Homework 3

Math 25a

Due September 20, 2017

Topics covered: vector subspaces, span and linear independence, finite dimensionality

Instructions:

- The homework is divided into one part for each CA. You will submit each part to the corresponding CA's mailbox on the second floor of the science center.
- If your submission to any one CA takes multiple pages, then staple them together. A stapler is available in the Cabot library in the science center.
- If you collaborate with other students, please mention this near the corresponding problems.
- Some problems from this assignment come from the 3rd edition of Axler's book. I've indicated this next to the problems. For example, Axler 1.B.4 means problem 4 from the exercises to Section B of Chapter 1. Sometimes the problem in Axler is slightly different, so make sure you do the problem as listed in the assignment.

1 For Michele

Problem 1 (Axler 1.C.10). *Show that if U_1 and U_2 are subspaces of V are subspaces, then $U_1 \cap U_2$ is also a subspace of V.*

Solution. □

Problem 2 (Axler 1.C.1). *Let F be a field. For each of the following subsets of F^3, determine whether it is a subspace.*

(a) $U_1 = \{(x_1, x_2, x_3) \in F^3 : x_1 + 2x_2 + 3x_3 = 0\}$;

(b) $U_2 = \{(x_1, x_2, x_3) \in F^3 : x_1 + 2x_2 + 3x_3 = 4\}$;

(c) $U_3 = \{(x_1, x_2, x_3) \in F^3 : x_1 x_2 x_3 = 0\}$;

(d) $U_4 = \{(x_1, x_2, x_3) \in F^3 : x_1 = 5x_3\}$.

Solution. □

Problem 3 (Axler 1.C.7-8). *(a) Give an example of a nonempty subset U of $\mathbb{R}^2$ such that U is closed under addition and taking additive inverses, but U is not a subspace of $\mathbb{R}^2$.*

(b) Give an example of a nonempty subset $U \subset \mathbb{R}^2$ such that U is closed under scalar multiplication, but U is not a subspace of $\mathbb{R}^2$.

Solution. □

2 For Ellen

Problem 4 (Axler 1.C.24)**.** *Let V be the vector space of all functions $f : \mathbb{R} \to \mathbb{R}$. A function $f : \mathbb{R} \to \mathbb{R}$ is called* even *if it satisfies $f(-x) = f(x)$ and* odd *if it satisfies $f(-x) = -f(x)$. Let U_e be the subset of even functions and U_o be the subset of odd functions.*

(a) Show U_e and U_o are subspaces of V.

(b) Show $V = U_e \oplus U_o$.

Solution. □

Problem 5 (Axler 1.9)**.** *Prove that the union of two subspaces of V is a subspace of V if and only if one of the subspaces is contained in the other.*

Solution. □

Problem 6 (c.f. Axler 1.C.9)**.** *Let V be the vector space of all functions $f : \mathbb{R} \to \mathbb{R}$. A function $f : \mathbb{R} \to \mathbb{R}$ is called* periodic *with period p if $f(x+p) = f(x)$ for all $x \in \mathbb{R}$. For example, for $a \neq 0$, the function $f(x) = \sin(ax)$ is periodic with period $2\pi/a$. Let $V_p \subset V$ be the subset of functions with period p. Note that $V_0 = V$.*

(a) Fix $p \neq 0$. Is V_p a subspace of V?

(b) Is $\bigcup_{p \in \mathbb{Q} \setminus \{0\}} V_p$ a subspace of V?

(c) Is $\bigcup_{p \in \mathbb{R} \setminus \{0\}} V_p$ a subspace of V?

In each case, prove or give a counterexample.

Solution. □

3 For Natalia

Problem 7. *Consider the vector space* $V = \mathbb{R}^2$.

(a) Verify directly that $(1,5)$ *and* $(2,1)$ *are linearly independent in* V *(set up a system of linear equations and solve).*

(b) Show that $(1,5)$ *and* $(2,1)$ *span in* V *by showing that* $(1,0)$ *and* $(0,1)$ *belong to the span of* $(1,5)$ *and* $(2,1)$.

Solution. □

Problem 8 (c.f. Axler 2.A.1 and 2.A.6)**.** *Let* $v_1, \ldots, v_n$ *be vectors in* V.

(a) Prove that if $(v_1, \ldots, v_n)$ *spans* V*, then so does*

$$(v_1 - v_2, v_2 - v_3, \ldots, v_{n-1} - v_n, v_n).$$

(b) Prove that if $(v_1, \ldots, v_n)$ *is linearly independent in* V*, then so is*

$$(v_1 - v_2, v_2 - v_3, \ldots, v_{n-1} - v_n, v_n).$$

Solution. □

Problem 9 (Axler 2.A.10)**.** *Suppose that* $(v_1, \ldots, v_n)$ *is linearly independent and* $w \in V$*. Prove that if* $(v_1 + w, \ldots, v_n + w)$ *is linearly dependent, then* $w \in span(v_1, \ldots, v_n)$.

Solution. □

4 For Charlie

Problem 10 (Axler 2.A.5)**.** *Let V be a vector space over $\mathbb{C}$.*

(a) Show that V is also a vector space over $\mathbb{R}$. Show that if V is finite dimensional over $\mathbb{C}$, then it is finite dimensional over $\mathbb{R}$.

(b) Show that for $V = \mathbb{C}$, the vectors $1 + i$ and $1 - i$ are linearly independent over $\mathbb{R}$, but linearly dependent over $\mathbb{C}$.

Solution. □

Problem 11 (Axler 2.A.14)**.** *Prove that V is infinite dimensional if and only if there is a sequence $v_1, v_2, \ldots$ of vectors in V such that $(v_1, \ldots, v_n)$ is linearly independent for every integer n.*

Solution. □

Problem 12. *Let S be a set and let F be a field. Prove that the vector space $V = \{f : S \to F\}$ of all maps from S to F is finite dimensional if and only if S is finite.*

Solution. □