Homework 10

Math 25a

Due November 29, 2017

Topics covered: inner product spaces, spectral theorem, positive operators, isometries, singular values, determinants

Instructions:

- The homework is divided into one part for each CA. You will submit each part to the corresponding CA's mailbox on the second floor of the science center.
- If your submission to any one CA takes multiple pages, then staple them together. A stapler is available in the Cabot library in the science center.
- If you collaborate with other students, please mention this near the corresponding problems.
- Some problems from this assignment come from the 3rd edition of Axler's book. I've indicated this next to the problems. For example, Axler 1.B.4 means problem 4 from the exercises to Section B of Chapter 1. Sometimes the problem in Axler is slightly different, so make sure you do the problem as listed in the assignment.

1 For Ellen

Problem 1 (Axler 7.B.14). *Let U be a finite dimensional real vector space and fix $T \in L(U)$. Prove that the following are equivalent.*

(a) There exists a basis $u_1, \ldots, u_n \in U$ of eigenvectors of T.

(b) There exists an inner product on U such that T is self-adjoint with respect to this inner product.

Solution. □

Problem 2 (Axler 7.C.13). *Let V be a finite-dimensional inner product space and $S \in L(V)$. For each statement, prove or give a counterexample:*

(a) If S is an isometry, then there exists an orthonormal basis $e_1, \ldots, e_n \in V$ such that $||Se_j|| = 1$ for every j.

(b) If there exists an orthonormal basis $e_1, \ldots, e_n \in V$ such that $||Se_j|| = 1$ for every j, then S is an isometry.

Solution. □

Problem 3. *Consider $V = \mathbb{R}^3$ with the standard inner product. Show that if $S \in L(V)$ is an isometry, then there exists $x \in V$ such that $S^2 x = x$. (It might first help to show that S has an eigenvector.)*

Solution. □

2 For Charlie

Problem 4 (Axler 7.C.9). *Prove or disprove: the identity $I : \mathbb{R}^2 \to \mathbb{R}^2$ has infinitely many square roots.*

Solution. □

Problem 5 (Axler 7.C.7). *Let V be a finite dimensional real inner product space. Assume $T \in L(V)$ is positive. Show that T is invertible if and only if $\langle Tv, v\rangle > 0$ for every $v \in V \setminus \{0\}$.*

Solution. □

Problem 6 (Axler 7.D.7). *Define $T \in L(\mathbb{R}^3)$ by $T(x_1, x_2, x_3) = (x_3, 2x_1, 3x_2)$. Find (explicitly) an isometry $S \in L(\mathbb{R}^3)$ such that $T = S\sqrt{T^*T}$. Write you final answer in the form $S(x_1, x_2, x_3) = (\cdots)$.*

Solution. □

3 For Michele

Problem 7 (Axler 7.D.12)**.** *Prove or give a counterexample: if $T \in L(V)$, then the singular values of T^2 equal the squares of the singular values of T.*

Solution. □

Problem 8 (Axler 7.D.13)**.** *Suppose $T \in L(V)$. Prove that T is invertible if and only if 0 is not a singular value of T.*

Solution. □

Problem 9 (Axler 7.D.15)**.** *Suppose $S \in L(V)$. Prove that S is an isometry if and only if all the singular values of S equal 1.*

Solution. □

4 For Natalia

Problem 10 (Axler 10.B.1)**.** *Let V be a real inner product space. Show that if $T \in L(V)$ has no eigenvalues, then $\det T$ is positive.*

Solution. □

Problem 11 (Axler 10.B.5)**.** *Prove or give a counterexample: if $S, T \in L(V)$, then $\det(S+T) = \det S + \det T$.*

Solution. □

Problem 12 (Axler 10.B.4)**.** *Suppose $T \in L(V)$ and $c \in F$. Prove that $\det(cT) = c^{\dim V} \det T$. How can you understand this geometrically?*

Solution. □