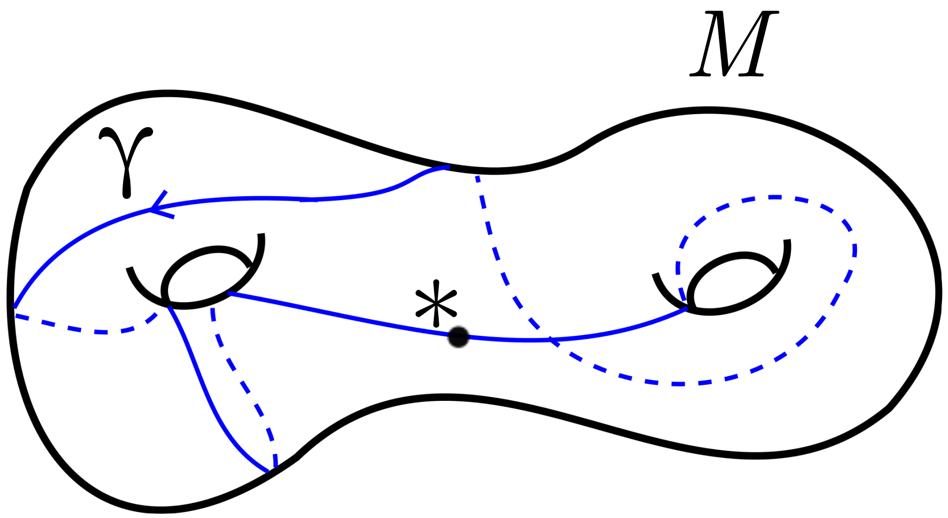
# Point-pushing and Nielsen realization

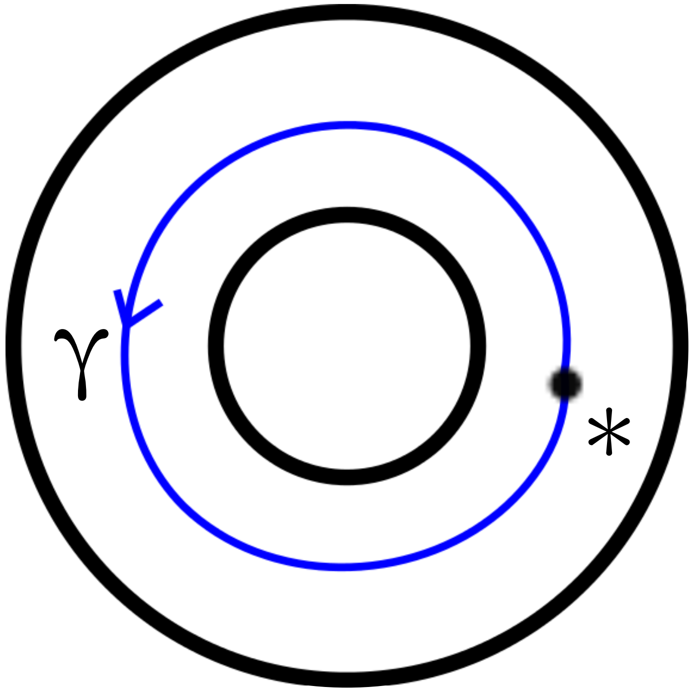
Bena Tshishiku
August 25, 2014

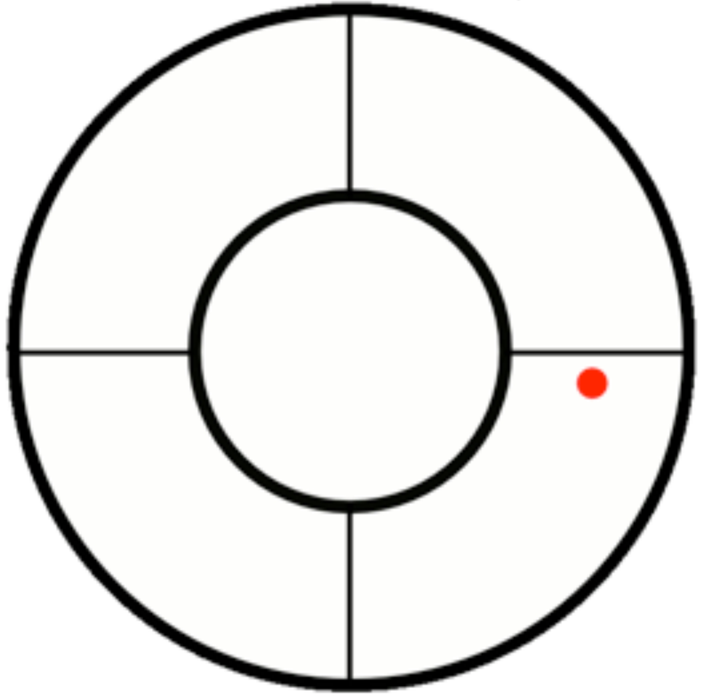
# Point-pushing diffeomorphisms

- $(M,*)$ manifold with basepoint



- $\mathrm{Diff}(M,*)$ group of diffeomorphisms fixing $*$
- $\gamma$ loop based at $*$
- $\mathrm{Push}(\gamma) \in \mathrm{Diff}(M,*)$ defined by "Pushing $*$ around $\gamma$"





# The Push homomorphism

- Mod($M$,$*$):= $\pi_0$Diff($M$,$*$) isotopy classes of diffeomorphisms
- Point-pushing homomorphism

$$\begin{aligned} \mathrm{Push} : \pi_1(M,*) &\to \mathrm{Mod}(M,*) \\ [\gamma] &\to [\mathrm{Push}(\gamma)] \end{aligned}$$

# Nielsen realization problem for point-pushes

Does there exist $\varphi : \pi_1(M,*) \to \mathrm{Diff}(M,*)$ making the following diagram commute?

$$\begin{tikzcd}
 & \mathrm{Diff}(M,*) \arrow[d, "p"] \\
\pi_1(M,*) \arrow[ur, dotted, "\varphi"] \arrow[r, "\mathrm{Push}"'] & \mathrm{Mod}(M,*)
\end{tikzcd}$$

If $\varphi$ exists, we say Push *is realized by diffeomorphisms.*

# Locally symmetric manifolds

- $M = \Gamma\backslash G/K$ is a locally symmetric manifold.
- $G$ real semisimple Lie group without compact factors (e.g. $\mathrm{Isom}(H^n)$, $\mathrm{SL}_n(\mathrm{R})$, $\mathrm{E}_{8(8)}$)
- $K \subset G$ maximal compact subgroup
- $\Gamma \subset G$ torsion-free lattice
- Example: $G = \mathrm{PSL}_2(\mathrm{R})$, $K = \mathrm{SO}(2)$, $\Gamma \cong \pi_1(S_g)$
  $M =$ hyperbolic surface

**Theorem 1.** (Bestvina-Church-Souto, 2009)

Let $M = S_g$ be a closed surface of genus $g \geq 2$. Then $\mathrm{Push} : \pi_1(M,*) \to \mathrm{Mod}(M,*)$ is not realized by diffeomorphisms.

$$\begin{tikzcd}
 & \mathrm{Diff}(M,*) \arrow[d, "p"] \\
\pi_1(M,*) \arrow[r, "\mathrm{Push}"'] \arrow[ur, dashed, "\varphi", "\times" marking] & \mathrm{Mod}(M,*)
\end{tikzcd}$$

**Theorem 2.** (T–, 2014)

Let $M$ be a locally symmetric manifold such that $(***)$. Then $\mathrm{Push} : \pi_1(M,*) \to \mathrm{Mod}(M,*)$ is not realized by diffeomorphisms.

# Nielsen realization problems

$$\begin{tikzcd}
 & \mathrm{Diff}(M) \arrow[d, "p"] \\
\Lambda \arrow[ur, dotted, "\varphi"] \arrow[r, hook, "i"] & \mathrm{Mod}(M)
\end{tikzcd}$$

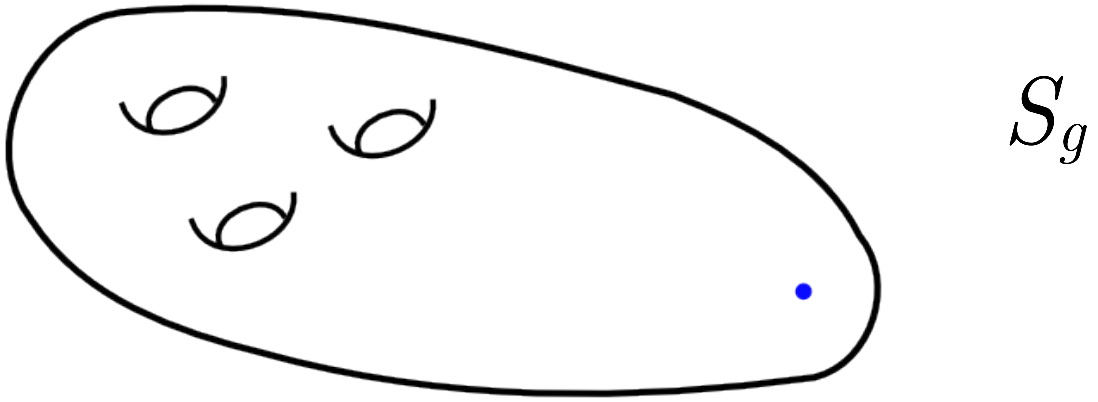
Theorem. (Kerckhoff, 1983) Fix $g \geq 2$. Any finite subgroup $\Lambda \subset \mathrm{Mod}(S_g)$ is realized by diffeomorphisms.

Theorem. (Morita, 1987; Franks-Handel, 2009) Fix $g \geq 3$. Then $\Lambda = \mathrm{Mod}(S_g)$ is not realized by diffeomorphisms.

# An open realization problem

- Handle-pushing subgroups

$$\pi_1(S_g) \subset \mathrm{Mod}(S_g,*)$$



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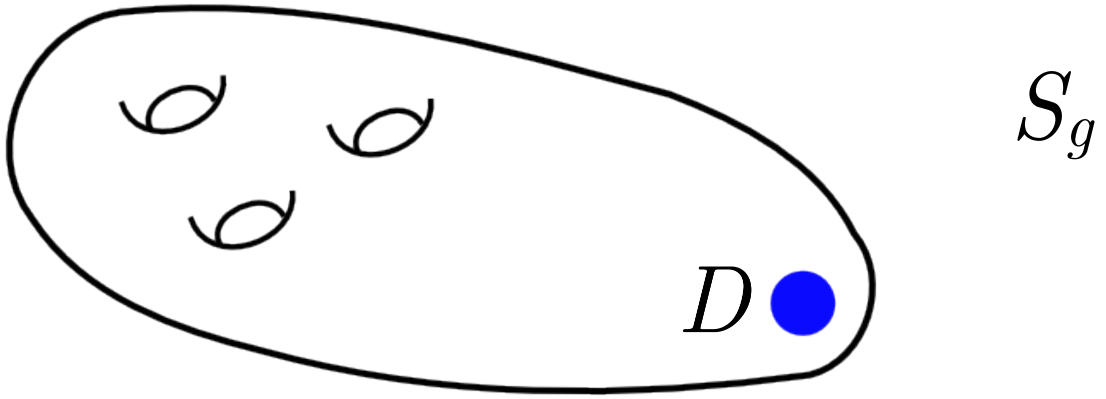

$$\pi_1(\mathrm{U}S_g) \subset \mathrm{Mod}(S_g,D)$$

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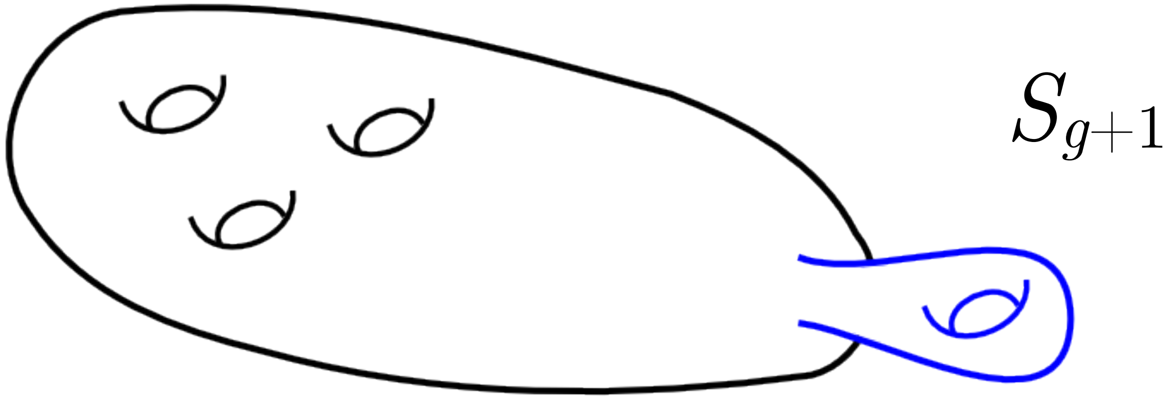

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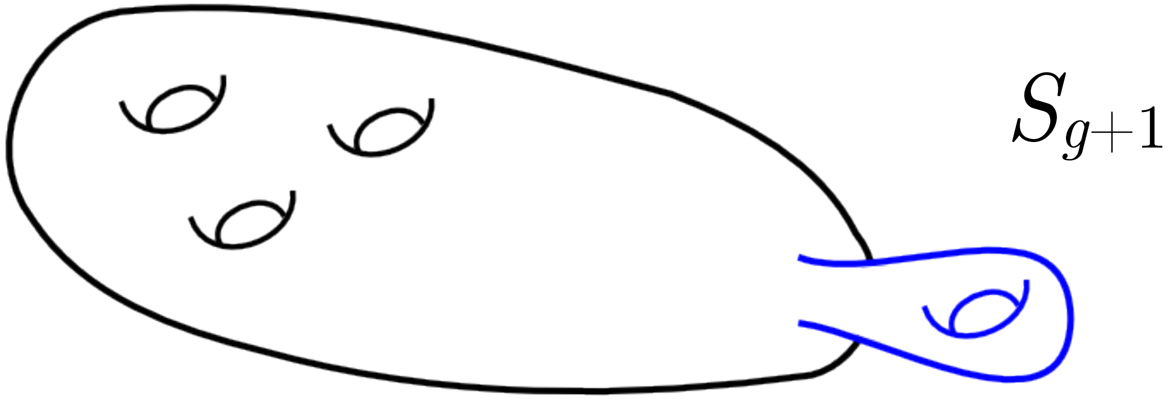

$$\pi_1(\mathrm{U}S_g) \subset \mathrm{Mod}(S_g,D) \subset \mathrm{Mod}(S_{g+1})$$

# An open realization problem

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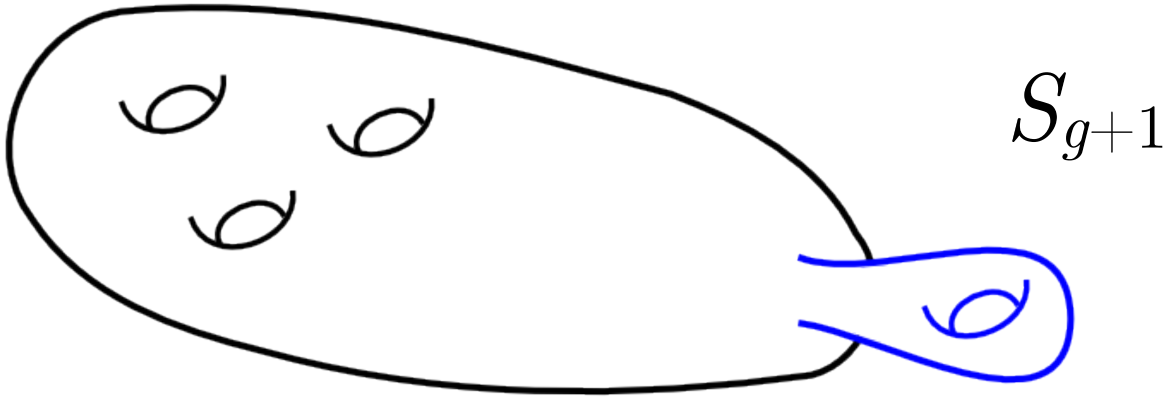


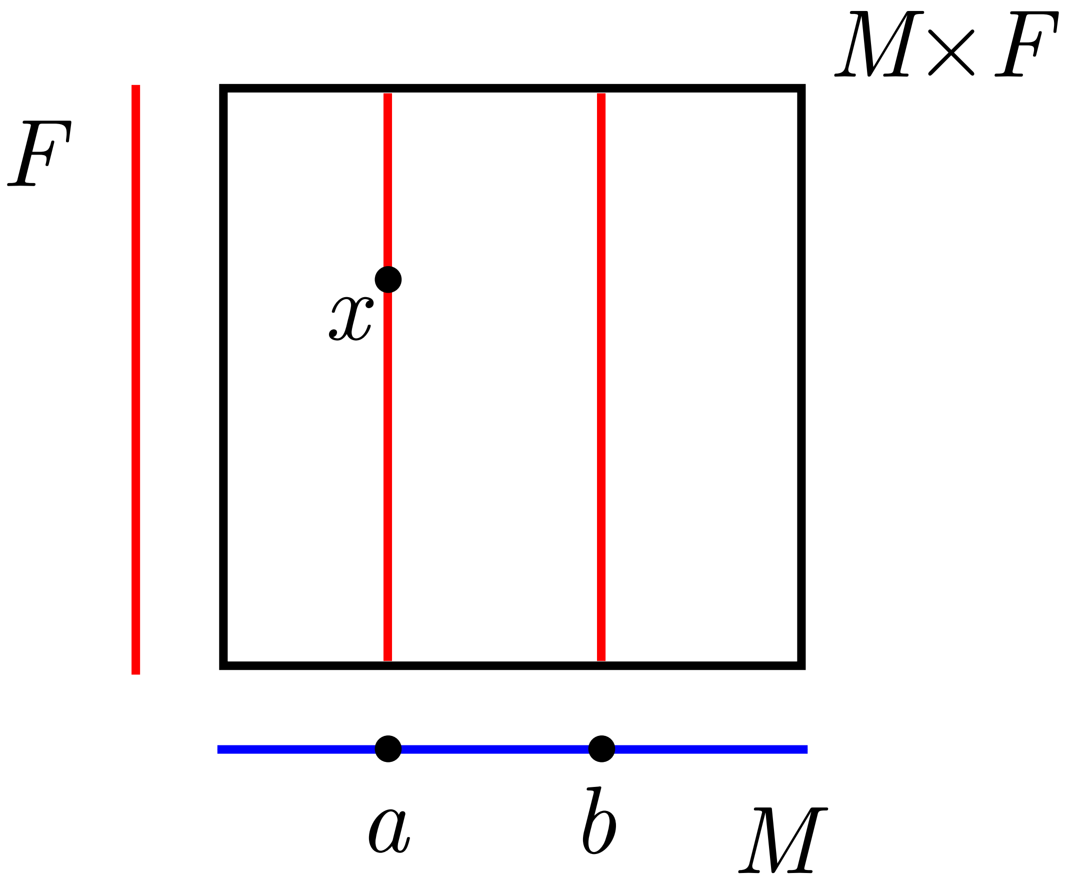
$$\pi_1(\mathrm{U}S_g) \subset \mathrm{Mod}(S_g,D) \subset \mathrm{Mod}(S_{g+1})$$

Question. Is $\pi_1(\mathrm{U}S_g) \subset \mathrm{Mod}(S_{g+1})$ realized by diffeomorphisms?

# Geometry and flat bundles

# Flat connections on fiber bundles

- $F$, $M^n$ manifolds
- $F$ bundle $E \to M$



Trivial bundle

$$E = M\times F$$

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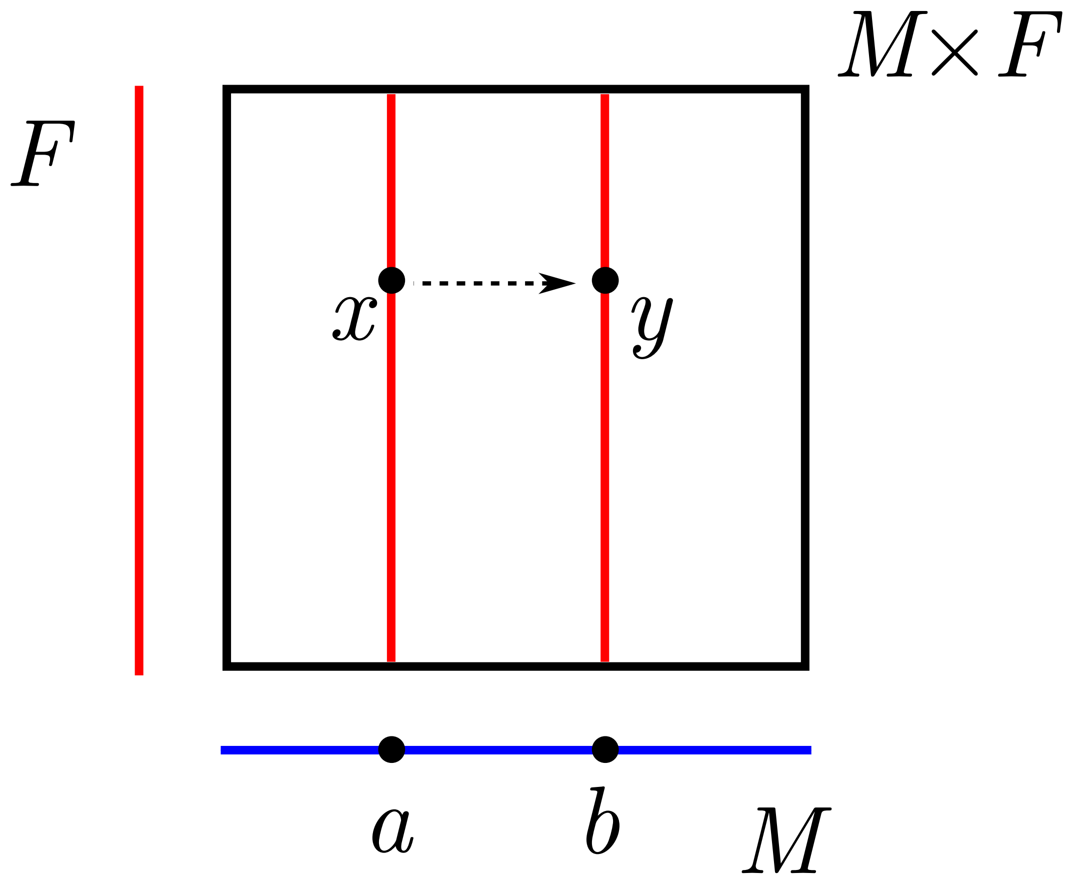

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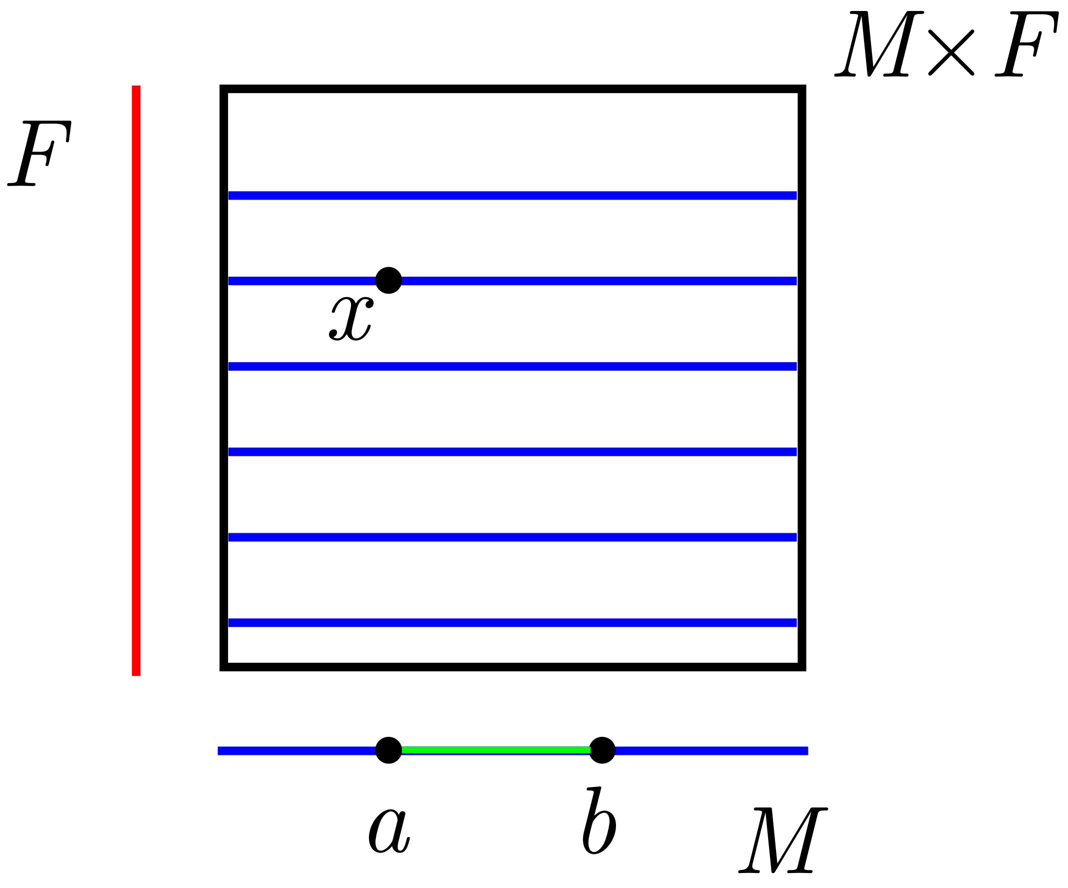

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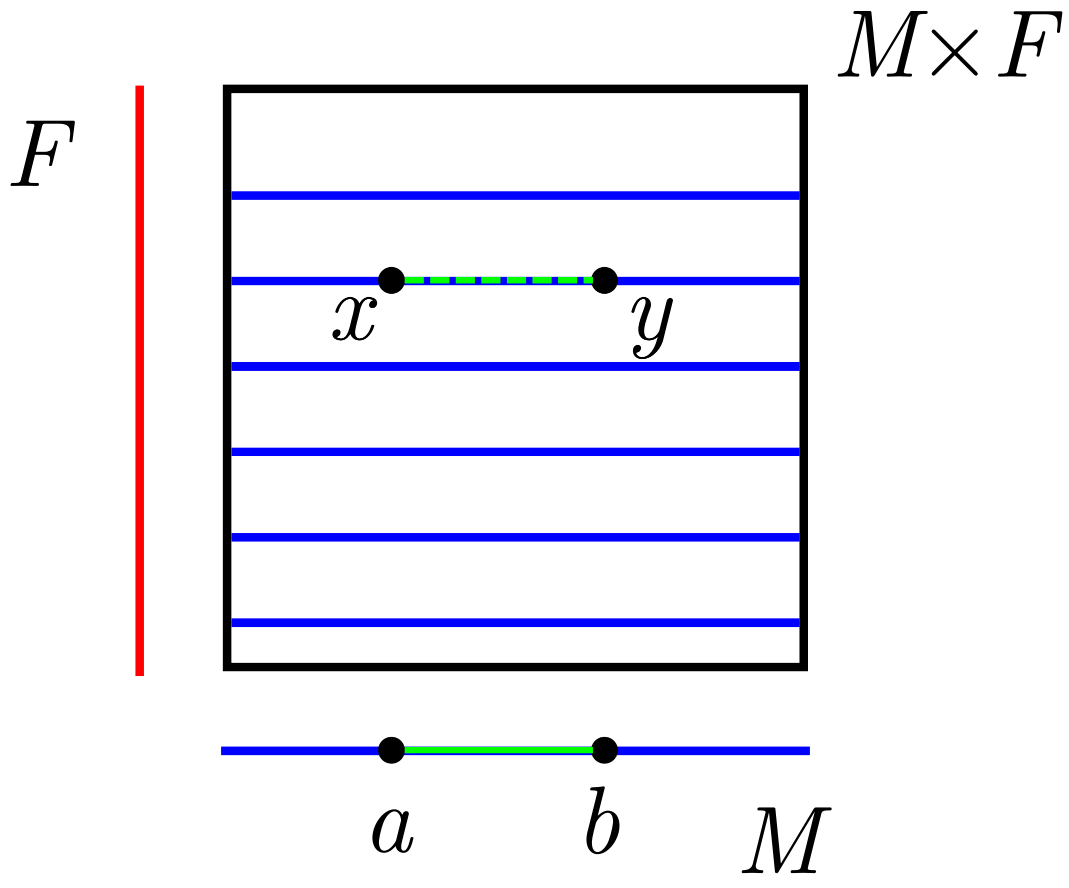

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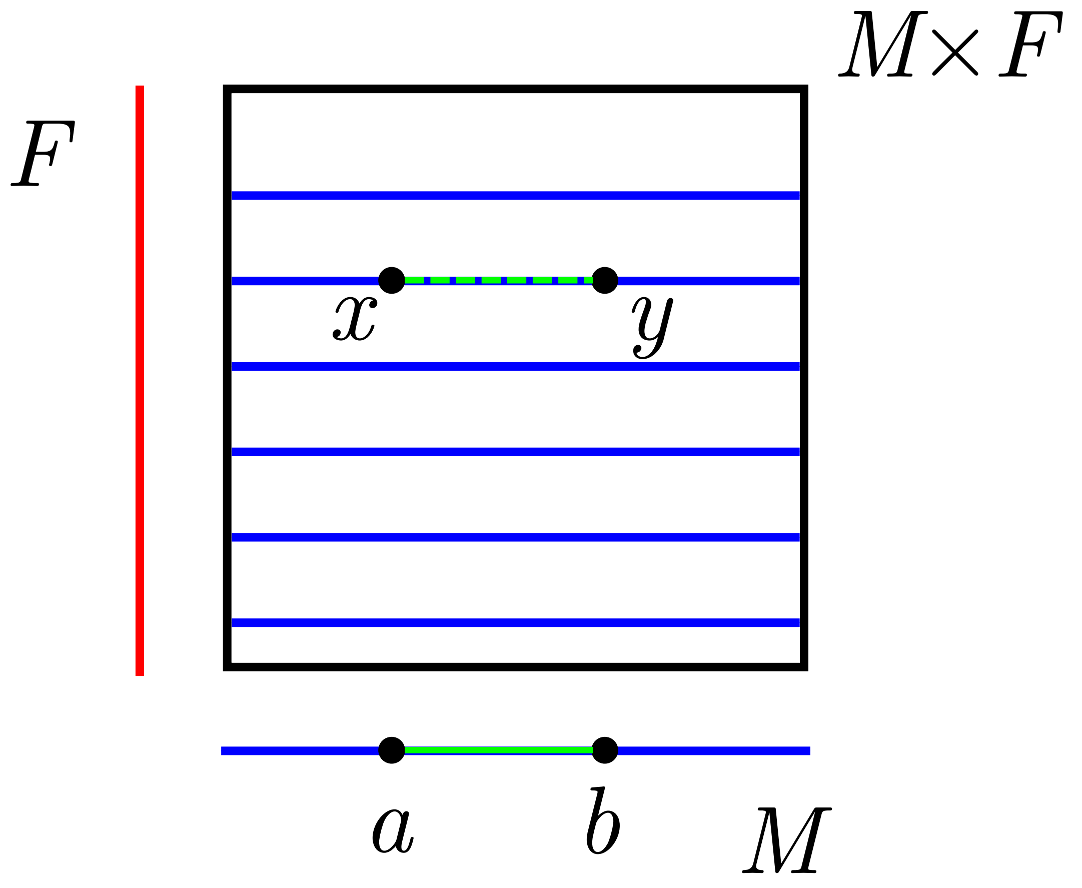
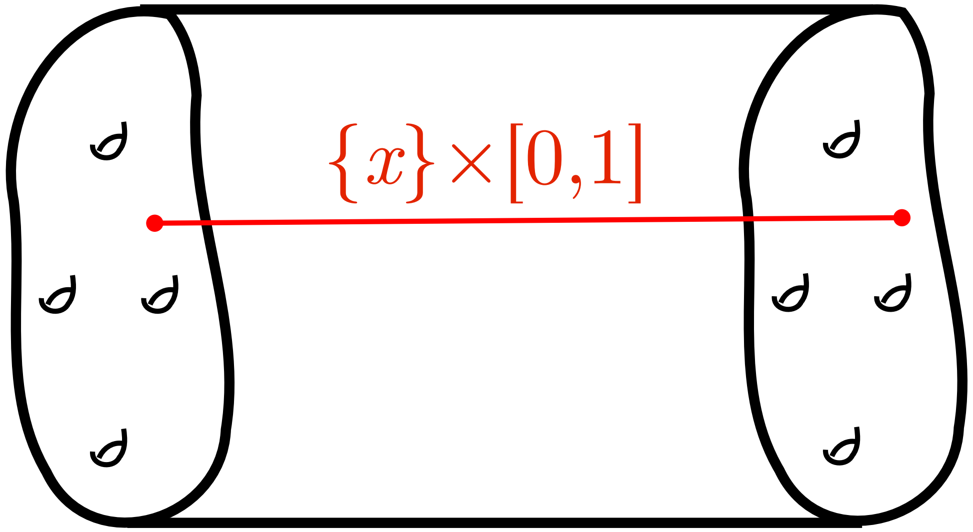


**Definition.** An $F$ bundle $E \to M$ admits a *flat connection* if $E$ has a foliation whose leaves are $n$-dimensional and transverse to the fibers of $p$.

# Flat surface bundles

Theorem. (Morita, 1987) For $g \geq 18$ there exists an $S_g$ bundle $E \to M^6$ that does not admit a flat connection.

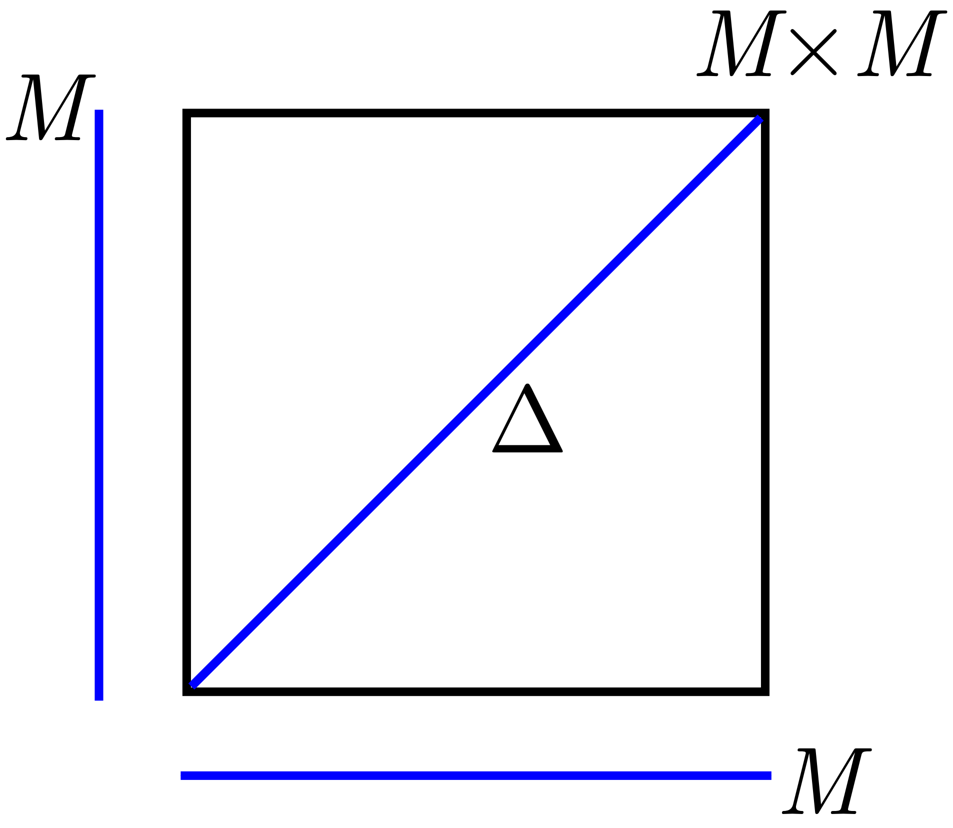
Remark. Every $S_g$ bundle $E \to S^1$ admits a flat connection.



Open Question. Does every $S_g$ bundle $E \to S_h$ admit a flat connection?

# A Basic Question

- $M$ manifold with $\pi_1(M) \neq \{e\}$.
- $M \times M \to M$ projection onto the first factor
- $\Delta : M \to M \times M$ diagonal section.



Question. Does $M \times M \to M$ admit a flat connection for which the diagonal is parallel?

# Monodromy and flat connections

$F$ bundle $E \to M$ $\rightsquigarrow$ *monodromy* $\mu : \pi_1(M) \to \mathrm{Mod}(F)$

$E \to M$ admits a flat connection $\Leftrightarrow$

$$\begin{tikzcd}
 & \mathrm{Diff}(F) \arrow[d, "p"] \\
\pi_1(M) \arrow[ur, dashed, "\varphi"] \arrow[r, "\mu"] & \mathrm{Mod}(F)
\end{tikzcd}$$

$M \times M \to M$

$\Delta : M \to M \times M$ $\rightsquigarrow$ *monodromy* $\mathrm{Push} : \pi_1(M,*) \to \mathrm{Mod}(M,*)$

# Monodromy and flat connections

$F$ bundle $E \to M$ $\rightsquigarrow$ *monodromy* $\mu : \pi_1(M) \to \mathrm{Mod}(F)$

$M \times M \to M$ admits flat connection where diagonal is parallel.

$\Leftrightarrow$

$$\begin{tikzcd} & \mathrm{Diff}(M,*) \arrow[d, "p"] \\ \pi_1(M,*) \arrow[r, "\mathrm{Push}"] \arrow[ur, dotted, "\varphi"] & \mathrm{Mod}(M,*) \end{tikzcd}$$

$M \times M \to M$

$\Delta : M \to M \times M$

$\rightsquigarrow$

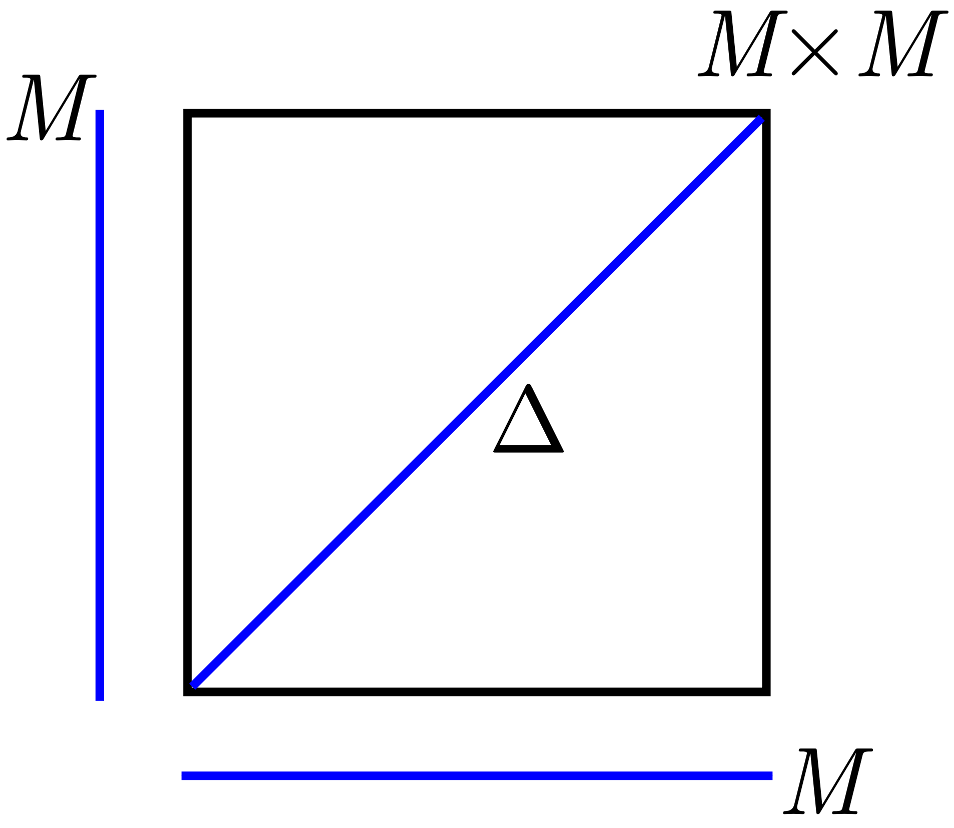
*monodromy*

$\mathrm{Push} : \pi_1(M,*) \to \mathrm{Mod}(M,*)$

## Corollary to Theorems 1 & 2.

Let $M$ be a locally symmetric manifold as in Theorems 1 and 2.

Then $M \times M \to M$ does not admit a flat connection for which the diagonal is parallel.



# Cohomology and flat bundles

# Characteristic classes

- Fix $F$.

- A characteristic class $c$

$$\left\{\begin{matrix}\text{Isomorphism classes of}\\ F\text{ bundles }E\to M\end{matrix}\right\} \xrightarrow{c} \mathrm{H}^*(M)$$

$$E \to M \quad \longmapsto \quad c(E)$$

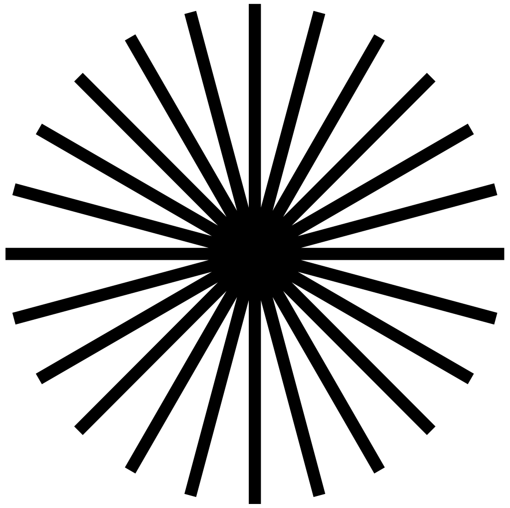
- Examples: Euler, Chern, Pontryagin, MMM

- Characteristic classes of flat bundles are often restricted.

# Characteristic classes of flat bundles

Example 1. $M^n$ manifold

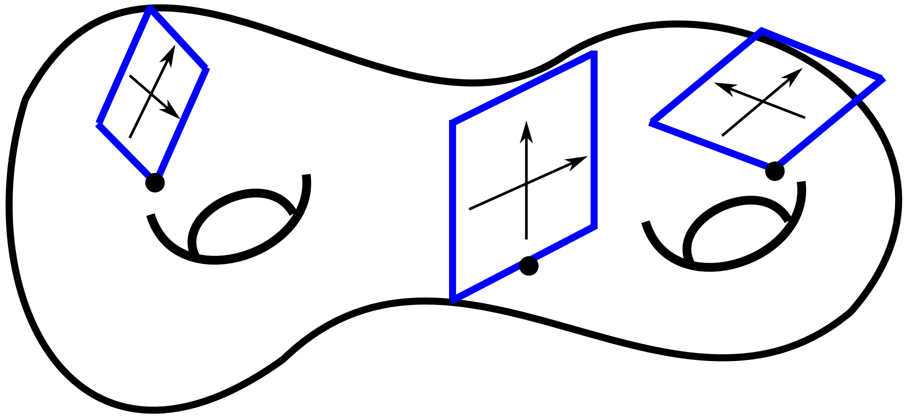
e.g. $M = \mathrm{CP}^2$



- $E \to M$ rank-$n$ vector bundle
- $p_i(E) \in H^{4i}(M)$ the $i$-th Pontryagin class
- Chern-Weil theory: If $E \to M$ admits a flat connection, then $p_i(E){=}0$ for all $i > 0$.
- Example. $M = \mathrm{CP}^2$, tangent bundle $TM \to M$. $p_1(TM){\neq}0$, so $TM{\to}M$ does not admit a flat connection.

# Characteristic classes of flat bundles

Example 2. $M = S_g$ closed surface, $g \geq 1$.



- $E \to M$ rank-2 vector bundle
- $e(E) \in H^2(M)$ the Euler class
- Milnor-Wood inequality (1958):
  If $E \to M$ admits a flat connection, then
  $$1-g \;\leq\; \langle \mathrm{e}(E),[M] \rangle \;\leq\; g-1$$
- Example. $E = TM$, $\langle \mathrm{e}(TM),[M] \rangle = \chi(M) = 2-2g$, so if $g \geq 2$, $TM \to M$ does not admit a flat connection.

# Characteristic classes of flat bundles

Example 3. $F = S_g, \quad g \geq 2.$

- $E \to M$ surface bundle
- $e_i(E) \in H^{2i}(M)$ the $i$-th MMM class
- Bott Vanishing Theorem (1970) ⇒ If $E \to M$ admits a flat connection, then $e_i(E)=0$ for $i \geq 3$.
- Example. (Morita) To show $E \to M^6$ does not admit a flat connection, show $e_3(E)\neq 0$.

# Main Theorem

Goal. Show that for any locally symmetric manifold $M = \Gamma\backslash G/K$, $\text{Push} : \pi_1(M,*) \to \text{Mod}(M,*)$ is not realized by diffeomorphisms.

$$\begin{tikzcd}
 & \text{Diff}(M,*) \arrow[d, "p"] \\
\pi_1(M,*) \arrow[r, "\text{Push}"'] \arrow[ur, dashed, "\varphi", "\times" description] & \text{Mod}(M,*)
\end{tikzcd}$$

- $M = \Gamma\backslash G/K$
- $p_i(M) \in \mathrm{H}^{4i}(M;\,\mathrm{R})$ $i$-th Pontryagin class of $\mathrm{T}M$

Theorem 2. (T−, 2014) Suppose one of the following holds

i) $M$ is a product of surfaces of genus $\geq 2$.

ii) $p_i(M) \neq 0$ for some $i > 0$.

iii) $\operatorname{rank} G \geq 2$ and every $\Gamma \to \mathrm{U}(n)$ has finite image.

Then Push : $\pi_1(M,*) \to \mathrm{Mod}(M,*)$ is not realized by diffeomorphisms.

$$\begin{tikzcd}
 & \mathrm{Diff}(M,*) \arrow[d, "p"] \\
\pi_1(M,*) \arrow[ur, dashed, "\varphi", "\times" marking] \arrow[r, "\mathrm{Push}"'] & \mathrm{Mod}(M,*)
\end{tikzcd}$$

# Elements of the proof

(i) Euler class and Milnor-Wood inequalities

(ii) Pontryagin classes, Chern-Weil theory, and classifying spaces of Lie groups

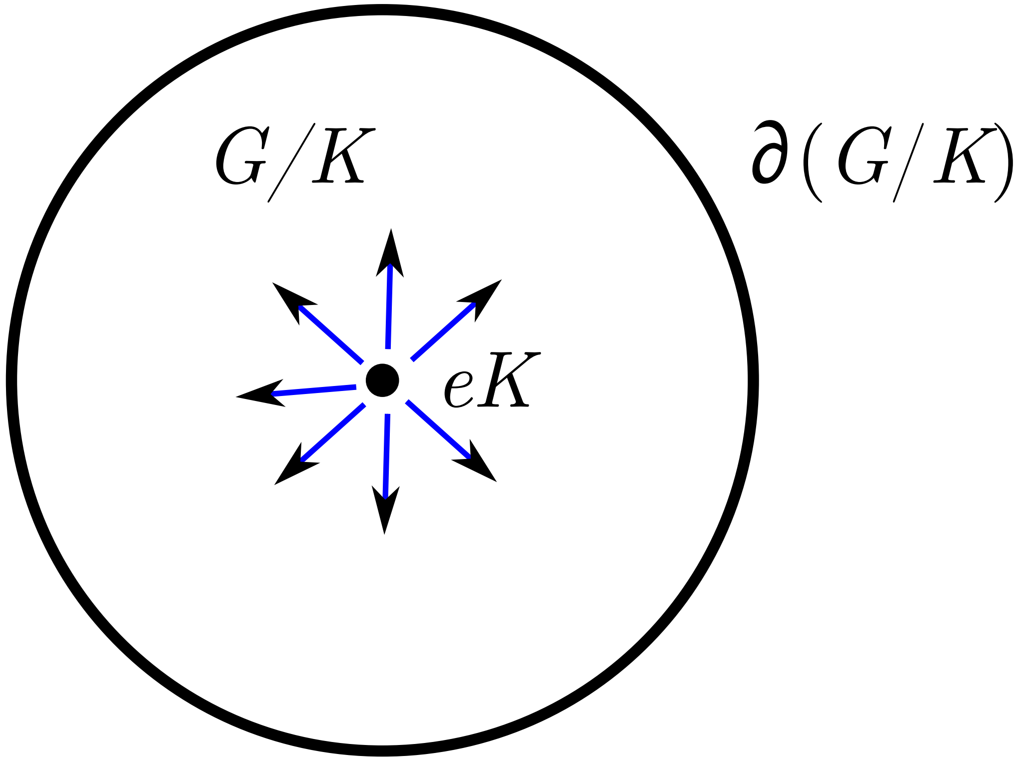
(iii) Margulis Superrigidity and representation theory of Lie algebras

Today. Explain (ii). Show if $p_i(M)\neq 0$ for some $i>0$, then

$$\mathrm{Push} : \pi_1(M,*) \to \mathrm{Mod}(M,*)$$

is not realized by diffeomorphisms.

# Geometry of symmetric spaces



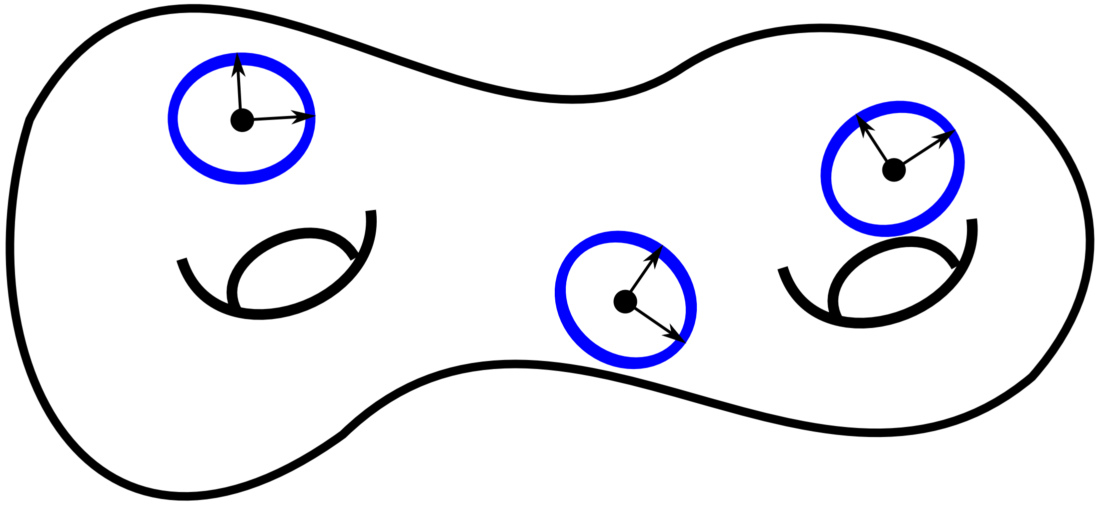
- $M = \Gamma \backslash G/K$
- $\widetilde{M} \cong G/K$
- $\partial(G/K)$ visual boundary
- $G$ acts on $\partial(G/K) \cong S^{n-1}$ by homeomorphisms

# Tangent bundle of a locally symmetric manifold

- $\Gamma \to G \to \mathrm{Homeo}(\partial(G/K)) \cong \mathrm{Homeo}(S^{n-1})$ induces an $S^{n-1}$ bundle

$$\frac{G/K \times S^{n-1}}{\Gamma} \longrightarrow \Gamma\backslash G/K$$

isomorphic to the unit tangent bundle of $\Gamma\backslash G/K$.



# Classifying spaces

- BHomeo$(S^{n-1})$ classifying space

$$\left\{\begin{matrix}\text{Isomorphism classes}\\ S^{n-1}\text{ bundles } E \to M\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{Homotopy classes}\\ M \to \text{BHomeo}(S^{n-1})\end{matrix}\right\}$$

- For $G \subset \text{Homeo}(S^{n-1})$

$$\left\{\begin{matrix}\text{isomorphism classes}\\ S^{n-1}\text{ bundles } E \to M\\ \text{structure group } G\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{homotopy classes}\\ M \to \text{B}G\end{matrix}\right\}$$

- $G^\delta$ denotes $G$ with discrete topology.

  B$G^\delta$ classifies flat $S^{n-1}$ bundles with holonomy in $G$

$M = \Gamma \backslash G / K$

Problem. Show $\varphi : \pi_1(M,*) \to \mathrm{Diff}(M,*)$ does not exist.

Step 1. Suppose $\varphi$ exists. Produce a commutative diagram:

$$\begin{tikzcd}
 & \mathrm{H}^*(\mathrm{B}G) \arrow[dl, "s"'] & \\
\mathrm{H}^*(M) & & \mathrm{H}^*(\mathrm{BHomeo}(\mathrm{S}^{n-1})) \arrow[ul, "t"'] \arrow[dl, "v"] \\
 & \mathrm{H}^*(\mathrm{BGL}_n(\mathrm{R})) \arrow[ul, "u"] &
\end{tikzcd}$$

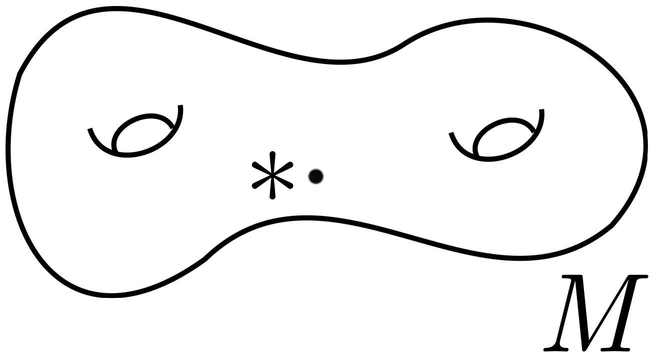
Step 2. Find $\alpha \in \mathrm{H}^*(\mathrm{BHomeo}(\mathrm{S}^{n-1}))$ so that $s \circ t(\alpha) \neq u \circ v(\alpha)$.

$\rightarrow\leftarrow$

# Step 1a. The two actions.

Suppose $\varphi : \pi_1(M,*) \to \mathrm{Diff}(M,*)$ exists.

$$\pi_1(M,*) \curvearrowright (M,*)$$

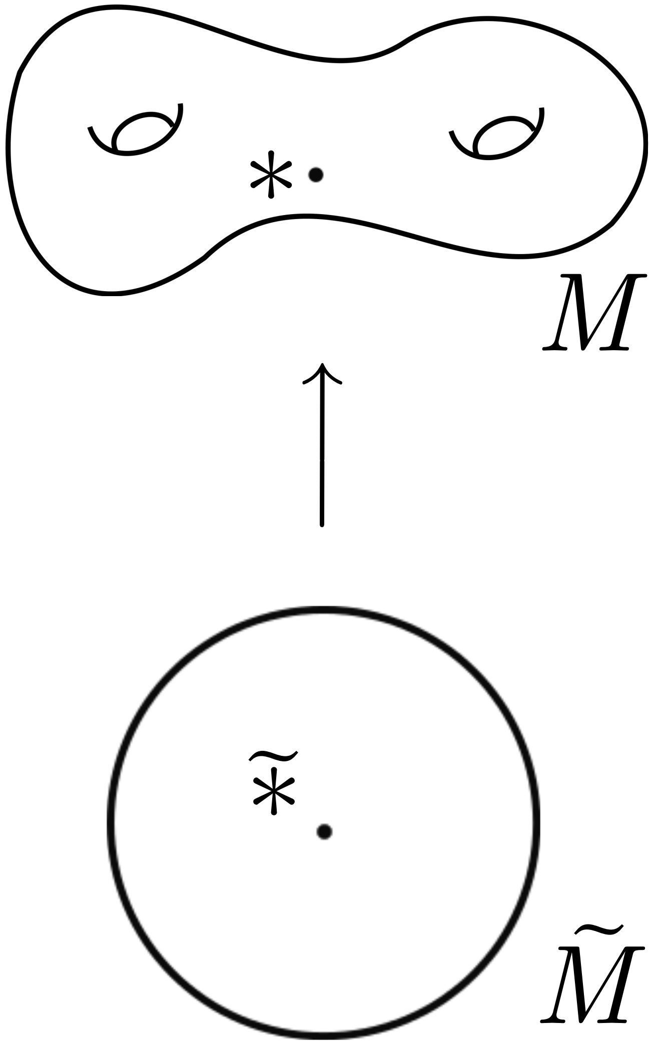


# Step 1a. The two actions.

Suppose $\varphi : \pi_1(M,*) \to \mathrm{Diff}(M,*)$ exists.

$$\pi_1(M,*) \curvearrowright (M,*)$$

*induces* $\pi_1(M,*) \curvearrowright (\tilde{M},\tilde{*})$



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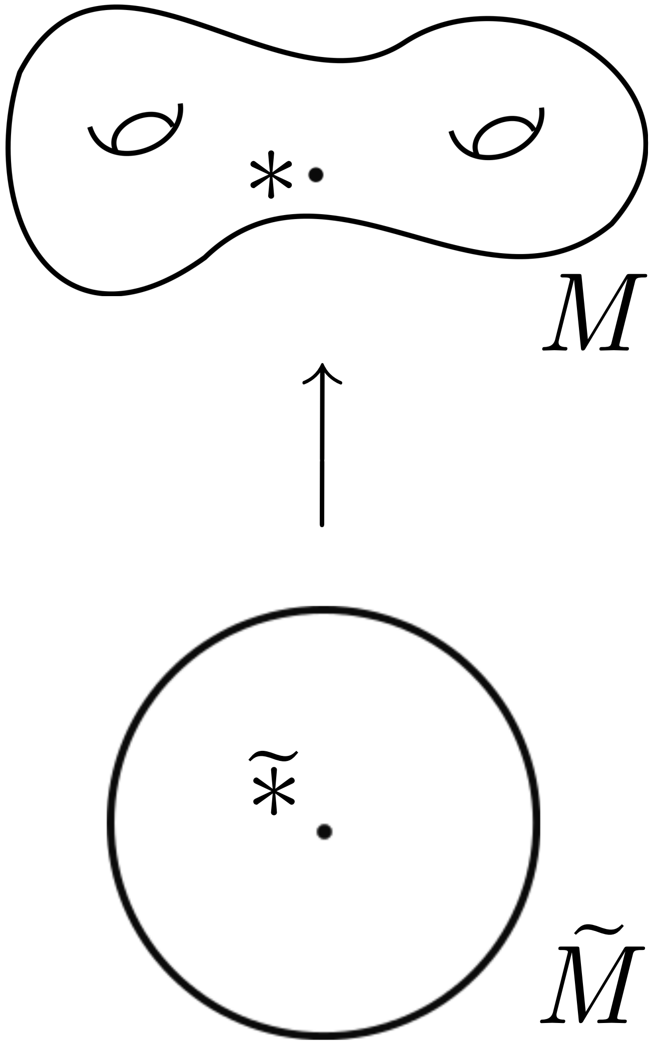

$$\pi_1(M,*) \curvearrowright (M,*)$$

*induces* $\quad \pi_1(M,*) \curvearrowright (\tilde{M},\tilde{*})$

**Action 1.** $\pi_1(M,*) \curvearrowright P(T_{\tilde{*}}\tilde{M}) \cong S^{n-1}$

**Action 2.** $\pi_1(M,*) \curvearrowright \partial \tilde{M} \cong S^{n-1}$

Step 1b. The diagram.

$$M = \Gamma \backslash G / K$$

## Step 1b. The diagram.

$M = \Gamma \backslash G / K$

$$\begin{tikzcd}
 & G \arrow[dr] & \\
\pi_1(M,*) \arrow[ur] \arrow[dr] & & \mathrm{Homeo}(\mathrm{S}^{n-1}) \\
 & \mathrm{GL}_n(\mathrm{R}) \arrow[ur] &
\end{tikzcd}$$

## Step 1b. The diagram.

$M = \Gamma \backslash G / K$

$$\begin{tikzcd}
 & G^{\delta} \arrow[r] & G \arrow[dr] & \\
\pi_1(M,*) \arrow[ur] \arrow[dr] & & & \mathrm{Homeo}(S^{n-1}) \\
 & \mathrm{GL}_n(\mathrm{R})^{\delta} \arrow[r] & \mathrm{GL}_n(\mathrm{R}) \arrow[ur] &
\end{tikzcd}$$

## Step 1b. The diagram.

$M = \Gamma \backslash G / K$

$$\begin{tikzcd}
 & \mathrm{B}G^{\delta} \arrow[r] & \mathrm{B}G \arrow[dr] & \\
\mathrm{B}\pi_1(M,*) \arrow[ur] \arrow[dr] & & & \mathrm{BHomeo}(\mathrm{S}^{n-1}) \\
 & \mathrm{BGL}_n(\mathrm{R})^{\delta} \arrow[r] & \mathrm{BGL}_n(\mathrm{R}) \arrow[ur] &
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\end{tikzcd}$$

## Step 1b. The diagram.

$M = \Gamma \backslash G / K$

$$\begin{tikzcd}
 & \mathrm{H}^*(\mathrm{B}G^\delta) \arrow[dl] & \mathrm{H}^*(\mathrm{B}G) \arrow[l] & \\
\mathrm{H}^*(M) & & & \mathrm{H}^*(\mathrm{BHomeo}(\mathrm{S}^{n-1})) \arrow[ul] \arrow[dl] \\
 & \mathrm{H}^*(\mathrm{BGL}_n(\mathrm{R})^\delta) \arrow[ul] & \mathrm{H}^*(\mathrm{BGL}_n(\mathrm{R})) \arrow[l] &
\end{tikzcd}$$

**Proposition.** This diagram commutes.

## Step 2. Characteristic classes.

$$\begin{tikzcd}
 & \mathrm{H}^*(\mathrm{B}G^\delta) \arrow[dl] & \mathrm{H}^*(\mathrm{B}G) \arrow[l] & \\
\mathrm{H}^*(M) & & & \mathrm{H}^*(\mathrm{BHomeo}(\mathrm{S}^{n-1})) \arrow[ul] \arrow[dl] \\
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 & & p_i &
\end{tikzcd}$$

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 & \mathrm{H}^*(\mathrm{BGL}_n(\mathrm{R})^\delta) \arrow[ul] & \mathrm{H}^*(\mathrm{BGL}_n(\mathrm{R})) \arrow[l] & \\
 & & p_i &
\end{tikzcd}$$

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$$\begin{tikzcd}
 & \mathrm{H}^*(\mathrm{B}G^{\delta}) \arrow[dl] & \mathrm{H}^*(\mathrm{B}G) \arrow[l] & p_i \\
\mathrm{H}^*(M) & & & \mathrm{H}^*(\mathrm{BHomeo}(\mathrm{S}^{n-1})) \arrow[ul] \arrow[dl, "\text{surjective}"] \\
 & \mathrm{H}^*(\mathrm{BGL}_n(\mathrm{R})^{\delta}) \arrow[ul] & \mathrm{H}^*(\mathrm{BGL}_n(\mathrm{R})) \arrow[l] &
\end{tikzcd}$$

## Step 2. Characteristic classes.

$$\begin{tikzcd}
 & \mathrm{H}^*(\mathrm{B}G^\delta) \arrow[dl, "p_i(M)"'] & \mathrm{H}^*(\mathrm{B}G) \arrow[l] & \\
\mathrm{H}^*(M) & & & \mathrm{H}^*(\mathrm{BHomeo}(\mathrm{S}^{n-1})) \arrow[ul] \arrow[dl, "\text{surjective}"] \\
 & \mathrm{H}^*(\mathrm{BGL}_n(\mathrm{R})^\delta) \arrow[ul] & \mathrm{H}^*(\mathrm{BGL}_n(\mathrm{R})) \arrow[l] &
\end{tikzcd}$$

# Step 2. Characteristic classes.

$$\begin{tikzcd}
 & & H^*(BG^\delta) \arrow[dl] & H^*(BG) \arrow[l] & & \\
p_i(M) & H^*(M) & & & H^*(BHomeo(S^{n-1})) \arrow[ul] \arrow[dl, "\text{surjective}"] & \\
 & & H^*(BGL_n(R)^\delta) \arrow[ul] & H^*(BGL_n(R)) \arrow[l] & & p_i
\end{tikzcd}$$

## Step 2. Characteristic classes.

$$\begin{tikzcd}
 & \mathrm{H}^*(\mathrm{B}G^\delta) \arrow[ddl] & \mathrm{H}^*(\mathrm{B}G) \arrow[l] & \\
p_i(M) & & & \\
\mathrm{H}^*(M) & & & \mathrm{H}^*(\mathrm{BHomeo}(\mathrm{S}^{n-1})) \arrow[uul] \arrow[dl, "\text{surjective}"] \\
 & \mathrm{H}^*(\mathrm{BGL}_n(\mathrm{R})^\delta) \arrow[ul] & \mathrm{H}^*(\mathrm{BGL}_n(\mathrm{R})) \arrow[l] & \\
 & & p_i &
\end{tikzcd}$$

## Step 2. Characteristic classes.

$$\begin{tikzcd}
 & & \mathrm{H}^*(\mathrm{B}G^\delta) \arrow[dl] & \mathrm{H}^*(\mathrm{B}G) \arrow[l] & \\
p_i(M) & \mathrm{H}^*(M) & & & \mathrm{H}^*(\mathrm{BHomeo}(\mathrm{S}^{n-1})) \arrow[ul] \arrow[dl, "\text{surjective}"] \\
 & & \mathrm{H}^*(\mathrm{BGL}_n(\mathrm{R})^\delta) \arrow[ul] & \mathrm{H}^*(\mathrm{BGL}_n(\mathrm{R})) \arrow[l, "\text{Chern-Weil}"'] & \\
 & & & p_i &
\end{tikzcd}$$

## Step 2. Characteristic classes.

$p_i(M)$

$$\begin{tikzcd}
 & \mathrm{H}^*(\mathrm{B}G^\delta) \arrow[dl] & \mathrm{H}^*(\mathrm{B}G) \arrow[l] & \\
\mathrm{H}^*(M) & & & \mathrm{H}^*(\mathrm{BHomeo}(\mathrm{S}^{n-1})) \arrow[ul] \arrow[dl, "\text{surjective}"] \\
 & \mathrm{H}^*(\mathrm{BGL}_n(\mathrm{R})^\delta) \arrow[ul] & \mathrm{H}^*(\mathrm{BGL}_n(\mathrm{R})) \arrow[l, "\text{Chern-Weil}"'] & \\
 & 0 & &
\end{tikzcd}$$

## Step 2. Characteristic classes.

$$\begin{tikzcd}
 & H^*(BG^\delta) \arrow[ddl] & H^*(BG) \arrow[l] & \\
p_i(M) & & & \\
H^*(M) & & & H^*(B\mathrm{Homeo}(S^{n-1})) \arrow[uul] \arrow[ddl, "\text{surjective}"] \\
0 & & & \\
 & H^*(BGL_n(\mathbb{R})^\delta) \arrow[uul] & H^*(BGL_n(\mathbb{R})) \arrow[l, "\text{Chern-Weil}"'] &
\end{tikzcd}$$

Conclusion. If $p_i(M)\neq 0$ for some $i>0$, then

Push : $\pi_1(M,*)\rightarrow \mathrm{Mod}(M,*)$

is not realized by diffeomorphisms.

## Question.

For which $\Gamma\backslash G/K$ is $p_i(\Gamma\backslash G/K) \neq 0$ for some $i > 0$?

# Pontryagin classes of locally symmetric manifolds

Assume $\Gamma\backslash G/K$ compact.

- (Borel-Hirzebruch, 1958):
  - algorithm to determine if $p_i(\Gamma\backslash G/K)\neq 0$ (depends only on $G$, not on particular $\Gamma$)
  - some examples ($G$ Hermitian)
- (T—): complete list of $G$ for which $p_i(\Gamma\backslash G/K)\neq 0$ for some $i$.

# $p_i(\Gamma\backslash \mathrm{G}/K)\neq 0$ for some $i$

| $G$ |
|---|
| $\mathrm{SU}(p,q)$ |
| $\mathrm{SP}(p,q)$ |
| $\mathrm{SO}(p,q)$ |
| $\mathrm{E}_{6(6)}$, $\mathrm{E}_{6(2)}$, $\mathrm{E}_{6(\text{-}14)}$ |
| $\mathrm{E}_{7(7)}$, $\mathrm{E}_{7(\text{-}5)}$, $\mathrm{E}_{7(\text{-}25)}$ |
| $\mathrm{E}_{8(8)}$, $\mathrm{E}_{8(\text{-}24)}$ |
| $\mathrm{F}_{4(4)}$, $\mathrm{F}_{4(\text{-}20)}$ |
| $\mathrm{G}_{2(2)}$ |

$p_i(\Gamma\backslash \mathrm{G}/K) \neq 0$ for some $i$

| $G$ |
|---|
| $\mathrm{SU}(p,q)$ |
| $\mathrm{SP}(p,q)$ |
| $\mathrm{SO}(p,q)$ |
| $\mathrm{E}_{6(6)}$, $\mathrm{E}_{6(2)}$, $\mathrm{E}_{6(-14)}$ |
| $\mathrm{E}_{7(7)}$, $\mathrm{E}_{7(-5)}$, $\mathrm{E}_{7(-25)}$ |
| $\mathrm{E}_{8(8)}$, $\mathrm{E}_{8(-24)}$ |
| $\mathrm{F}_{4(4)}$, $\mathrm{F}_{4(-20)}$ |
| $\mathrm{G}_{2(2)}$ |

$p_i(\Gamma\backslash \mathrm{G}/K) = 0$ for all $i$

| $G$ |
|---|
| $\mathrm{SL}_n(\mathrm{R})$ |
| $\mathrm{SU}^*(2n)$ |
| $\mathrm{SO}(p,1)$ |
| $\mathrm{SO}(2,2)$, $\mathrm{SO}(3,3)$ |
| $\mathrm{E}_{6(-26)}$ |
| $\mathrm{SL}_n(\mathrm{C})$ |
| $\mathrm{SO}_n(\mathrm{C})$, $\mathrm{SP}_{2n}(\mathrm{C})$ |
| complex exceptional |

# Thank you.

- B. Tshishiku, Cohomological obstructions to Nielsen realization, `arxiv:1402.0472`. Jan. 2014.
- B. Tshishiku, Pontryagin classes of locally symmetric spaces, `arxiv:1404.1115`. Apr. 2014.