RESEARCH STATEMENT

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My research area is geometric topology. I primarily study manifolds, fiber bundles, and group actions. My research also has ties to geometric group theory and arithmetic groups. Below I focus on three aspects of my work.

1. Group actions on manifolds/Nielsen realization. A basic form of the Nielsen realization problem asks, for a manifold M, when a group of symmetries of $\pi_1(M)$ can be "realized" by a group of symmetries of M. Versions of this problem were originally posed by Nielsen and Thurston. Very little is known for infinite groups, a problem that relates to flat connections on bundles. My work is primarily focused on (1) finding new examples of groups that are not realizable [Tsh15, ST16, GKT21] and (2) classifying special families of realizations that do exist [BT23, CT22, BCT23, BKKT23].

2. Arithmetic groups and manifold bundles. Arithmetic groups (e.g. $\mathrm{SL}_n(\mathbb{Z})$) arise in my work via the monodromy of manifold bundles. I have used arithmetic groups to (1) understand geometric, and topological properties of bundles [Tsh15, GKT21, ST20] and (2) to produce new characteristic classes of manifold bundles [Tsh21].

3. Aspherical manifolds and hyperbolic groups. The Wall conjecture predicts that every finitely-presented Poincaré duality group G is the fundamental group of a closed aspherical manifold $\pi_1(M) = G$. In addition, if G is hyperbolic and 3-dimensional, the Cannon conjecture predicts $G \hookrightarrow \mathrm{PSL}_2(\mathbb{C})$ is Kleinian. With collaborators, I have established new cases of Wall's conjecture for certain hyperbolic groups [LT19], and studied when a relatively hyperbolic group is a relative Poincaré duality group, which relates to a relative version of Cannon's conjecture [TW20].

1. Group actions and Nielsen realization

For a manifold M, there is a natural surjection $\mathrm{Homeo}(M) \twoheadrightarrow \mathrm{Mod}(M)$ from the homeomorphism group $\mathrm{Homeo}(M)$ to the mapping class group $\mathrm{Mod}(M) := \pi_0 \mathrm{Homeo}(M)$. The Nielsen realization problem asks, for each subgroup $G < \mathrm{Mod}(M)$, if there is a solution to the following lifting problem.

$$\begin{array}{ccc} & & \mathrm{Homeo}(M) \\ & \overset{L}{\nearrow} & \downarrow \\ G & \hookrightarrow & \mathrm{Mod}(M) \end{array}$$

When a lift exists, we say $G < \mathrm{Mod}(M)$ is *realizable*.

The optimist's conjecture.

For a surface S_g, Nielsen originally asked if every finite $G < \mathrm{Mod}(S_g)$ is realized by a group of isometries of S_g with respect to some hyperbolic metric. This was proved by Kerckhoff [Ker83].

A finite-order homeomorphism is an example of a *Nielsen–Thurston representative*, which are particularly simple elements in each isotopy class. We propose the following conjecture that would generalize Kerckhoff's theorem.

Conjecture 1 (optimist's conjecture)**.** If $G < \mathrm{Mod}(S_g)$ is realizable in $\mathrm{Homeo}(S_g)$, then there is a realization by Nielsen–Thurston representatives.

Conjecture 1 holds for every realizable $G < \mathrm{Mod}(S_g)$ known to the author. This includes finite groups, free groups, abelian groups, Veech groups, certain right-angled Artin groups, ... In each case, the relations in G are simple enough that they can be satisfied by Nielsen–Thurston representatives.

For $G = \mathrm{Mod}(S_g)$, Conjecture 1 predicts that $\mathrm{Homeo}(S_g) \to \mathrm{Mod}(S_g)$ does not split for $g \geq 2$. This was originally asked by Thurston in Kirby's problem list and proved by Markovic [Mar07].

Problem 2. Find new examples of infinite $G < \mathrm{Mod}(S_g)$ that are not realizable.

For example, Salter and I [ST16] consider surface braid groups.

Theorem 3 (Realizing braid groups)**.** *Let $B_n \cong \mathrm{Mod}(\mathbb{D}^2, n)$ denote the n-stranded braid group. For $n \geq 5$, the braid group B_n is not realizable by diffeomorphisms. Furthermore, surface braid groups $B_n(S_g) < \mathrm{Mod}(S_{g,n})$ are not realizable by diffeomorphisms when $n \geq 6$.*

This has since been improved by L. Chen who shows B_n is not realizable by homeomorphisms [Che19]. When $g \geq 2$ and $n \geq 6$, Theorem 3 gives an alternative proof of a result of Bestvina–Church–Souto [BCS13].

There are many examples of Problem 2 to consider. Some that I find interesting are (i) braid subgroups generated by a "chain" of Dehn twists, (ii) the handle-dragging subgroup $\pi_1(US_g) < \mathrm{Mod}(S_{g+1})$, (iii) the purely pseudo-Anosov surface subgroups constructed by Kent–Leininger [KL24]. This is an area where more techniques are needed.

3-manifolds.

For general 3-manifolds M^3, it's unclear how to formulate a version of Conjecture 1 even for finite $G < \mathrm{Mod}(M^3)$.

Problem 4. Give a criterion, applying to all 3-manifolds M, that characterizes when finite $G < \mathrm{Mod}(M)$ is realizable.

Currently Problem 4 is only solved for special families of M. It would be interesting to solve this problem for special G (e.g. cyclic) and arbitrary M. L. Chen and I [CT22] solve Problem 4 for G generated by sphere twists (the 3D analogue of Dehn twists for surfaces) for any M.

Theorem 5 (Realizing sphere twists)**.** *Let M be a closed, oriented 3-manifold. A subgroup $G < \mathrm{Mod}(M)$ generated by sphere twists is realizable if and only if G is cyclic and M is a connected sum of lens spaces.*

Combining Theorem 5 with older works, it should be possible to solve Problem 4.

Nielsen realization and the topology of bundles.

For $G = \pi_1(B)$, a homomorphism $G \to \mathrm{Diff}(M)$ determines an M-bundle $E \to B$ with a *flat connection* (a foliation with certain properties). Whether or not a given bundle admits a flat connection is a poorly understood problem.

Problem 6. Give new examples of M-bundles that are not flat (i.e. have no flat connection).

This can be approached with Nielsen realization: if $G \to \mathrm{Mod}(M)$ is not realizable in $\mathrm{Diff}(M)$, then no bundle with this monodromy is flat.

Morita [Mor87] gave the first examples of non-flat S_g-bundles. Giansiracusa–Kupers and I [GKT21] apply similar ideas to the K3 4-manifold.

Theorem 7 (Non-flat K3 bundles)**.** *Let M^4 be the* K3 *surface.*

- *Finite-index $G < \mathrm{Mod}(M)$ are not realizable by diffeomorphisms.*
- *The tautological M-bundle over the moduli space of Einstein metrics on M is not flat.*

It would be interesting to extend Morita's argument to other 4-manifolds.

The following theorem from [Tsh15] also addresses Problem 6. It builds on work of Bestvina–Church–Souto [BCS13] who solved the surface case.

Theorem 8. *Let $M = \Gamma\backslash G/K$ be locally symmetric of noncompact type. If $\mathbb{Q}$-rank$(\Gamma) \geq 1$ or M has a nonzero Pontryagin class, then*

- *The pointpushing group $\pi_1(M, *) < \mathrm{Mod}(M, *)$ is not realizable in* $\mathrm{Diff}(M, *)$.
- *The product bundle $M \times M \to M$ is not flat relative to the diagonal $\Delta \subset M \times M$.*

The theorem applies, for example, to compact complex-hyperbolic manifolds $\Gamma\backslash\mathbb{C}H^n$ and finite manifold covers of $\mathrm{SL}_n(\mathbb{Z})\backslash \mathrm{SL}_n(\mathbb{R})/\mathrm{SO}(n)$, $n \geq 3$.

Classifying realizations.

There are very few known examples of natural actions of mapping class groups $\mathrm{Mod}(M)$ and outer automorphism groups $\mathrm{Out}(\pi_1(M))$ on manifolds. Examples include:

(1) (Cheeger [Gro00]) $\mathrm{Mod}(S_g)$ acts on the unit tangent bundle US_g.
(2) $\mathrm{Out}(\mathbb{Z}^n) \cong \mathrm{GL}_n(\mathbb{Z})$ acts on the n-torus T^n.
(3) (Mostow rigidity) For hyperbolic M^n, $n \geq 3$, $\mathrm{Out}(\pi_1 M)$ acts on M.

We view each of these as a rare gem and seek to prove rigidity results that quantify this; see Conjectures 9 and 11.

Mod(S_g) acting on 3-manifolds. The lack of examples of actions of $\mathrm{Mod}(S_g)$ on 3-manifolds leads us to the following conjecture.

Conjecture 9. If $\mathrm{Mod}(S_g)$ acts faithfully on a 3-manifold M^3, then $M = US_g$ and the action is conjugate to Cheeger's construction.

Conjecture 9 appears to be out of reach in general, but it contains interesting cases that are tractable. For example, the action $\mathrm{Mod}(S_g) \curvearrowright US_g$ is not smooth, so Conjecture 9 implies, in particular, that this action is not homotopic to a smooth action. This was proved by Souto [Sou10] (for the extended mapping class group).

As another special case of Conjecture 9, for a circle bundle $M \to S_g$, the natural surjection

$$\mathrm{Homeo}(M) \twoheadrightarrow \mathrm{Mod}(M) \twoheadrightarrow \mathrm{Mod}(S_g)$$

should split only for $M = US_g$. Evidence for this is provided by the following theorem, proved in joint works with L. Chen and my student Alina al Beaini [CT23, BCT23].

Theorem 10 (Realizing $\mathrm{Mod}(S_g)$ on circle bundles)**.** *Fix an oriented circle bundle $M \to S_g$ and let $e(M) \in H^2(S_g;\mathbb{Z}) \cong \mathbb{Z}$ be its Euler class/number.*

(i) $\mathrm{Mod}(M) \twoheadrightarrow \mathrm{Mod}(S_g)$ *splits* $\Leftrightarrow 2-2g = \chi(S_g)$ *divides* $e(M)$.
(ii) $\mathrm{Homeo}(S_g \times S^1) \twoheadrightarrow \mathrm{Mod}(S_g)$ *does not split for infinitely many* g.

Actions of $\mathrm{SL}_n(\mathbb{Z})$ on n-manifolds. Similar to Souto's result [Sou10], for any smooth structure $\mathfrak{T}$ on T^n, we can ask whether the action $\mathrm{GL}_n(\mathbb{Z}) \curvearrowright T^n$ is homotopic to a smooth action on $\mathfrak{T}$, i.e. whether one can split the map

$$\mathrm{Diff}(\mathfrak{T}) \to \mathrm{Out}(\pi_1\mathfrak{T}) \cong \mathrm{GL}_n(\mathbb{Z}). \tag{1}$$

Conjecture 11. The map (1) splits only for the standard torus $\mathfrak{T} = T^n$.

Conjecture 11 is implied by a conjecture of Fisher–Melnick [FM22] that proposes a classification of actions $\mathrm{SL}_n(\mathbb{Z}) \curvearrowright M^n$ (as part of Zimmer program).

Bustamante–Krannich–Kupers and I [BKKT23] prove a partial result. Below Σ is a smooth homotopy n-sphere and $\eta(\Sigma) \in \Theta_{n+1}$ is a group-valued invariant defined by Milnor–Munkres–Novikov.

Theorem 12 (Actions of $\mathrm{SL}_n(\mathbb{Z})$ on homotopy tori)**.** *Let* $\mathfrak{T} = T^n \# \Sigma$*,* $n \geq 5$*.*

(i) $\mathrm{Mod}(\mathfrak{T}) \to \mathrm{SL}_n(\mathbb{Z})$ *splits if and only if* $\eta(\Sigma)$ *is divisible by* 2*.*
(ii) If $\eta(\Sigma)$ *is not divisible by* 2*, then* $\mathrm{Diff}(\mathfrak{T}) \to \mathrm{GL}_n(\mathbb{Z})$ *does not split. In addition, every homomorphism* $\mathrm{SL}_n(\mathbb{Z}) \to \mathrm{Diff}(\mathfrak{T})$ *is trivial.*

Actions of $\mathrm{Out}(\pi_1 M)$ on hyperbolic manifolds. Let $\mathfrak{M}$ be a smooth structure on a hyperbolic manifold M. We ask whether a finite group action $G \curvearrowright M$ is homotopic to a smooth action on $\mathfrak{M}$. This relates to splitting

$$\mathrm{Diff}(\mathfrak{M}) \to \mathrm{Out}(\pi_1 \mathfrak{M}) \cong \mathrm{Out}(\pi_1 M),$$

a problem posed for negatively-curved $\mathfrak{M}$ by Schoen–Yau [SY79], generalizing Nielsen's question. A negative answer was given by Farrell–Jones [FJ90] with examples of the form $\mathfrak{M} = M \# \Sigma$ where Σ is a homotopy n-sphere.

Bustamante and I [BT23] (building on [BT22]) extend Farrell–Jones [FJ90] (with a stronger conclusion) in dimension 7.

Theorem 13. *Let* M *is a hyperbolic* 7*-manifold, and assume* $\mathrm{Isom}(M)$ *acts freely on* M*. Let* Σ *be a homotopy* 7*-sphere.*

(1) An action $G \curvearrowright M$ *is homotopic to a smooth action on* $M \# \Sigma$ *if and only if* Σ *is divisible by* $|G|$ *in* Θ_n*.*
(2) Each action $G \curvearrowright M \# \Sigma$ *is obtained from an action on* M *by equivariant connected sum.*

2. Arithmetic groups, monodromy, and cohomology

Monodromy of holomorphic bundles.

A surface bundle $S_g \to E \to B$ has a monodromy representation

$$\rho : \pi_1(B) \to \mathrm{Mod}(S_g) \to \mathrm{Sp}_{2g}(\mathbb{Z}).$$

In general $\mathrm{Image}(\rho) < \mathrm{Sp}_{2g}(\mathbb{Z})$, called the *monodromy group*, can be any subgroup, but if $E \to B$ is a *holomorphic* fibration, then Deligne [Del87] proved that the Zariski closure of its monodromy group $\Gamma_E < \mathrm{Sp}_{2g}(\mathbb{Z})$ is semi-simple, and Griffiths–Schmid [GS75] asked:

Question 14 (Griffiths–Schmid)**.** When is the monodromy group of a holomorphic S_g-fibration an arithmetic group?

Both arithmetic and non-arithmetic monodromy groups occur [DM86, Ven14].

There is an instance of Question 14 for every cover $S_g \to S_h$ of surfaces (possibly branched): there is a holomorphic S_g-bundle $E \to \mathcal{M}'_h$, where

$\mathcal{M}'_h$ is a finite cover of the moduli space $\mathcal{M}_h$ of genus-h Riemann surfaces. For these examples, when $h \geq 3$, the arithmeticity question is related to a conjecture of Putman–Wieland [PW13] that would imply that $\mathrm{Mod}(S_g)$ does not virtually surject to $\mathbb{Z}$.

My student Trent Lucas has studied Question 14 for all covers $S_g \to S_h$ with $g \leq 3$. His analysis includes 17 previously unstudied cases, and he finds that the monodromy group is always arithmetic when $g \leq 3$ [Luc24].

Salter and I [ST20] answer Question 14 for certain holomorphic S_g-bundles constructed by Atiyah–Kodaira. As a topological consequence, we compute the number of fiberings of these examples, a result motivated by Thurston's theory of fibering 3-manifolds.

Theorem 15 (Atiyah–Kodaira bundles)**.** *Let $S_g \to E \to S_h$ be one of the classical holomorphic families constructed by Atiyah and Kodaira. If h is sufficiently large, then*

(i) *the image of the monodromy $\pi_1(S_h) \to \mathrm{Mod}(S_g) \to \mathrm{Sp}_{2g}(\mathbb{Z})$ is an arithmetic group;*
(ii) *the 4-manifold E fibers as a surface bundle in exactly two ways.*

Unstable cohomology.

In studying M-bundles, a fundamental problem is to compute the ring of characteristic classes $H^*(B\,\mathrm{Diff}(M))$. When $M_g^{2d} = \#_g(S^d \times S^d)$, this ring is known in a range $* \ll g$ (Mumford's conjecture) [GRW14, MW07, Mum83]. Little is known about $H^*(B\,\mathrm{Diff}(M_g))$ when $* \geq g$, although there have been recent important results [CGP18].

Problem 16. Give new constructions of characteristic classes, i.e. elements in $H^*(B\,\mathrm{Diff}(M))$.

In [Tsh21], I produce new classes in $H^g(B\,\mathrm{Diff}'(M_g^{2d}))$ and for certain finite-index "congruence" subgroups $\mathrm{Diff}'(M_g) < \mathrm{Diff}(M_g)$, when $d \gg g$ is even. This is related to arithmetic groups via the following theorem of [Tsh21].

Theorem 17 (New cohomology for lattices in $\mathrm{SO}(p,q)$)**.** *Fix $1 \leq p \leq q$ with $p + q \geq 3$. Let $\Lambda \subset \mathbb{R}^{p+q}$ be a lattice with an integral bilinear form of signature (p,q). There exists a finite-index subgroup $\Gamma < \mathrm{SO}(\Lambda)$ so that $\dim H^p(\Gamma;\mathbb{Q}) \neq 0$.*

The cohomology in Theorem 17 comes from flat p-dimensional tori in the associated locally symmetric manifolds. D. Studenmund and I [ST22] compute lower bounds on the dimension of the subspace generated by these classes. For example, for SL_{n+1} we show:

Theorem 18 (Cohomology growth, congruence subgroups of $\mathrm{SL}_{n+1}(\mathbb{Z})$)**.** *Fix $n \geq 2$, and let $\Gamma(s) < \mathrm{SL}_{n+1}(\mathbb{Z})$ denote the level-s principal congruence* *subgroup. Then for each prime p,*

$$\dim H_n(\Gamma(p^\ell);\mathbb{Q}) \gtrsim |\operatorname{SL}_{n+1}(\mathbb{Z}):\Gamma(p^\ell)|^{\frac{n+1}{n^2+2n}} \quad \textit{for } \ell \gg 0.$$

The above approach for constructing characteristic classes suggests a way to find new cohomology in finite-index subgroups of $\operatorname{Mod}(S_g)$. I plan to explore this in future work.

Arithmetic mapping tori.

By a theorem of Margulis [Mar91], a lattice Γ in a semisimple Lie group is arithmetic if and only if Γ has infinite index in its commensurator. In contrast, no general arithmeticity characterization for lattices in *solvable* Lie groups is known. In [Tsh22] for solvable lattices of the form $\Gamma = \mathbb{Z}^n \rtimes_A \mathbb{Z}$, I provide an arithmeticity criterion in terms of the eigenvalues of A, building in particular on work of Grunewald–Platonov [GP98].

Theorem 19 (Arithmeticity criterion)**.** *Fix $A \in \operatorname{GL}_n(\mathbb{Z})$ hyperbolic and semisimple. Then $\mathbb{Z}^n \rtimes_A \mathbb{Z}$ is arithmetic if and only if $\log(\mu)$ and $\log(\nu)$ are commensurable for any real monomials μ, ν in the eigenvalues of A and their inverses.*

It would be interesting to prove an analogous theorem that characterizes arithmeticity for groups $\Gamma = \pi_1(S_g) \rtimes_\phi \mathbb{Z}$ with $\phi \in \operatorname{Out}(\pi_1(S_g)) \cong \operatorname{Mod}(S_g)$ pseudo-Anosov, in terms of some property ϕ.

3. Aspherical manifolds and hyperbolic groups

Wall and Cannon conjectures.

In the classification of aspherical manifolds, the basic existence and uniqueness problems are as follows.

Conjecture 20.
1. (Wall) If G is a finitely-generated Poincaré duality group, then there exists a closed aspherical manifold M with $\pi_1(M) \cong G$.
2. (Borel) Two closed aspherical manifolds M, M' with $\pi_1(M) \cong \pi_1(M')$ are homeomorphic.

These conjectures hold for many groups/manifolds coming from geometry. For example, Bartels–Lück–Weinberger [BLW10] prove the Wall conjecture for hyperbolic groups with sphere boundary $\partial G \cong S^n$, $n \geq 5$. Lafont and I [LT19] prove a relative version that extends [BLW10].

Theorem 21 (Wall conjecture, special case)**.** *Let G be a hyperbolic group whose Gromov boundary is an $(n-2)$-dimensional Sierpinski space. If $n \geq 7$, then $G \cong \pi_1(M)$ where M is a compact aspherical manifold with aspherical boundary.*

The Cannon conjecture, a version of Wall's conjecture in geometric group theory and low-dimensional topology, predicts that a torsion-free hyperbolic group G with boundary $\partial G \cong S^2$ the 2-sphere is the fundamental group of a closed hyperbolic 3-manifold. Bestvina–Mess [BM91] showed that $\partial G \cong S^2$ implies that G is a Poincaré duality group (a necessary condition for $G \cong \pi_1(M^3)$).

In connection to a relative version of Cannon's conjecture, G. Walsh and I [TW20] show the following result, also proved by Manning–Wang [MW20].

Theorem 22 (Duality for relatively hyperbolic groups)**.** *Let $(G, \mathcal{P})$ be a relatively hyperbolic group. Then $(G, \mathcal{P})$ is a 3-dimensional Poincaré duality pair if and only if the Bowditch boundary $\partial(G, \mathcal{P})$ is the 2-sphere.*

Hyperbolization of groups.

Thurston's geometrization implies that a closed aspherical 3-manifold is hyperbolic if its fundamental group does not contain $\mathbb{Z}^2$. Gromov proposed a group-theoretical analogue: a group G (with a finite $K(G,1)$) that contains no Baumslag–Solitar subgroup is necessarily hyperbolic. A counterexample has been found [IMM23] via a construction of hyperbolic 5-manifolds, but Gromov's conjecture might be correct for e.g. surface group extensions

$$1 \to \pi_1(S_g) \to G \to \Gamma \to 1. \tag{2}$$

For such G, Gromov's conjecture specializes to a conjecture of Farb–Mosher [FM02].

Conjecture 23 (Farb–Mosher)**.** If $\Gamma < \mathrm{Mod}(S_g)$ and every nontrivial element of $\mathrm{Mod}(S_g)$ is pseudo-Anosov, then G is convex cocompact in $\mathrm{Mod}(S_g)$ (and therefore the extension group G in (2) is hyperbolic).

New examples of [KL24] could (as of this writing) be counterexamples to Conjecture 23, but the conjecture is known for many classes of groups, e.g. [KLS09, DKL14, KMT17]. In [Tsh24], I verify Conjecture 23 for subgroups of the genus-2 *Goeritz group* $\mathcal{G}$, the subgroup of $\mathrm{Mod}(S_2)$ of mapping classes that extend to the genus-2 Heegaard splitting of S^3. In the process, I also characterize reducible elements in $\mathcal{G}$.

Theorem 24 (Pseudo-Anosovs in the Goeritz group)**.** *Let $\mathcal{G} < \mathrm{Mod}(S_2)$ be the genus-2 Goeritz group.*

(i) *Conjecture 23 is true for subgroups of $\mathcal{G}$.*
(ii) *An element of $\mathcal{G}$ is reducible if and only if it stabilizes one of the following: (a) a primitive multi-disk, (b) a reducing sphere, or (c) an embedding of the figure-8 knot on $S \subset S^3$.*

Combined with a known presentation for $\mathcal{G}$, (ii) gives an effective way to test if an element of $\mathcal{G}$ is pseudo-Anosov and to construct explicit purely pseudo-Anosov subgroups.

References

[BCS13] M. Bestvina, T. Church, and J. Souto, *Some groups of mapping classes not realized by diffeomorphisms*, Comment. Math. Helv. **88** (2013), no. 1, 205–220. MR 3008918

[BCT23] A. Al Beaini, L. Chen, and B. Tshishiku, *Surface mapping class group actions on 3-manifolds*, 2023.

[BKKT23] M. Bustamante, M. Krannich, A. Kupers, and B. Tshishiku, *Mapping class groups of exotic tori and actions by* $\mathrm{sl}_d(F)$, 2023.

[BLW10] A. Bartels, W. Lück, and S. Weinberger, *On hyperbolic groups with spheres as boundary*, J. Differential Geom. **86** (2010), no. 1, 1–16. MR 2772544

[BM91] M. Bestvina and G. Mess, *The boundary of negatively curved groups*, J. Amer. Math. Soc. **4** (1991), no. 3, 469–481. MR 1096169

[BT22] M. Bustamante and B. Tshishiku, *Symmetries of exotic negatively curved manifolds*, J. Differential Geom. **120** (2022), no. 2, 231–250. MR 4385117

[BT23] Mauricio Bustamante and Bena Tshishiku, *Symmetries of exotic aspherical space forms*, 2023.

[CGP18] M. Chan, S. Galatius, and S. Payne, *Tropical curves, graph homology, and top weight cohomology of* M_g, arxiv:1805.10186, May 2018.

[Che19] L. Chen, *On the nonrealizability of braid groups by homeomorphisms*, Geom. Topol. **23** (2019), no. 7, 3735–3749. MR 4047651

[CT22] L. Chen and B. Tshishiku, *Nielsen realization for sphere twists on 3-manifolds*, 2022.

[CT23] ______, *Mapping class groups of circle bundles over a surface*, 2023.

[Del87] P. Deligne, *Un théorème de finitude pour la monodromie*, Discrete groups in geometry and analysis (New Haven, Conn., 1984), Progr. Math., vol. 67, Birkhäuser Boston, Boston, MA, 1987, pp. 1–19. MR 900821

[DKL14] S. Dowdall, A. Kent, and C. Leininger, *Pseudo-Anosov subgroups of fibered 3-manifold groups*, Groups Geom. Dyn. **8** (2014), no. 4, 1247–1282. MR 3314946

[DM86] P. Deligne and G. D. Mostow, *Monodromy of hypergeometric functions and nonlattice integral monodromy*, Inst. Hautes Études Sci. Publ. Math. (1986), no. 63, 5–89. MR 849651

[FJ90] F. T. Farrell and L. E. Jones, *Smooth nonrepresentability of* $\mathrm{Out}\,\pi_1 M$, Bull. London Math. Soc. **22** (1990), no. 5, 485–488. MR 1082021

[FM02] B. Farb and L. Mosher, *Convex cocompact subgroups of mapping class groups*, Geom. Topol. **6** (2002), 91–152. MR 1914566

[FM22] D. Fisher and K. Melnick, *Smooth and analytic actions of* $sl(n,\mathbf{r})$ *and* $sl(n,\mathbf{z})$ *on closed* n*-dimensional manifolds*, 2022.

[GKT21] J. Giansiracusa, A. Kupers, and B. Tshishiku, *Characteristic classes of bundles of K3 manifolds and the Nielsen realization problem*, Tunis. J. Math. **3** (2021), no. 1, 75–92. MR 4103767

[GP98] F. Grunewald and V. Platonov, *Non-arithmetic polycyclic groups*, C. R. Acad. Sci. Paris Sér. I Math. **326** (1998), no. 12, 1359–1364. MR 1649174

[Gro00] M. Gromov, *Three remarks on geodesic dynamics and fundamental group*, Enseign. Math. (2) **46** (2000), no. 3-4, 391–402. MR 1805410

[GRW14] S. Galatius and O. Randal-Williams, *Stable moduli spaces of high-dimensional manifolds*, Acta Math. **212** (2014), no. 2, 257–377. MR 3207759

[GS75] P. Griffiths and W. Schmid, *Recent developments in Hodge theory: a discussion of techniques and results*, Discrete subgroups of Lie groups and applications to moduli (Internat. Colloq., Bombay, 1973) (1975), 31–127. MR 0419850

[IMM23] G. Italiano, B. Martelli, and M. Migliorini, *Hyperbolic 5-manifolds that fiber over* S^1, Invent. Math. **231** (2023), no. 1, 1–38. MR 4526820

[Ker83] S. Kerckhoff, *The Nielsen realization problem*, Ann. of Math. (2) **117** (1983), no. 2, 235–265. MR 690845

[KL24] A. E. Kent and C. J. Leininger, *Atoroidal surface bundles*, 2024.

[KLS09] A. Kent, C. Leininger, and S. Schleimer, *Trees and mapping class groups*, J. Reine Angew. Math. **637** (2009), 1–21. MR 2599078

[KMT17] T. Koberda, J. Mangahas, and S. Taylor, *The geometry of purely loxodromic subgroups of right-angled Artin groups*, Trans. Amer. Math. Soc. **369** (2017), no. 11, 8179–8208. MR 3695858

[LT19] J.-F. Lafont and B. Tshishiku, *Hyperbolic groups with boundary an n-dimensional Sierpinski space*, J. Topol. Anal. **11** (2019), no. 1, 233–247. MR 3918068

[Luc24] T. Lucas, *Homological representations of low genus mapping class groups*, 2024.

[Mar91] G. A. Margulis, *Discrete subgroups of semisimple Lie groups*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 17, Springer-Verlag, Berlin, 1991. MR 1090825 (92h:22021)

[Mar07] V. Markovic, *Realization of the mapping class group by homeomorphisms*, Invent. Math. **168** (2007), no. 3, 523–566. MR 2299561

[Mor87] S. Morita, *Characteristic classes of surface bundles*, Invent. Math. **90** (1987), no. 3, 551–577. MR 914849

[Mum83] D. Mumford, *Towards an enumerative geometry of the moduli space of curves*, Arithmetic and geometry, Vol. II, Progr. Math., vol. 36, Birkhäuser Boston, Boston, MA, 1983, pp. 271–328. MR 717614

[MW07] I. Madsen and M. Weiss, *The stable moduli space of Riemann surfaces: Mumford's conjecture*, Ann. of Math. (2) **165** (2007), no. 3, 843–941. MR 2335797

[MW20] J. F. Manning and O. H. Wang, *Cohomology and the Bowditch boundary*, Michigan Math. J. **69** (2020), no. 3, 633–669. MR 4132607

[PW13] A. Putman and B. Wieland, *Abelian quotients of subgroups of the mappings class group and higher Prym representations*, J. Lond. Math. Soc. (2) **88** (2013), no. 1, 79–96. MR 3092259

[Sou10] J. Souto, *A remark on the action of the mapping class group on the unit tangent bundle*, Ann. Fac. Sci. Toulouse Math. (6) **19** (2010), no. 3-4, 589–601. MR 2790810

[ST16] N. Salter and B. Tshishiku, *On the non-realizability of braid groups by diffeomorphisms*, Bull. Lond. Math. Soc. **48** (2016), no. 3, 457–471. MR 3509906

[ST20] ———, *Arithmeticity of the monodromy of some Kodaira fibrations*, Compos. Math. **156** (2020), no. 1, 114–157. MR 4036450

[ST22] D. Studenmund and B. Tshishiku, *Counting flat cycles in the homology of locally symmetric spaces*, 2022.

[SY79] R. Schoen and S. T. Yau, *Compact group actions and the topology of manifolds with nonpositive curvature*, Topology **18** (1979), no. 4, 361–380. MR 551017

[Tsh15] B. Tshishiku, *Cohomological obstructions to Nielsen realization*, J. Topol. **8** (2015), no. 2, 352–376. MR 3356765

[Tsh21] ———, *Geometric cycles and characteristic classes of manifold bundles*, Comment. Math. Helv. **96** (2021), no. 1, 1–45, With an appendix by M. Krannich. MR 4228612

[Tsh22] ———, *Arithmeticity of groups $\mathbb{Z}^n \rtimes \mathbb{Z}$*, Indiana Univ. Math. J. **71** (2022), no. 4, 1797–1818. MR 4481100

[Tsh24] ———, *Convex-compact subgroups of the Goeritz group*, Trans. Amer. Math. Soc. **377** (2024), no. 1, 271–322. MR 4684593

[TW20] B. Tshishiku and G. Walsh, *On groups with S^2 Bowditch boundary*, Groups Geom. Dyn. **14** (2020), no. 3, 791–811. MR 4167021

[Ven14] T. N. Venkataramana, *Image of the Burau representation at d-th roots of unity*, Ann. of Math. (2) **179** (2014), no. 3, 1041–1083. MR 3171758