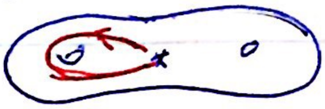
Point-pushing & Nielsen Realization

Goal Use cohomology to study group actions
(or flat connections on fiber bundles)

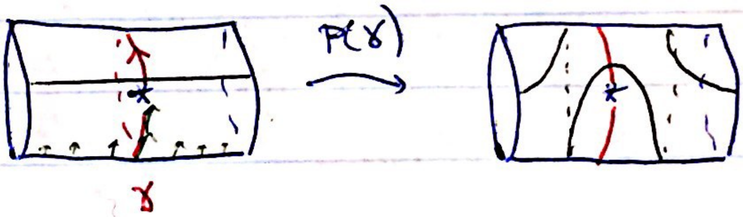
I. Nielsen realization problem for point-pushing.



Setup
- M mfld, $* \in M$ basepoint
- $\mathrm{Diff}(M,*)$ diffeos fixing $*$. (C^1, or. pres)
- $\mathrm{Mod}(M,*) := \pi_0 \mathrm{Diff}(M,*)$ isotopy classes.

Push homomorphism $\mathrm{Push}: \pi_1(M,*) \longrightarrow \mathrm{Mod}(M,*)$.

- γ loop based at $*$ $\rightsquigarrow$ $P(\gamma) \in \mathrm{Diff}(M,*)$



- ~~$\mathrm{Push}([\gamma]) = [P(\gamma)]$~~.

$$\begin{aligned} \mathrm{Push}: \pi_1(M) &\longrightarrow \mathrm{Mod}(M,*) \\ [\gamma] &\longmapsto [P(\gamma)] \end{aligned}$$

Rmk Push is connecting homomorphism assoc. to fibration

$$\begin{tikzcd} \mathrm{Diff}(M,*) \arrow[r] & \mathrm{Diff}(M) \arrow[r] & M \\ & f \arrow[r, mapsto] & f(*) \end{tikzcd}$$

Question 1 Does there exist $\varphi: \pi_1(M) \longrightarrow \mathrm{Diff}(M,*)$ s.t.

$$\begin{tikzcd} & \mathrm{Diff}(M,*) \arrow[d] \\ \pi_1(M) \arrow[ru, "\varphi"] \arrow[r, "\mathrm{Push}"'] & \mathrm{Mod}(M,*) \end{tikzcd}$$

Commutes?

If φ exists, say Push is <u>realized by diffeos</u>.

<u>Significant case</u> $M = \Gamma \backslash G / K$ locally symmetric manifold, noncompact type

- G real semisimple Lie group w/ no compact factors (e.g. $\mathrm{Isom}(\mathbb{H}^n)$, $SL_n(\mathbb{R})$, $E_{8(8)}$)
- $K < G$ maximal compact
- $\Gamma < G$ torsion free lattice

e.g. $G = PSL_2\mathbb{R}$ $K = SO(2)$ $\Gamma = \pi_1(S_g)$ $g \geq 2$

$\rightsquigarrow$ M = hyperbolic surface.

<u>Thm</u> (Bestvina-Church-Souto 2009, Tsh 2014)

If $M = S_g$ closed surface $g \geq 2$ or a loc. sym mfld st (***)

then Push is <u>not</u> realized by diffeos

<u>Rough idea</u> (BCS) Use Euler class and Milnor-Wood inequalities as obstruction to existence of φ.

II. Flat connections on fiber bundles

- Fix F, M mflds

~~Defn An F bundle E → M is flat if~~

<u>Defn</u> An F bundle $E \to M$ <u>admits a flat connection</u> (or is <u>flat</u>) if E has a foliation $\mathcal{F}$ whose leaves project to ~~B~~M as covering spaces.

Central example Fix $\rho : \pi_1(M) \longrightarrow \mathrm{Diff}(F)$.

Define $E_\rho \longrightarrow M$ where

$$E_\rho = \frac{\tilde{M} \times F}{\pi_1(M)} .$$

The foliation with leaves

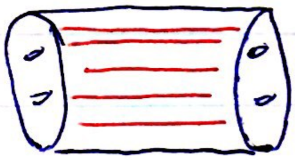
$$\mathrm{im}\left[\tilde{M} \times \{x\} \hookrightarrow \tilde{M} \times F \longrightarrow \frac{\tilde{M} \times F}{\pi_1(M)} \right]$$

defines a flat connection on $E_\rho \longrightarrow M$.

Rmk Every example is of this form.

Flat connections on surface bundles

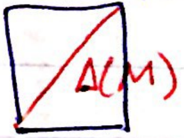
- (Morita) $\exists \quad S_g \to E \to M^6$ not flat.
- **Rmk** Every $S_g \longrightarrow E \longrightarrow S^1$ is flat



$S_g \times [0,1]$.

- Open Q: Is every $S_g \to E \to S_n$ flat?

Question 2



- M $\pi_1(M) \neq 1$
- $M \times M \longrightarrow M$ proj 1st factor
- $\Delta : M \longrightarrow M \times M$

Does $M \times M \to M$ admit a flat connection where Δ is parallel?

Monodromy & flat connections

- $F \to E \to M$ ⇝ monodromy $\mu: \pi_1(M) \longrightarrow \mathrm{Mod}(F)$ ~~$\pi_0 \mathrm{Diff}(F$~~

- $E \to M$ flat $\Longrightarrow$

$$\begin{tikzcd} & \mathrm{Diff}(F) \arrow[d] \\ \pi_1(M) \arrow[ur, dashed, "\varphi"] \arrow[r, "\mu"'] & \mathrm{Mod}(F) \end{tikzcd}$$

- $M \times M \to M$, $\Delta: M \to M \times M$ ⇝ monodromy $\mathrm{Push}: \pi_1(M) \longrightarrow \mathrm{Mod}(M, *)$

- $M \times M \to M$ flat wrt Δ $\Longrightarrow$ Push realized by diffeos.

Rmks on converse

(i) False for $\pi_1(M) = 1$.

(ii) True ~~for~~ when $\dim(M) = 2$
because $B\mathrm{Diff}(M) \sim B\mathrm{Mod}(M)$
but not true in general. (realization might define flat cnxn on different bdl.)

III. Characteristic classes of flat bundles

- Fix F.

Defn A characteristic class c is a map

$$(F \to E \to M) \longmapsto c(E) \in H^k(M)$$

that is natural wrt bundle pullbacks.

Trend Characteristic classes of flat bundles are often restricted.

Examples

Pontryagin classes

① $\mathbb{R}^n \to E \to M^n$ vector bundle.

- $p_i(E) \in H^{4i}(M;\mathbb{R})$ ith Pontryagin class.
- Chern-Weil theory: $E \to M$ flat $\Rightarrow$ $p_i(E)=0$ $\forall i > 0$.
- Ex $M = \mathbb{C}P^2$ $T\mathbb{C}P^2 \to \mathbb{C}P^2$

  $p_1(T\mathbb{C}P^2) \neq 0$ $\Rightarrow$ $TM \to M$ not flat.

② Euler class

$\mathbb{R}^2 \to E \to S_g$ vector bundle $g \geq 1$. (S_g closed)

- $e(E) \in H^2(S_g;\mathbb{Z}) \cong \mathbb{Z}$ Euler class
- Milnor-Wood inequality (1958):

  $E \to S_g$ flat $\Rightarrow$ $1-g \leq e(E) \leq g-1$

- Cor $TS_g \to S_g$ does not have a flat (~~$GL_2\mathbb{R}$~~) linear connection.

  $e(TS_g) = \chi(S_g)$

  $g \geq 2$

IV. Main theorem

- $M^n = \Gamma \backslash G / K$.

Thm (T) Suppose one of the following.

(i) M product of closed surfaces, genus ≥ 2.

(ii) $p_i(TM) \neq 0$ some $i > 0$.

(iii) $\mathbb{R}$-rank$(G) \geq 2$, Γ irreducible & nonuniform

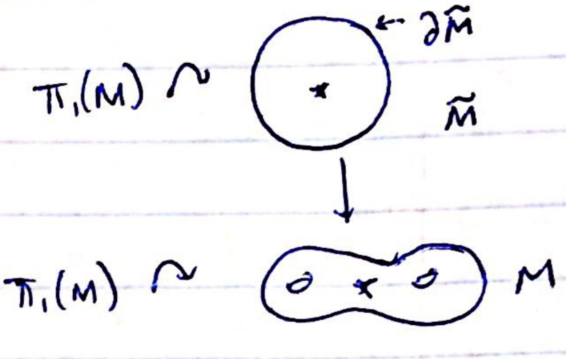
(e.g. $\Gamma = SL_n\mathbb{Z}$ $G = SL_n\mathbb{R}$)

Then Push is not realized by diffeos.

Proof Outline

Step1 (Push realized) + $\begin{pmatrix} M \text{ nonpos.} \\ \text{curved} \end{pmatrix}$ $\Longrightarrow$ (TM → M has same cc's as flat ~~$GL_n\mathbb{R}$~~ vector bundle) * (‡)

$$\begin{tikzcd} & \mathrm{Diff}(M,*) \arrow[d] \\ \pi_1(M) \arrow[ur, "\varphi"] \arrow[r, "\mathrm{Push}"'] & \mathrm{Mod}(M,*) \end{tikzcd} \rightsquigarrow$$



Induces

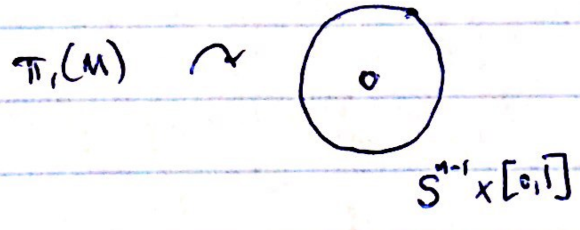
$$\rho_1 : \pi_1(M) \longrightarrow GL(T_*\tilde{M}) \longrightarrow \mathrm{Homeo}(S^{n-1})$$

$$\rho_2 : \pi_1(M) \longrightarrow G \longrightarrow \mathrm{Homeo}(S^{n-1})$$

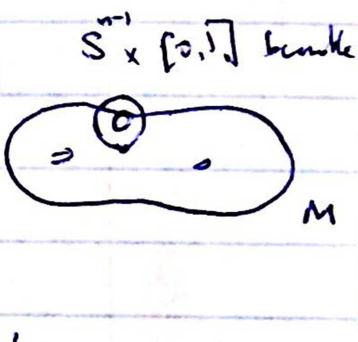
ρ_1, ρ_2 induce flat S^{n-1} bundles $E_1 \to M$ $E_2 \to M$.

s.t.
- $E_2 \simeq T^1M$ unit tangent
- E_1 has flat linear connection.

E_1 & E_2 have same cc's b/c they're fiberwise bordant.



induces



Step2 Show (‡) false using Milnor-Wood ineq's, Chern-Weil Theory, or Margulis superrigidity

□

Q: Which examples does Thm apply to?

Short Ans: For any $G \neq SO(n,1)$ thm either

applies to
- all cocompact Γ

or
- all noncocompact Γ

(possibly both)

Pontryagin classes When is $p_i(T(\Gamma\backslash G/K)) \neq 0$ for some i?

Assume Γ cocompact.

- (Borel - Hirzebruch '58) gave algorithm (answer depends only on G when Γ cocompact)
- (T) implement algorithm $\forall\, G$:

$T'M \to M$ flat induced by

$$\Gamma \longrightarrow G \longrightarrow \mathrm{Homeo}(\underset{\approx S^{n-1}}{\partial(G/K)})$$

must compute

$$M \approx B\Gamma \longrightarrow BG^{\delta} \longrightarrow BG \longrightarrow \underbrace{B\,\mathrm{Homeo}(S^{n-1})}_{p_i \text{ live here}}$$

on $H^*(-)$.

transfer (Γ cocpt)

Chern-Weil (Milnor)

reduces to rep theory $K \curvearrowright \partial(G/K)$ linear.

Some nonzero Pontryagin classes

$SU(p,q)$ $p,q \geq 1$ $p+q \geq 2$

$SP(2n,\mathbb{R})$ $n \geq 2$

$SO(p,q)$ $p,q \geq 2$ $(p,q) \neq \begin{matrix}(2,2)\\(3,3)\end{matrix}$

$SO^*(2n)$ $n \geq 3$

$G_{2(2)}$ $F_{4(4)}$ $F_{4(-20)}$

$E_{6(6)}$ $E_{6(2)}$ $E_{6(-14)}$

$E_{7(7)}$ $E_{7(-5)}$ $E_{7(-25)}$

$E_{8(8)}$ $E_{8(-24)}$

All Pontryagin classes zero.

$SL_n\mathbb{R}$ $n \geq 2$

$SO(n,1)$ $n \geq 2$

$SU^*(2n)$ $n \geq 2$

$E_{6(-26)}$

$SL_n\mathbb{C}$, $SO(n,\mathbb{C})$, $SP(2n,\mathbb{C})$

$G_2(\mathbb{C})$, $F_4(\mathbb{C})$, $E_6(\mathbb{C})$

$E_7(\mathbb{C})$, $E_8(\mathbb{C})$.

Superrigid case WTS $TM \to M$ not flat. Suppose it is.

Then

$$\begin{tikzcd}
& K \arrow[d, hook] & \\
& G \arrow[dd, dashed] \arrow[dr] & \\
\Gamma \arrow[ur] \arrow[dr] & & \mathrm{Homeo}(S^{n-1}) \\
& GL_n\mathbb{R} \arrow[ur] &
\end{tikzcd}$$

commutes on $H^\bullet(B-)$

$\Rightarrow$ Isotropy rep $K \longrightarrow \mathrm{Aut}(T_{eK}\, G/K)$ extends (‡) to rep of G.

Use rep theory to show (‡) false. □