ellen

A square matrix $A = (a_{ij}) \in M_n(F)$ is called lower-triangular if it has no nonzero entries above the diagonal, i.e. if $i < j$, then $a_{ij} = 0$. For example, for $n = 3$, A looks like

$$\begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Problem 4. *Read and understand the proof of Proposition 5.30 in Axler Section 5.B. Let* $A \in M_n(F)$ *be an lower triangular matrix. Show that* A *is invertible if and only if all the diagonal entries are nonzero. (You can adapt Axler's argument to this case. If you do this, improve on his argument for the "backward direction" by giving a rigorous proof by induction.)*

Solution. □

Problem 5. *Is there basis for* $\mathbb{R}^2$ *so that the matrix of*

$$T(x, y) = (5x + y, 3y - x)$$

is upper triangular? If so, give such a matrix. Is there a basis for $\mathbb{R}^2$ *so that the matrix of* T *is diagonal? If so, give such a matrix.*

Solution. □

3 For Natalia

Problem 6 (Axler 5.B.6)**.** *Suppose $T \in L(V)$ and $U \subset V$ is a subspace invariant under T. Prove that U is invariant under $p(T)$ for every $p \in \mathrm{Poly}(F)$.*

Solution. □

Problem 7 (Axler 5.B.3)**.** *Suppose $T \in L(V)$ and $T^2 = i_V$ and -1 is not an eigenvalue of T. Prove that $T = i_V$.*

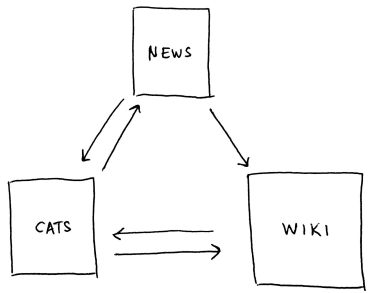
Solution. □

Problem 8 (Axler 5.B.8)**.** *Give an example of $T \in L(\mathbb{R}^2)$ such that $T^4 = -i_V$.*

Solution. □

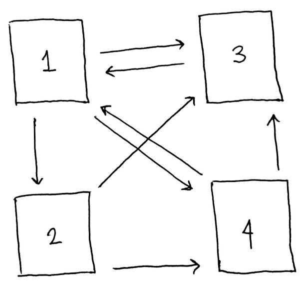
4 For Charlie

Problem 9. *Determine the weighted adjacency matrix and importance scores for the web pictured below. Which page is the most important?*



Solution. □

Problem 10 (Page-rank article, Exercise 1)**.** *In class we considered the web*



and showed that the importance scores are $(\frac{12}{31}, \frac{4}{31}, \frac{9}{31}, \frac{6}{31})$*. In attempt to boost their page's score, the people who own page 3, create a page 5 that links to page 3 and also link page 3 to page 5. Does this boost page 3?s score above that of page 1?*

Solution. □

Problem 11 (Page-rank article, Exercise 5)**.** *Prove that in any web the importance score of a page with no backlinks is zero.*

Solution. □