Wall conjecture and hyperbolic groups

Spring Topology and Dynamics Conference

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Joint with Jean Lafont

Part 1:
Wall conjecture

Conjecture (Cannon, 1994). Γ finitely presented.

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Γ torsion-free hyperbolic group, $\partial\Gamma \simeq S^{n\text{-}1}$.

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No. complex hyperbolic manifolds,
or Gromov-Thurston branched covers

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A. Is $\Gamma = \pi_1\Big($ hyperbolic n-manifold$\Big)$? Not necessarily.

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Theorem (Bestvina-Mess). Γ hyperbolic.

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Theorem (Bestvina-Mess). Γ hyperbolic.

$$\Gamma \text{ is PD}(n) \quad \leftrightarrow \quad H^*(\partial\Gamma) \simeq H^*(S^{n-1})$$

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Problem D. (Wall conjecture for hyperbolic groups)

Show if Γ torsion-free hyperbolic group, $\partial\Gamma \simeq S^{n\text{-}1}$, then

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Problem D'. (relative version)

Show if Γ torsion-free hyperbolic group, $\partial\Gamma \simeq \mathcal{S}^{n\text{-}1}$ Sierpinski space, then

$$\Gamma = \pi_1 \left(\begin{array}{c} \text{aspherical topological} \\ (n+1)\text{-manifold with boundary} \end{array} \right).$$

Definition (Cannon). An $(n\text{-}1)$-dimensional Sierpinski space is

$$\mathcal{S}^{n\text{-}1} = \mathrm{S}^{\mathrm{n}} \setminus \bigcup_{i=1}^{\infty} D_i$$

where $\{D_i\}$ dense collection of disjoint open disks,

$\mathrm{S}^{\mathrm{n}} \setminus D_i$ n-cell, $\mathrm{diam}(D_i) \to 0$.

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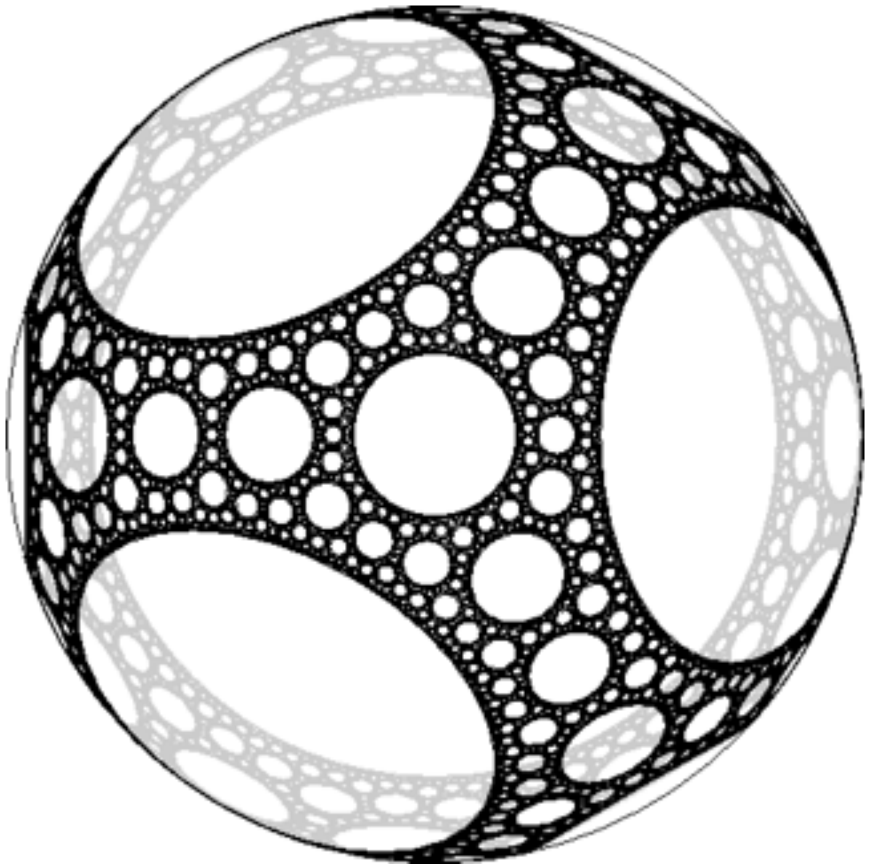

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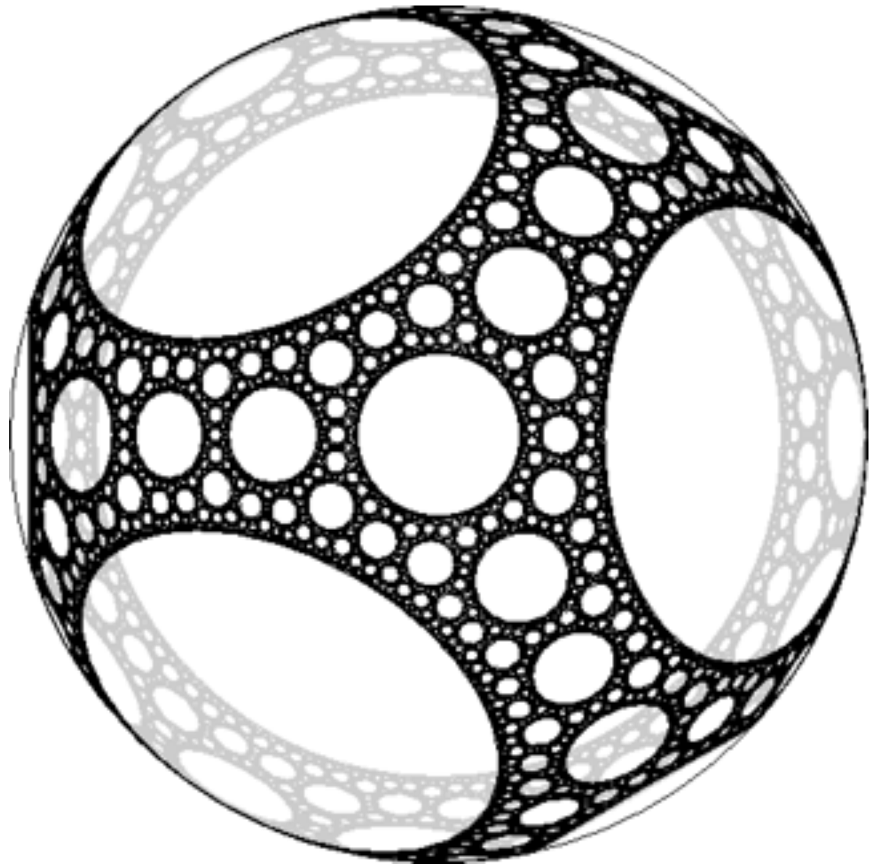

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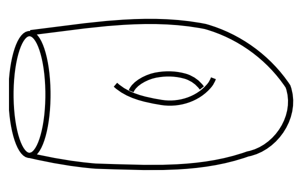
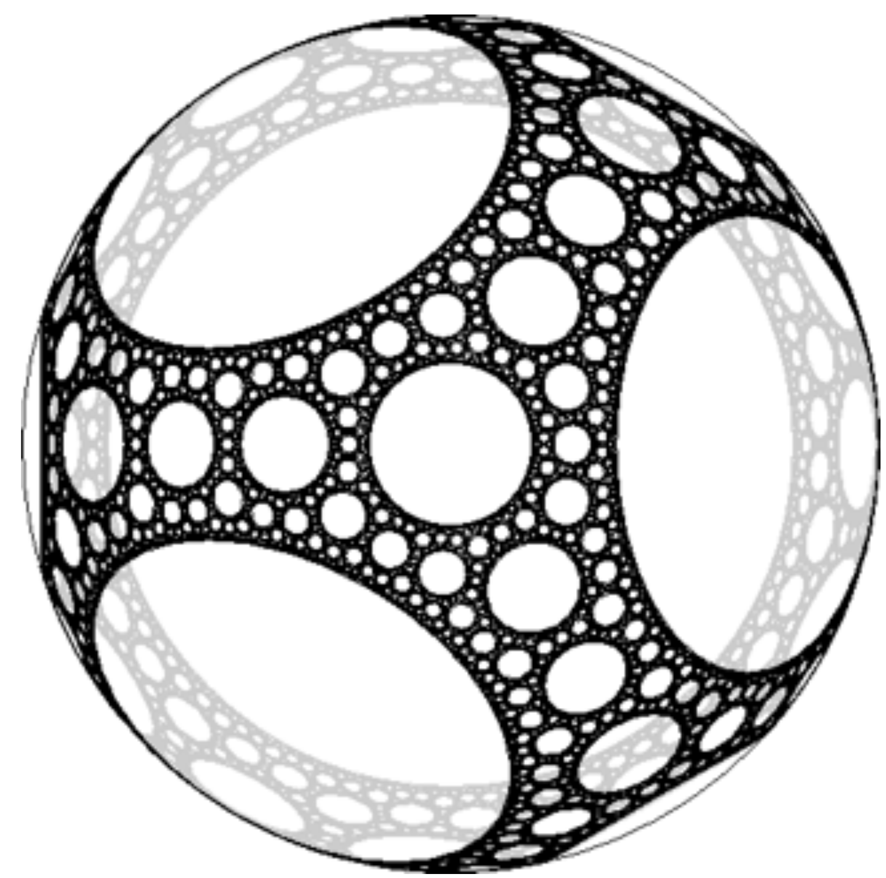
$\Gamma = \pi_1$ (), $\quad \partial\Gamma \simeq \mathcal{S}^0$



Part 2:
Main Theorem

Theorem. $n \geq 6$. Γ torsion-free hyperbolic group

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- (Bartels-Lueck-Weinberger, 2010)

$$\partial\Gamma \simeq S^{n\text{-}1} \quad \Rightarrow \quad \Gamma = \pi_1 \begin{pmatrix} \text{closed aspherical} \\ \text{topological } n\text{-manifold} \end{pmatrix}.$$

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- (Lafont-T, 2015)

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- $\{\Lambda_i\}$ fall into finitely many conjugacy classes
- $\mathrm{G} = \Gamma *_{\{\Lambda i\}} \Gamma$ hyperbolic, $\partial\mathrm{G} \simeq \mathrm{S}^{\mathrm{n}}$
  $\Rightarrow$ $(\Gamma,\{\Lambda_i\})$ is PD(n) pair

Thanks!