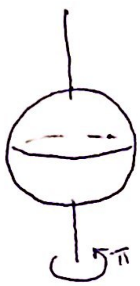
I. Locally Symmetric Manifolds

<u>Defn</u> A Riemannian mfld M is <u>locally symmetric</u> if $\tilde{M}$ is symmetric: (i) $\mathrm{Isom}(\tilde{M}) \curvearrowright M$ transitively

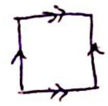
(ii) $\exists$ involutive isometry $\phi: \tilde{M} \to \tilde{M}$ ($\phi^2 = id$) w/ isolated fixed pt.



S^2 symmetric

$\underbrace{\qquad\qquad}_{K>0}$ "cpt type"

$\cancel{\phi}\ \phi = -\mathbb{1} \in \mathrm{Isom}\,\mathbb{E}^2$

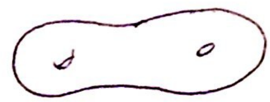


flat $\mathbb{T}^2$ locally symmetric

$\underbrace{\qquad\qquad}_{K=0}$ "Euclidean"

$\mathbb{H}^2$

$z \mapsto -\frac{1}{z}$



(S_g, hyp) $g \geq 2$ locally symmetric

$\underbrace{\qquad\qquad}_{K \leq 0}$ "noncpt type"

~~Noncompact locally symmetric mflds~~ <u>Recipe for noncpt type examples:</u>

- G simple, noncpt, real Lie group w/ finite center
- $K \subset G$ maximal cpt subgp
- $\Gamma \subset G$ ~~lattice~~ torsion free lattice (discrete in G, $\Gamma \backslash G$ finite vol)

$\Rightarrow$ • $X = G/K$ has G invar Riem metric, $K \leq 0$ and is symmetric space. (have involution $\tau \in \mathrm{Aut}(G)$ w/ $K \subset \mathrm{Stab}(\tau)$)

- $M = \Gamma \backslash X$ is a locally symmetric space.

<u>Examples</u> $G = SO(2,1) = SU(1,1) = SL_2\mathbb{R} = Sp_2\mathbb{R}$

$$\begin{array}{ccccc} & \downarrow & \downarrow & \downarrow & \downarrow \\ G & SO(n,1) & SU(n,1) & SL_n\mathbb{R} & Sp_{2n}\mathbb{R} \\ X & \mathbb{H}^n & \mathbb{H}^n_{\mathbb{C}} & & \mathfrak{S}_n \end{array}$$

Easiest lattices: $\Gamma_n(k) = \ker\left(SL_n\mathbb{Z} \longrightarrow SL_n(\mathbb{Z}/k)\right)$

torsion free lattice in $SL_n\mathbb{R}$ $k \geq 3$

<u>Fact</u> $\Gamma = G \cap \Gamma_n(k) \subset G$ is a lattice (torsion free)

II. Pontryagin classes "invariants of mflds that live in coho"

<u>Defn</u> Grassmannian $Gr_n = \{n\text{-planes in } \mathbb{R}^\infty \text{ through } 0.\}$ "∞-diml CW cplx w/ k-skel. a cpt mfld $\forall k$"

$$H^\bullet(Gr_n;\mathbb{Q}) = \begin{cases} \mathbb{Q}[p_1,\dots,p_{k-1},e] & n=2k \\ \mathbb{Q}[p_1,\dots,p_k] & n=2k+1. \end{cases}$$

• Let M^n mfld w/ $M \hookrightarrow \mathbb{R}^\infty$ embedding.

<u>Gauss map</u>: $g: M \longrightarrow Gr_n$
$q \longmapsto T_qM \subset \mathbb{R}^\infty$

$\rightsquigarrow g^*: H^\bullet(Gr_n) \longrightarrow H^\bullet(M)$.

<u>Defn</u> $p_i(M) := g^*(p_i)$ ith Pont class (well-defined)

<u>Prob</u> For $M = \Gamma\backslash G/K$, determine if $p_i(M) \neq 0$.

Open Prob(?) For $M = \mathcal{M}_g$ (moduli space of genus g Riem surf) (or finite mfld cover) determine if $p_i(M) \neq 0$.

III. Pontryagin classes of ~~loc sym mflds~~ $M=\Gamma\backslash G/K$.

A. Approach 1 (Classical)

- $G_{\mathbb{C}}$ complexification (e.g. $SL_n(\mathbb{R})_{\mathbb{C}} = SL_n(\mathbb{C})$.)
- $U \subset G_{\mathbb{C}}$ maximal compact

$$\begin{array}{ccc} G & \hookrightarrow & G_{\mathbb{C}} \\ \cup & & \cup \\ K & \hookrightarrow & U \end{array}$$

Rmk U/K called compact dual.

Assume $M = \Gamma\backslash G/K$ compact

Step1 (Proportionality Principle) $p_i(M) \neq 0 \iff p_i(U/K) \neq 0$.
"fixing G, all cpt loc sym mflds have (non) zero pont classes together"

Step2 (Borel-Hirzebruch) ~~concrete~~ algorithm to determine if $p_i(U/K) \neq 0$.

Examples (Step1 goes a long way)

① $G = SO(n,1)$ $\quad G_{\mathbb{C}} = SO(n+1,\mathbb{C})$ $\quad U = SO(n+1,\mathbb{R})$

$U/K = SO(n+1)/SO(n) \simeq S^n$

$\Rightarrow p_i(U/K) = 0 \ \forall i$ (b/c S^n is a boundary and doesn't have cohomology)

$\Rightarrow p_i(\Gamma\backslash \mathbb{H}^n) = 0 \ \forall i$

② $G = SU(n,1)$ $\quad$ ~~G~~ $G_{\mathbb{C}} = SL_{n+1}(\mathbb{C})$ $\quad U = SU_{n+1}$

$U/K = SU_{n+1}/S(U_n \times U_1) \simeq \mathbb{CP}^n$

$p_i(U/K)$ not hard (Milnor-Stasheff)

$\Rightarrow p_1(\Gamma\backslash \mathbb{H}^n_{\mathbb{C}}) \neq 0$ for Γ cocpt. $n \geq 2$.

Rmk Not always so simple.

E.g. $G = E_{8(-24)}$ $\quad K \doteq E_7 \times SU_2$

$G_{\mathbb{C}} = E_8^{\mathbb{C}}$ $\quad U = E_8$

$$p_i\left(E_8 / E_7 \times SU_2\right) \neq 0 \;?$$

B. Approach 2

Rough idea: Use that $X = G/K$ has nonpositive curvature and determine if $p_i(\Gamma \backslash X) \neq 0$ by studying Γ action on $\partial_\infty X \simeq S^{n-1}$ $(n = \dim X)$

IV. Longest proof that $\chi(S_g) \neq 0$ $\quad g \geq 2$.

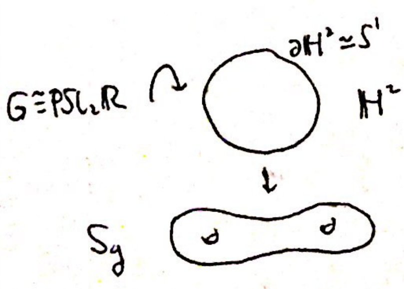
Euler class $\quad B\,\mathrm{Homeo}(S^1)$ classifying space.

$$\left\{\begin{matrix}\text{htpy classes} \\ M \longrightarrow B\mathrm{Homeo}(S^1)\end{matrix}\right\} \xleftrightarrow{1-1} \left\{\begin{matrix}\text{isomorphism classes} \\ S^1 \longrightarrow E \to M\end{matrix}\right\}$$

$H^\bullet(B\mathrm{Homeo}(S^1); \mathbb{Z}) \simeq \mathbb{Z}[e]$ $\quad e \in H^2(\;)$ euler class.

- $T^1 S_g \to S_g$ classified by $f : S_g \longrightarrow B\mathrm{Homeo}(S^1)$ and $\chi(S_g) = \langle f^*(e), [S_g] \rangle \in \mathbb{Z}$.

Geometry of $T^1 S_g$

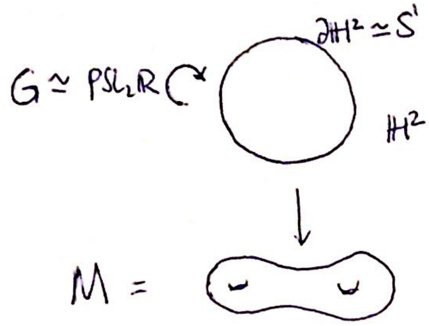


$\rho : \pi_1(S_g) \longrightarrow G \longrightarrow \mathrm{Homeo}(\partial \mathbb{H}^2)$

defines an S^1 bundle

$$\partial \mathbb{H}^2 \longrightarrow E = \frac{\mathbb{H}^2 \times \partial \mathbb{H}^2}{\pi_1(S_g)} \longrightarrow \frac{\mathbb{H}^2}{\pi_1(S_g)} = M$$

IV. The longest proof that S_g $(g \geq 2)$ has nonzero Euler Characteristic.



$\pi_1(M) =: \Gamma \subset G.$

$$\rho : \pi_1(M) \hookrightarrow G \longrightarrow \mathrm{Homeo}(\partial\mathbb{H}^2)$$

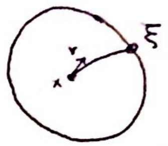
- ρ defines a circle bundle

$$E = \frac{\mathbb{H}^2 \times \partial\mathbb{H}^2}{\pi_1(M)} \xrightarrow{q} \pi_1(M)\backslash\mathbb{H}^2 = M.$$

<u>Claim</u> $E \simeq T^1M$ (unit tangent bundle)

<u>Pf</u>: $T^1\mathbb{H}^2 \xrightarrow{\simeq} \mathbb{H}^2 \times \partial\mathbb{H}^2$ G equivariant

$(x, v) \longmapsto (x, \exp_x tv)$



$\xi = \exp_x(tv)$

$$\Rightarrow T^1M = \pi_1(M)\backslash T^1\mathbb{H}^2 \simeq \frac{\mathbb{H}^2 \times \partial\mathbb{H}^2}{\pi_1(M)} = E. \qquad \square$$

- Foliation on E: leaves are images $\mathbb{H}^2 \times \{\theta\} \in \mathbb{H}^2 \times \partial\mathbb{H}^2 \longrightarrow E.$

  Leaves transverse to fibers of q.

  Such a foliation is called a flat connection on E. Holonomy in $PSL_2\mathbb{R} \subset \mathrm{Homeo}\,S^1$.

- The Euler class

  $B\mathrm{Homeo}(S^1)$ classifying space $\left\{\begin{array}{c}\text{htpy classes} \\ M \to B\mathrm{Homeo}\,S^1\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{iso classes} \\ S^1 \to E \to M\end{array}\right\}$

  $H^\bullet(B\mathrm{Homeo}(S^1); \mathbb{Q}) \simeq \mathbb{Q}[e]$ $\quad |e| = 2$ $\quad$ obstruction to section.

$$B\pi_1(S_g) \longrightarrow BG^\delta \longrightarrow BG \longrightarrow B\,\mathrm{Homeo}(S^1)$$

$$H^\cdot(S_g) \xleftarrow{1} H^\cdot(BG^\delta) \xleftarrow{2} H^\cdot(BG) \xleftarrow{3} H^\cdot(B\,\mathrm{Homeo}\,S^1) \;;\; \rho^*$$

$$\rho^*(e) = \chi(S_g)$$

1. injective (transfer argument)
3. isomorphism b/c $PSL_2\mathbb{R} \to \mathrm{Homeo}\,S^1$ is htpy equiv.
2. Computed w/ Chern-Weil theory.

<u>Prop</u> (Milnor) Let G be a real semisimple Lie group. Then

$$H^\cdot(BG^\delta) \xleftarrow{\alpha} H^\cdot(BG) \xleftarrow{\beta} H^\cdot(BG_{\mathbb{C}})$$

is exact: $\quad \mathrm{Ker}\,\alpha = \mathrm{Ideal}\left[\beta\left(H^{\geq 1}(BG_{\mathbb{C}})\right)\right]$

<u>Application</u> $\quad G = PSL_2\mathbb{R} \qquad G_{\mathbb{C}} = PSL_2\mathbb{C}$.

$$\mathbb{Q}[c_2] \simeq H^\cdot(BSU_2) \simeq H^\cdot(BG_{\mathbb{C}}) \xrightarrow{\beta} H^\cdot(BG) \simeq H^\cdot(BSO_2) \simeq \mathbb{Q}[\omega]$$

$$\beta(c_2) = e^2 \Rightarrow e \notin \mathrm{im}\,\beta \Rightarrow \alpha(\omega) \neq 0$$

$$\Rightarrow \rho^*(e) \neq 0$$

$$\Rightarrow \chi(S_g) \neq 0.$$