Homework 6

Math 25a

Due October 18, 2017

Topics covered: Matrices, polynomials, eigenvectors, eigenvalues, invariant subspaces

Instructions:

- The homework is divided into one part for each CA. You will submit each part to the corresponding CA's mailbox on the second floor of the science center.
- If your submission to any one CA takes multiple pages, then staple them together. A stapler is available in the Cabot library in the science center.
- If you collaborate with other students, please mention this near the corresponding problems.
- Some problems from this assignment come from the 3rd edition of Axler's book. I've indicated this next to the problems. For example, Axler 1.B.4 means problem 4 from the exercises to Section B of Chapter 1. Sometimes the problem in Axler is slightly different, so make sure you do the problem as listed in the assignment.

1 For Charlie

Problem 1. *A matrix $A \in M_n(F)$ is invertible if there exists a matrix B so that $AB = BA = I$. Prove that if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(F)$ and $ad - bc \neq 0$, then A is invertible. Find the inverse of A. (Write $B = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$ and set up a system of equations to solve for the coefficients of B.)*

Solution. □

Problem 2. *Let $F = \mathbb{Z}/2\mathbb{Z}$.*

(a) Write down all the invertible matrices in $M_2(F)$.

(b) Find an invertible matrix $A \in M_2(F)$ so that $A^2 \neq I$, but $A^3 = I$.

Solution. □

Problem 3 (Axler 4.5)**.** *(a) Suppose $z_1, \ldots, z_{m+1} \in F$ are distinct and $w_1, \ldots, w_{m+1} \in F$ are arbitrary. Prove that there exists a unique polynomial $p \in \mathrm{Poly}_m(F)$ such that $p(z_j) = w_j$. (Define a linear map that helps you study this problem.)*

(b) For $m = 2$ and $F = \mathbb{Z}/5\mathbb{Z}$, give a polynomial such that $p(1) = 2$, $p(2) = -1$, and $p(-1) = 4$.

Solution. □

2 For Ellen

Problem 4 (Axler 4.6)**.** *Let $p \in \text{Poly}_m(F)$. Let $D : \text{Poly}(F) \to \text{Poly}(F)$. Show that p has m distinct roots if and only if $D(p)$ and p have no common roots. (You may use the "product rule" $D(pq) = D(p)q + pD(q)$.)*

Solution. □

Problem 5 (Axler 4.10)**.** *For a complex number $z = a + bi$, the complex conjugate, denoted $\bar{z}$, is the complex number $\bar{z} = a - bi$. A complex number is called real if $z = \bar{z}$ (i.e. z has the form $z = a + 0i$ for some $a \in \mathbb{R}$).*

(a) For $p = a_n x^n + \cdots + a_1 x + a_0 \in \text{Poly}(\mathbb{C})$, denote $\bar{p} = \bar{a}_n x^n + \cdots + \bar{a}_1 x + \bar{a}_0$ (the polynomial whose coefficients are the complex conjugates of the coefficients of p). Show that p has real coefficients if and only if $p = \bar{p}$.

(b) Suppose $p \in \text{Poly}_m(\mathbb{C})$ and there exists $x_0, x_1, \ldots, x_m \in \mathbb{R}$ such that $p(x_i) \in \mathbb{R}$ for each i. Show that p has real coefficients. (Hint: It may help you to use one of the previous problems on this assignment.)

Solution. □

Problem 6 (Axler 4.11)**.** *Let $V = \text{Poly}(F)$. Suppose $p \in V$ is nonzero. Let $U = \{pq : q \in V\}$. Find a basis for V/U and show that $\dim V/U = \deg p$.*

Solution. □

3 For Natalia

Definition. Let $T \in L(V)$ be a linear operator. A subspace $U \subset V$ is said to be invariant under T if whenever $u \in U$ then also $Tu \in U$.

Problem 7 (Axler 5.A.2-3)**.** *Let $S, T \in L(V)$ and suppose that $ST = TS$ (in this case we say T and S commute). Prove that $\ker S$ and $\operatorname{Im} S$ are both invariant under T.*

Solution. □

Problem 8 (Axler 5.A.6)**.** *Prove or give a counterexample: if V is finite dimensional and U is a subspace of V that is invariant under every operator on V, then $U = \{0\}$ or $U = V$.*

Solution. □

Problem 9 (Axler 5.A.7)**.** *Consider $T \in L(\mathbb{R}^2)$ defined by $T(x, y) = (-3y, x)$. Find the eigenvalues of T.*

Solution. □

4 For Michele

Problem 10 (Axler 5.A.24). *Let $A \in M_n(F)$. Let $T \in L(F^n)$ be the linear operator given by $Tx = Ax$.*

(a) Suppose the sum of the entries in each row of A equals 1. Prove that 1 is an eigenvalue of T.

(b) Suppose the sum of the entries in each column of A equals 1. Prove that 1 is an eigenvalue of T.

Solution. □

Problem 11 (Axler 5.A.20). *Find all eigenvalues and eigenvectors of the operator $T \in L(F^\infty)$ defined by $T(x_1, x_2, \ldots) = (x_2, x_3, \ldots)$.*

Solution. □

Problem 12 (Axler 5.A.23). *Suppose V is finite dimensional and $S, T \in L(V)$. Prove that ST and TS have the same eigenvalues. (Hint: You will need to use the assumption that V is finite dimensional!)*

Solution. □