Pont Classes for (a) Loc. Sym. Mfld.

Two Similar Problems

- Problem 1: $-\ X = SU_{6,1} / S(U_6 \times U_1) = \mathbb{C}H^6$

  $-\ \Gamma < SU_{6,1}$ acts on visual boundary $\partial X \simeq S^{11}$

  Compute $H^\bullet(B\mathrm{Homeo}\, S^{11}) \longrightarrow H^\bullet(B\Gamma)$.

- Problem 2: $-\ X = \mathrm{Teich}(S_3)$

  $-\ \Gamma = \mathrm{Mod}(S_3)$ acts on boundary $\partial X = PMF \simeq S^{6(3)-7} = S^{11}$.

  Compute $H^\bullet(B\mathrm{Homeo}\, S^{11}) \longrightarrow H^\bullet(B\Gamma)$

- Prob 1 easy, Prob 2 hard.

Characteristic Classes of a Representation

$-\ K$ cpt Lie, $T \subset K$ max torus, $W = N_K(T)/T$ weyl gp.

Thm 1 (Borel) $H^\bullet(BK) \longrightarrow H^\bullet(BT)$ injective w/ image $H^\bullet(BT)^W$

Example $K = U_n$ $\quad T = \begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix}$ $\quad W = S_n$

$$H^\bullet(BU_n) = H^\bullet(BT)^W = \mathbb{Q}[x_1, \dots, x_n]^{S_n} = \mathrm{Sym}(x_1, \dots, x_n)$$

generated by terms of fixed degree in $\prod(1+x_i)$

Chern classes. $c = 1 + c_1 + c_2 + \dots + c_n = \prod(1+x_i)$.

Note $H^\bullet(BGL_n\mathbb{C}) \simeq H^\bullet(BU_n)$.

- Let $\rho : K \to GL_n\mathbb{C}$ be a representation.

Induces $\rho^* : H^\cdot(BGL_n\mathbb{C}) \to H^\cdot(BK)$.

- Compute ρ^*. $\rho|_T$ diagonal $\Rightarrow$ get $\lambda_i : T \to \mathbb{C}^\times$ $i = 1, \dots, n$

or equivalently $\lambda_i \in H^1(T; \mathbb{Z})$.

These are the <u>weights</u> of ρ. ~~Via transgression~~

$\tau : H^1(T) \to H^2(BT)$ trangression

<u>Chern class of ρ</u> $\quad c(\rho) = \prod_{i=1}^{n} (1 + \tau(\lambda_i)) \in H^\cdot(BK)$.

is an invariant of ρ.

<u>Rmk</u> $c(\rho) = \rho^*(c)$.

Pont class of ρ $\quad p(\rho) = \prod (1 - \tau(\lambda_i)^2)$

<u>Example</u> $K = U_2 \times U_1$ $\quad T = \begin{pmatrix} a_1 & & \\ & a_2 & \\ & & a_3 \end{pmatrix}$ $\quad H^1(T) = \mathbb{Z}\{\varphi_1, \varphi_2, \varphi_3\}$.

$K \subset U(2,1) = \{ A \in GL_3(\mathbb{C}) \mid A^* \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix} A = \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix} \}$.

$K \curvearrowright \mathfrak{u}_{2,1}$ decomposes $\mathfrak{u}_{2,1} = (\mathfrak{u}_2 \oplus \mathfrak{u}_1) \oplus V$.

$\dim_{\mathbb{C}} V = 2 \Rightarrow \rho : K \to \mathrm{Aut}(V) \simeq GL_2\mathbb{C}$.

$$\rho|_T : \begin{pmatrix} a_1 & & \\ & a_2 & \\ & & a_3 \end{pmatrix} \longmapsto \begin{pmatrix} a_1 a_3 & \\ & a_2 a_3 \end{pmatrix}.$$

$\Rightarrow \rho$ has weights $\varphi_1 + \varphi_3$, $\varphi_2 + \varphi_3$.

$$\Rightarrow c(\rho) = (1 + (x_1 + x_3))(1 + (x_2 + x_3)).$$

$$= 1 + (x_1 + x_2 + 2x_3) + (x_1 x_2 + x_2 x_3 + x_1 x_3 + x_3^2).$$

$$p(\rho) = 1 + (-x_3^2 - (x_1^2 + x_2^2 + x_3^2) - 2(x_1 + x_2)x_3).$$

~~Defn An bundle~~

<u>Construction</u> F, B spaces; $\varphi : \pi_1 B \to \mathrm{Homeo}(F)$.

$$F \to \frac{\tilde{B} \times F}{\pi_1 B} \to B \quad \text{defines an } F\text{-bundle over } B.$$

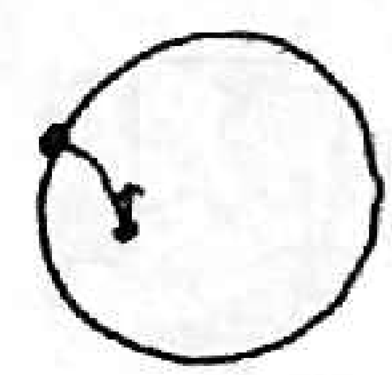
<u>Defn</u> $F \to E \to B$ is <u>flat</u> if it is isomorphic to a bundle as above.

<u>Examples</u>

1. $B = S^1$ Every F-bundle over S^1 is flat.
2. $B = S_g$, $g \geq 2$. Consider $\varphi : \pi_1 S_g \to PSL_2\mathbb{R} \to \mathrm{Homeo}(S^1)$.
   defines flat bundle $E_\varphi \to S_g$.

<u>Claim</u> $E_\varphi \simeq T^1(S_g)$

<u>Pf</u> Let $\phi : T^1\mathbb{H}^2 \to \partial\mathbb{H}^2$ via exponential
$\pi : T^1\mathbb{H}^2 \to \mathbb{H}^2$



Define
$$\begin{aligned} T^1\mathbb{H}^2 &\xrightarrow{\Phi} \mathbb{H}^2 \times \partial\mathbb{H}^2 \\ z &\longmapsto (\pi(z), \phi(z)). \end{aligned}$$

~~Thus~~ Φ is $\pi_1(S_g)$ equivariant so descends to

$$\begin{array}{c} \pi_1 S_g \backslash T^1\mathbb{H}^2 \longrightarrow \dfrac{\mathbb{H}^2 \times \partial\mathbb{H}^2}{\pi_1 S_g} \\ \| \\ T^1(S_g) \end{array}$$

∎

Char Classes for Flat Bundles

- For Lie G, let $G^\delta = G$ w/ discrete top.
- BG classifies G bundles, BG^δ classifies flat G-bundles.

have $H^\cdot(BG) \to H^\cdot(BG^\delta)$.

Thm 2 Let G real ss Lie w/ complexification $G_{\mathbb{C}}$.

Consider $G^\delta \to G \to G_{\mathbb{C}}$.

$$H^\cdot(BG^{\mathbb{C}}) \xrightarrow{\alpha} H^\cdot(BG) \xrightarrow{\beta} H^\cdot(BG^\delta) \qquad \text{exact:}$$

$\ker\beta$ = ideal generated by $\operatorname{im}\alpha$ in deg ≥ 1.

Pf comes from Chern-Weil Theory.

Examples

1) $G = SL_2\mathbb{R}$ (you). euler class $e \in H^2(BSL_2\mathbb{R})$ is cc of flat...

2) $G = U(2,1)$ $G_{\mathbb{C}} = GL_3\mathbb{C}$.

$$\begin{array}{ccc}
H^\cdot(BG_{\mathbb{C}}) & \longrightarrow & H^\cdot(BG) \\
\cong & & \cong \\
H^\cdot(BU_3) & & H^\cdot(B(U_2\times U_1)) \\
\cong & & \cong \\
H^\cdot(BT)^{W(U_3)} & & H^\cdot(BT)^{W(U_2)\times W(U_1)} \\
\parallel & & \parallel \\
\mathbb{Q}[x_1,x_2,x_3]^{S_3} & & \mathbb{Q}[x_1,x_2,x_3]^{S_2\times S_1} \\
\parallel & & \parallel \\
\mathrm{Sym}(x_1,x_2,x_3) & & \mathrm{Sym}(x_1,x_2)\otimes\mathrm{Sym}(x_3).
\end{array}$$

$$\Rightarrow \ker\left(H^\cdot(BU_{2,1}) \to H^\cdot(BU_{2,1}^\delta)\right) = \mathrm{Sym}^{>0}(x_1,x_2,x_3)$$

Computing Pont Classes

Thm3 $O_n \longrightarrow \mathrm{Homeo}\, S^{n-1}$ induces rational surjection

$$H^*(B\,\mathrm{Homeo}\, S^{n-1}) \longrightarrow H^*(BO_n) \cong \mathbb{Q}[p_1, \dots, p_{[n/2]}, e]/\sim.$$

$|p_i| = 4i$

$|e| = n.$

- $\Gamma < G$ $\qquad M^n = \Gamma \backslash G/K.$
- ~~$\Gamma \curvearrowright \partial(G/K)$ and~~ $\varphi : \Gamma \longrightarrow \mathrm{Homeo}(\partial(G/K))$ defines

$$\begin{tikzcd} E_\varphi \arrow[r, phantom, "\cong"] \arrow[dr] & T^1 M \arrow[d] \\ & M \end{tikzcd}$$

$\Rightarrow$ under $\quad \varphi^* : H^*(B\,\mathrm{Homeo}\, S^{n-1}) \longrightarrow H^*(B\Gamma) = H^*(M)$

$$p_i \longmapsto p_i(M).$$

- Goal: Compute $p_i(M)$. (nontrivial?)
- Have factoring $\quad \Gamma \longrightarrow G^\delta \longrightarrow G \longrightarrow \mathrm{Homeo}\, S^{n-1}$

$$H^*(B\,\mathrm{Homeo}) \xrightarrow{1} H^*(BG) \xrightarrow{2} H^*(BG) \xrightarrow{3} H^*(B\Gamma)$$

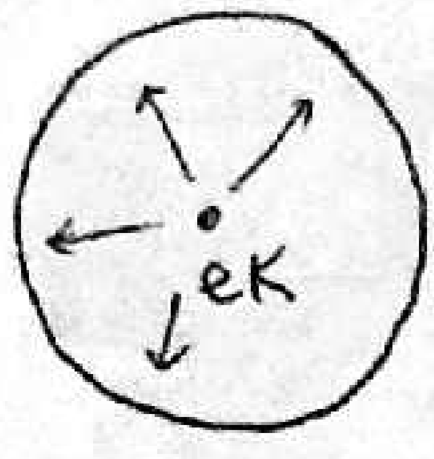
3 injective for Γ cocompact by transfer argument

2 computed w/ Thm 2 above

1 $BG \sim BK$ so compute $H^*(B\,\mathrm{Homeo}) \longrightarrow H^*(BK)$

Key: K acts on $\partial(G/K)$ linear!

Pf



G/K

$\phi : T'_{eK}(G/K) \longrightarrow \partial(G/K)$

isomorphic as K reps.

K acts on $T_{eK}(G/K)$:

K acts on $\mathfrak{g}$ decomposes $\mathfrak{g} \simeq \mathfrak{k} \oplus \mathfrak{p}$

$T_{eK}(G/K) \simeq \mathfrak{p}$. ∎

<u>Example</u> $G = U(2,1)$ $K = U_2 \times U_1$ $\Gamma < G$ cocompact lattice.

We computed

$$p_1(p) = -x_3^2 - (x_1^2 + x_2^2 + x_3^2) - 2(x_1 + x_2) \cdot x_3.$$

Under $H^\cdot(BU_{2,1}) \xrightarrow{\beta} H^\cdot(BU_{2,1}{}^\delta)$ $\qquad x_1^2 + x_2^2 + x_3^2 \mapsto 0$

$x_1 + x_2 + x_3 \mapsto 0.$

$$\Rightarrow \beta(p_1(p)) = -\beta(x_3^2) - 0 - 2\beta(-x_3^2)$$

$$= \beta(x_3)^2 . \neq 0.$$

$$\Rightarrow p_1(M) = \beta(x_3)^2 \neq 0.$$