Homework 2

Math 25a

Due September 13, 2017

Topics covered: cardinality, equivalence relations

Instructions:

- The homework is divided into one part for each CA. You will submit each part to the corresponding CA's mailbox on the second floor of the science center. The assignment is due 9:59am.
- If your submission to any one CA takes multiple pages, then staple them together. A stapler is available in the Cabot library in the science center.
- If you collaborate with other students, please mention this near the corresponding problems.
- For the first two assignments, we would like you to keep track of how long it takes you to complete the entire assignment. Please write that somewhere on your solutions, so we can adjust the difficulty accordingly.
- Some problems from this assignment come from Simmon's book *Introduction to topology and modern analysis*. I've indicated this next to the problems (e.g. Simmons 1.2.3 means problem 3 from the exercises to Section 2 of Chapter 1. You can find a digital copy of Simmons' book on the course website.

1 For Charlie

Problem 1 (Simmons 1.6.1)**.** *Prove the set of all rational numbers (positive, negative, and zero) is countable.*

Solution. □

Problem 2 (Simmons 1.6.2)**.** *Let $X_1, X_2, \ldots$ be a countable collection of sets. Show that if X_i is countable for each i, then the union $\cup_i X_i$ is also countable.*

Solution. □

Problem 3. *Suppose X is countable and $f : X \to Y$ is surjective. Prove that Y is countable.*

Solution. □

2 For Ellen

Problem 4 (Simmons 1.5.4)**.** *Determine which of the three properties reflexivity, symmetry, transitivity is true for the following relations on the positive integers: m is less than or equal to n ($m \leq n$), m is less than n ($m < n$), m divides n ($m \mid n$). (You don't need to give proofs.)*

Solution. □

Problem 5 (Simmons 1.5.5)**.** *Give an example of a nonempty set X with a relation $\sim$ that is*

(a) reflexive, but not symmetric or transitive;

(b) symmetric, but not reflexive or transitive;

(c) transitive, but not reflexive or symmetric;

(d) reflexive and symmetric, but not transitive;

(e) reflexive and transitive, but not symmetric;

(f) symmetric and transitive, but not reflexive.

(Remark: The hardest part about this problem is figuring out what set to consider.)

Solution. □

Problem 6 (Simmons 1.5.6)**.** *Let X be a nonempty set with a relation $\sim$. Consider the following argument.*

Claim: If $\sim$ is symmetric and transitive, then it is also reflexive.

Proof: $x \sim y \Rightarrow y \sim x$; also $x \sim y$ and $y \sim x \Rightarrow x \sim x$. Therefore $x \sim x$ for every $x \in X$.

This argument is at odds with part (f) of the previous problem. Where is the flaw?

Solution. □

3 For Natalia

Problem 7. *Fix an an integer $n \geq 2$. Consider the relation on $\mathbb{Z}$ defined by*

$$a \sim b \text{ if } (a-b) \text{ is an integer multiple of } n.$$

Show that $\sim$ is an equivalence relation. Let $[a]$ denote the equivalence class of an integer a, and let $\mathbb{Z}/n\mathbb{Z}$ denote the set of equivalence classes. Show that $\mathbb{Z}/n\mathbb{Z}$ is equal to the set $\{[0],[1],\dots,[n-1]\}$.

Solution. □

Problem 8. *Define addition and multiplication on $\mathbb{Z}/n\mathbb{Z}$ by the rules*

$$[a]+[b]=[a+b] \quad \text{and} \quad [a]\cdot[b]=[ab].$$

(a) *Prove addition is well-defined on $\mathbb{Z}/n\mathbb{Z}$. Conclude that $[0]$ is the additive identity (i.e. $[a]+[0]=[a]$ for each $[a]$), and that the additive inverse to $[a]$ is $[n-a]$.*

(b) *Prove that multiplication $[a][b]=[ab]$ is well-defined on $\mathbb{Z}/n\mathbb{Z}$. Conclude that $[1]$ is the multiplicative identity.*

(c) *Prove the distributive law holds $[a]([b]+[c])=[a][b]+[a][c]$. You may use that the distributive law holds for $\mathbb{Z}$.*

Solution. □

Problem 9. (a) *Show that for $n=4$, the element $[2]$ does not have a multiplicative inverse. (In fact if $n=ab$ for some $a,b\in\mathbb{Z}$ and $a,b>1$, then the class $[a]$ is nonzero and does not have a multiplicative inverse.)*

(b) *Show that for $n=7$ every class $[a]\neq[0]$ has a multiplicative inverse (work out the multiplication explicitly).*

Solution. □

4 For Michele

Problem 10. *Let a and b be integers. The greatest common divisor d of a and b is defined as the largest positive integer that divides both a and b. For example, if $b = p$ is a prime that does not divide a, then the greatest common divisor of a and b is 1. The Euclidean algorithm gives a way to find the greatest common divisor. Using this algorithm one can show that if d is the greatest common divisor of a and b, then $d = ax + by$ for some integers x and y. For more information google "Euclidean algorithm" and "Bezout's identity".*

Use these facts to show that if $n = p$ is prime, then every nonzero element $[a]$ of $\mathbb{Z}/n\mathbb{Z}$ has a multiplicative inverse.

Solution. $\square$

Problem 11. *Fix a set X, and let $P(X)$ denote the power set. In class we showed that there is no surjection $f : X \to P(X)$ if A is a countable set (using Cantor's diagonalizable argument). Prove that there is no surjection $f : X \to P(X)$ for* any *set X (countable or not). (Hint: the argument is similar in flavor to the argument of Russell's paradox.)*

Solution. $\square$

5 LaTeX guide

If you want to make a table, here is an example

	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1