Homework 4

Math 25a

Due October 4, 2017

Topics covered: bases, linear maps, kernel, image, rank-nullity

Instructions:

- The homework is divided into one part for each CA. You will submit each part to the corresponding CA's mailbox on the second floor of the science center.
- If your submission to any one CA takes multiple pages, then staple them together. A stapler is available in the Cabot library in the science center.
- If you collaborate with other students, please mention this near the corresponding problems.
- Some problems from this assignment come from the 3rd edition of Axler's book. I've indicated this next to the problems. For example, Axler 1.B.4 means problem 4 from the exercises to Section B of Chapter 1. Sometimes the problem in Axler is slightly different, so make sure you do the problem as listed in the assignment.

1 For Ellen

Problem 1 (Axler 3.A.7). *Suppose that V is 1-dimensional. Show that for every linear map $T : V \to V$ there exists $a \in F$ so that $Tv = av$ for all $v \in V$.*

Solution. □

Problem 2 (Axler 3.A.8). *Give an example of a function $f : \mathbb{R}^2 \to \mathbb{R}$ such that $f(av) = af(v)$ for all $a \in \mathbb{R}$ and all $v \in \mathbb{R}^2$ but f is not linear.*

Solution. □

Problem 3 (Axler 3.A.11). *Let V be a finite dimensional vector space with a subspace U. Show that for every linear map $T : U \to W$ there exists a linear map $S : V \to W$ so that $S(u) = T(u)$ for $u \in U$. In this case we say that S is an* extension *of T.*

Solution. □

2 For Charlie

Problem 4 (Axler 3.B.9-10)**.** *Let $T : V \to W$ be a linear map.*

(a) Show that if T is injective and $v_1, \ldots, v_n$ are linearly independent in V, then $Tv_1, \ldots, Tv_n$ are linearly independent in W.

(b) Show that if T is surjective and $v_1, \ldots, v_n$ span V, then $Tv_1, \ldots, Tv_n$ span W.

Solution. □

Problem 5 (Axler 3.B.16)**.** *Suppose there exists a linear map on V whose kernel and image are both finite dimensional. Show that V is finite dimensional. (Hint: You may not use the rank-nullity theorem.)*

Solution. □

Problem 6 (Axler 3.B.12)**.** *Let V be finite dimensional and let $T : V \to W$ be a linear map. Show there exists a subspace $U \subset V$ so that $U \cap \ker T = \{0\}$ and $\operatorname{Im} T = \{Tu : u \in U\}$.*

Solution. □

3 For Natalia

Problem 7 (Axler 3.B.20). *Assume V is finite dimensional and $T : V \to W$ is linear. Show that T is injective if and only if there exists a linear map $S : W \to V$ so that ST is the identity map on V.*

Solution. □

Problem 8 (Axler 3.B.21). *Assume V is finite dimensional and $T : V \to W$ is linear. Show that T is surjective if and only if there exists a linear map $S : W \to V$ so that TS is the identity map on W.*

Solution. □

Problem 9 (Axler 3.B.22-23). *Assume U and V are finite dimensional and $U \xrightarrow{S} V \xrightarrow{T} W$ are linear maps. Show that*

(a) $\dim \ker TS \leq \dim \ker S + \dim \ker T$.

(b) $\dim \operatorname{Im} TS \leq \min\{\dim \operatorname{Im} S, \dim \operatorname{Im} T\}$.

Solution. □

4 For Michele

Problem 10. *Let (v_1, v_2, v_3) be a basis for V over F, and let $T(v_1) = v_2$, $T(v_2) = v_1$, and $T(v_3) = v_1 + v_2$.*

(a) *Show that there is a unique linear map $T : V \to V$ taking these values on the basis. Compute the matrix of T with respect to this basis.*

(b) *Is T a linear isomorphism? Why or why not?*

Solution. □

Problem 11. *Let Poly(F) be the vector space of all polynomials with coefficients in F, and let $V =$ Fun(F, F) be the vector space of all functions $f : F \to F$. Define a map of sets $T :$ Poly$(F) \to V$ by $T(p)(a) = p(a)$ the function mapping $a \in F$ to $p(a) \in F$.*

(a) *Show that T is a linear map.*

(b) *For $F = \mathbb{R}$ show that T is injective but not surjective.*

(c) *Give an example of a field F where T is surjective but not injective, and prove your claim.*

Solution. □

Problem 12. *Let $F = \mathbb{Z}/p\mathbb{Z}$ for a prime number p. What is the probability that a linear map $T : F^2 \to F^2$ is a linear isomorphism when randomly choosing out of all such maps? (Note that a linear isomorphism sends a basis of F^2 to another basis of F^2 by Problem 4.) As p increases is one more or less likely to choose a linear isomorphism at random?*

Solution. □