Symmetries of exotic negatively curved manifolds

Bena Tshishiku

Spring Topology and Dynamics Conference

5/12/2021

joint with Mauricio Bustamante

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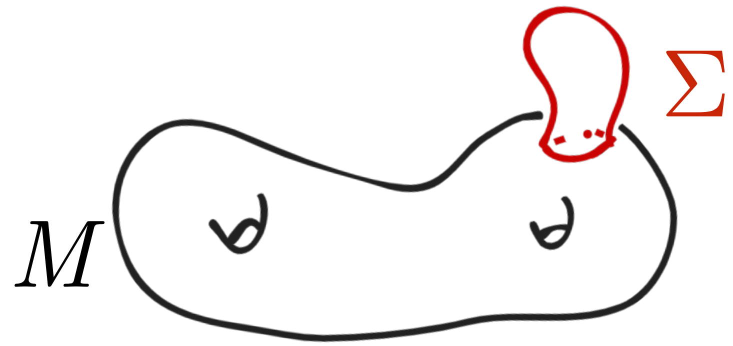

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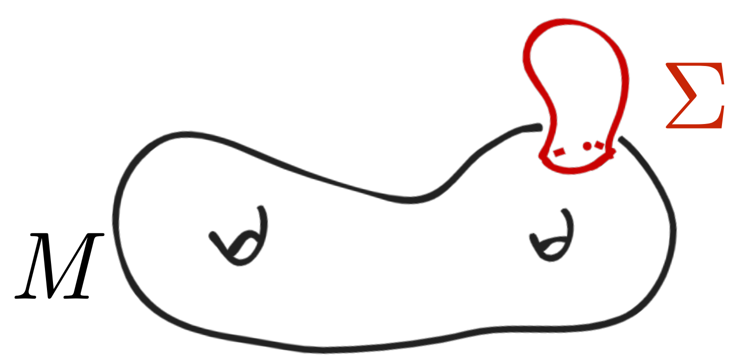

Main Question: How much symmetry does N have?

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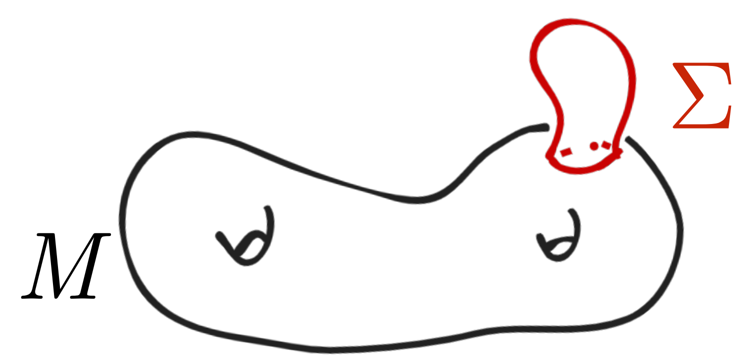

Main Question: How much symmetry does N have?

Specifically, what is the maximum size of a finite subgroup $G < \mathrm{Diff}(N)$?

An “upper bound”

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(sample) Question: Does $\mathrm{Isom}(M)$ act (faithfully) on N?

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- If $\Sigma \in \Theta_n \setminus bP_{n+1}$, then $\dim G \lesssim \frac{1}{8} \dim \mathrm{Isom}(S^n)$

Connection to Nielsen realization

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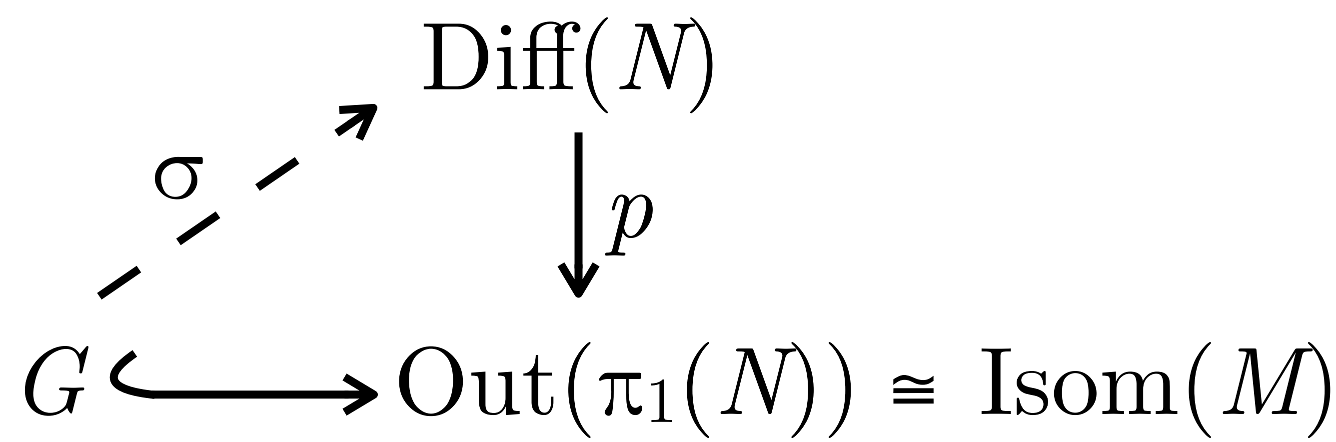

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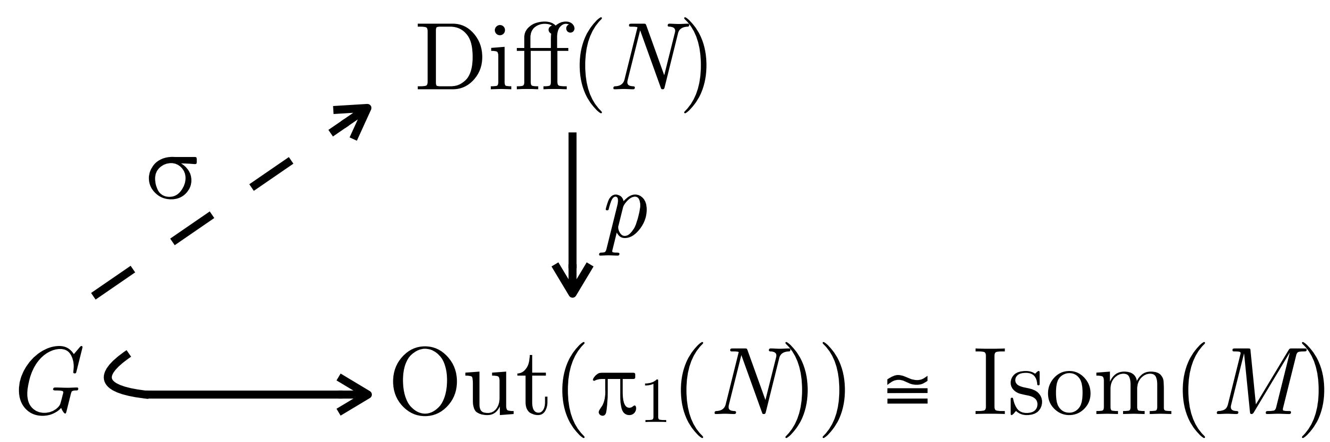

(Equivalent) Question: What is the maximum size of a subgroup of $\mathrm{Out}(\pi_1(N))$ that lifts to $\mathrm{Diff}(N)$?

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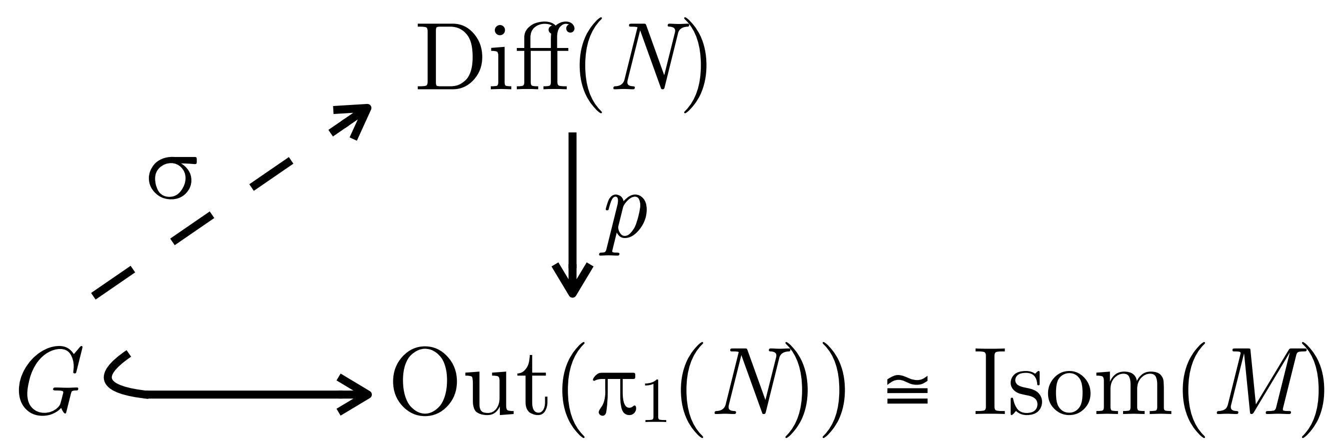
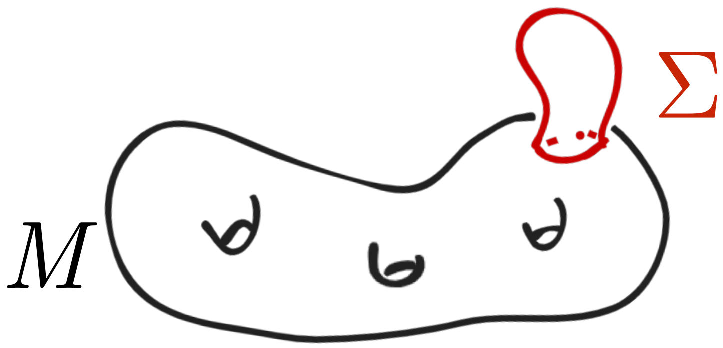


(Equivalent) Question: What is the maximum size of a subgroup of $\mathrm{Out}(\pi_1(N))$ that lifts to $\mathrm{Diff}(N)$?

E.g. does $p : \mathrm{Diff}(N) \to \mathrm{Out}(\pi_1(N))$ split?

Example: $N = M\#\Sigma$



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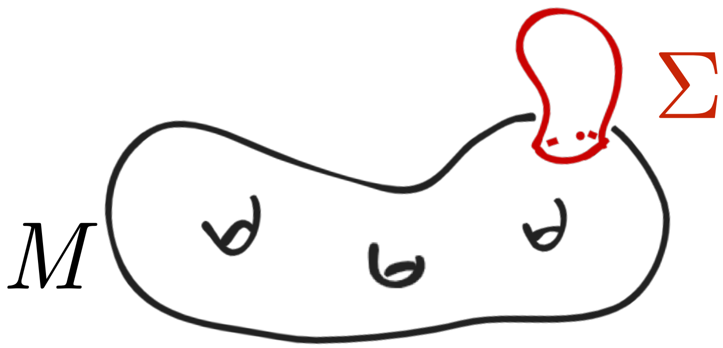

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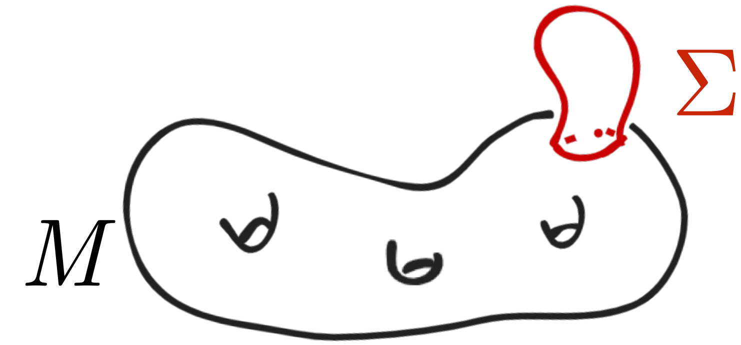

(Farrell-Jones): Fix n with $\Theta_n \neq 0$.

Then $\exists$ M^n and $N = M\#\Sigma$ so that

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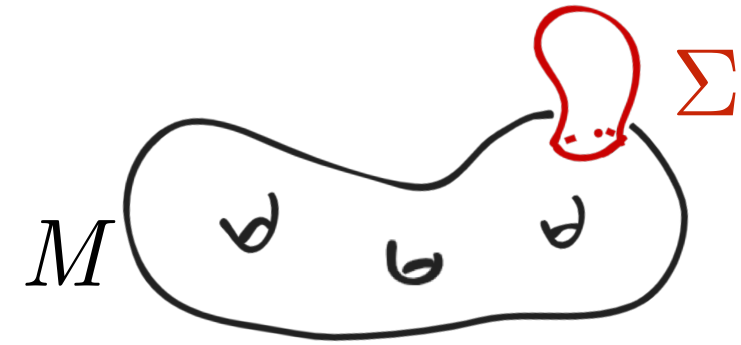

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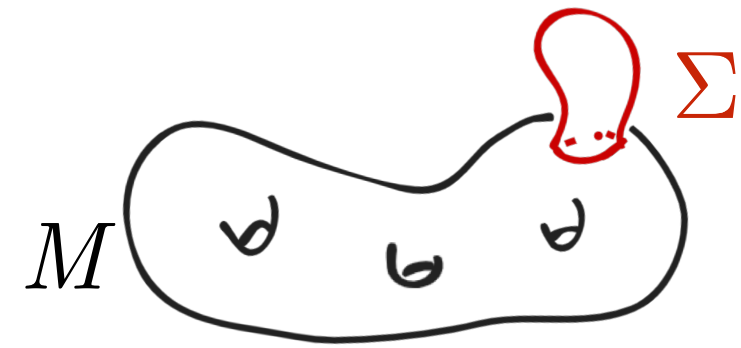

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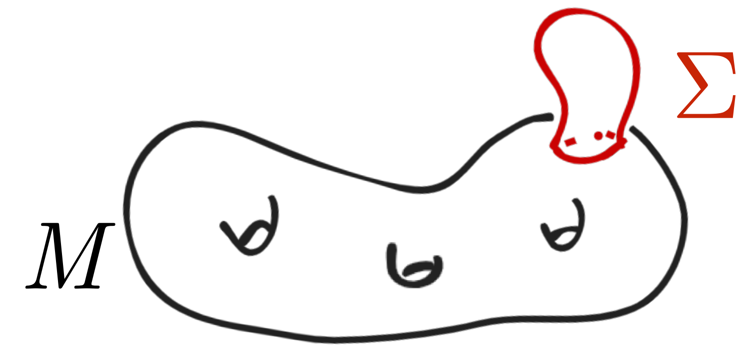

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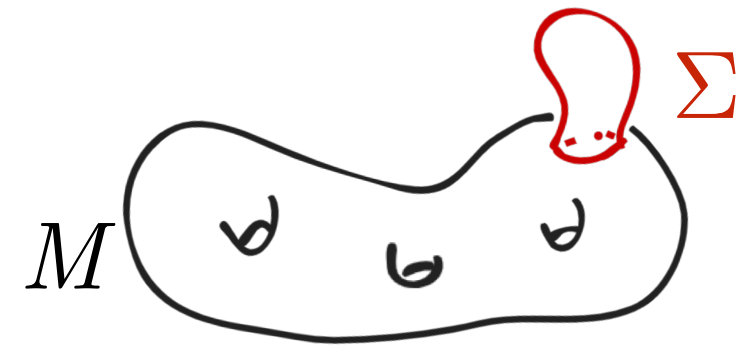

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$$G < \mathrm{Isom}(M) \text{ acts on } N = M\#\Sigma \iff |G| \text{ divides } |\Sigma|.$$

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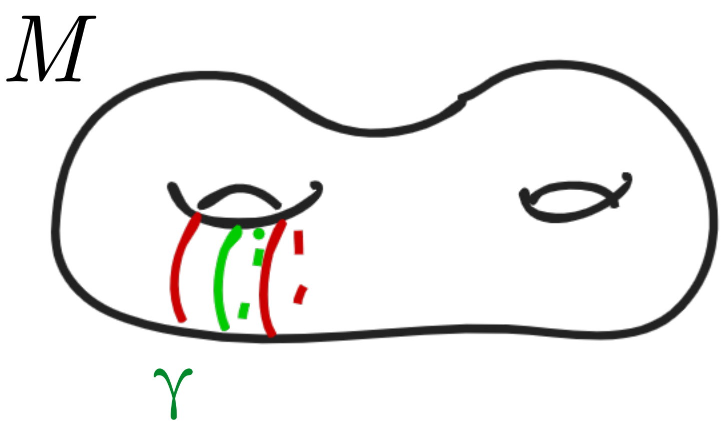

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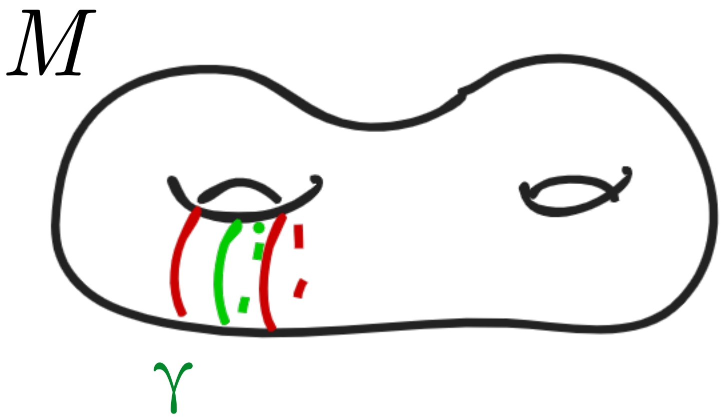

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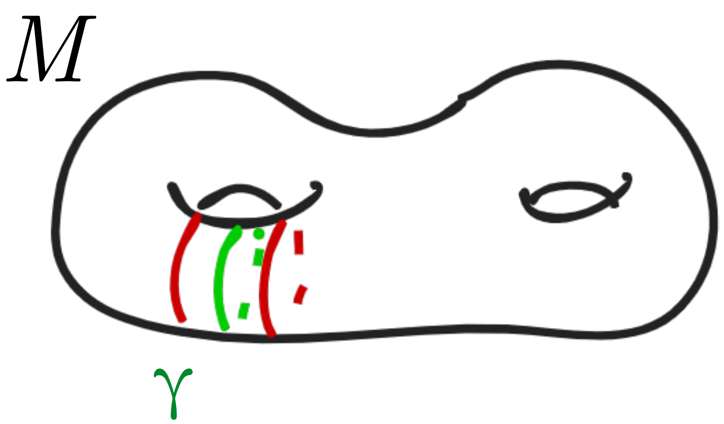

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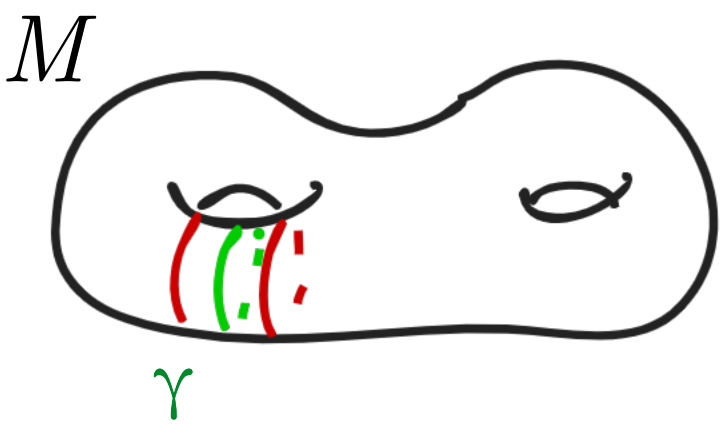

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Theorem (Bustamante-T). Fix n with $\Theta_{n-1} \neq 0$.

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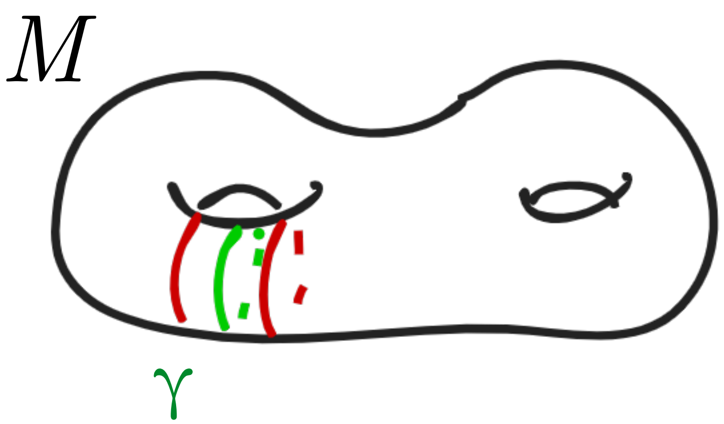

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Theorem (Bustamante-T). Fix n with $\Theta_{n-1} \neq 0$.

For each $d \geq 2$, $\exists$ M^n and $N{=}M_{\gamma,\varphi}$ so that for every finite $G{<}\mathrm{Diff}(N)$, $|G| \leq \frac{1}{d}\,|\mathrm{Isom}(M)|$.

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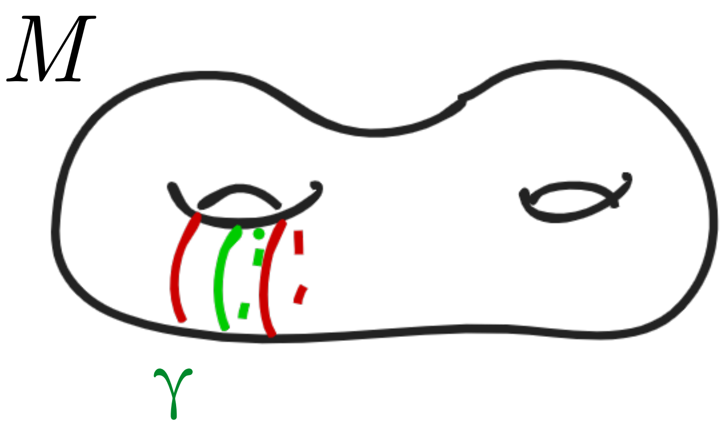

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<u>Theorem</u> (Bustamante-T). Fix n with $\Theta_{n-1} \neq 0$.

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Proved by showing $\mathrm{Im}(p)$ has index $\geq d$ in $\mathrm{Isom}(M)$

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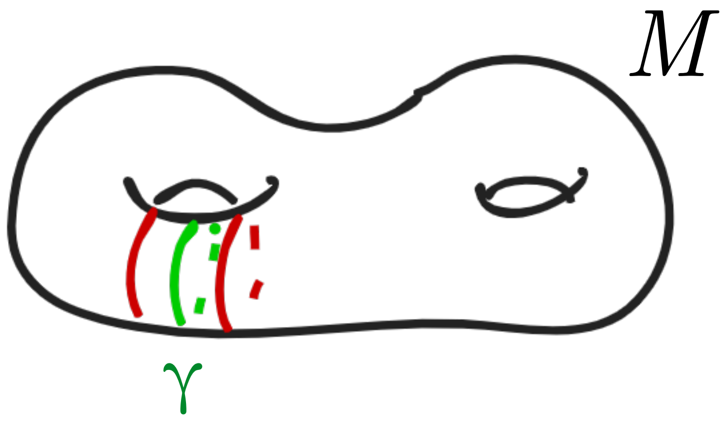
Proof Sketch

$N = M_{\gamma,\varphi}$



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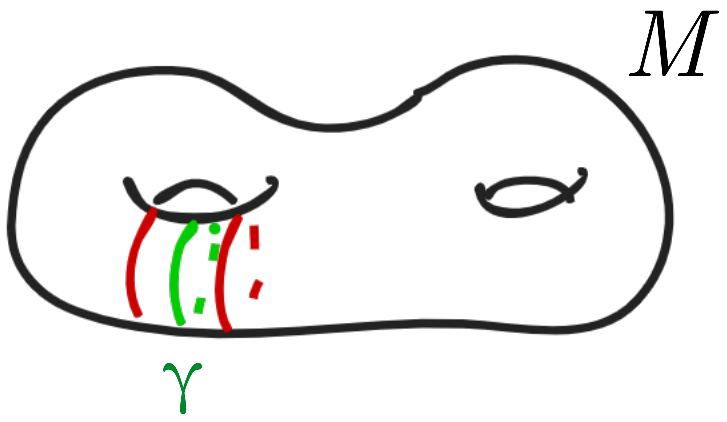

- Assume M has an order-d isometry α.

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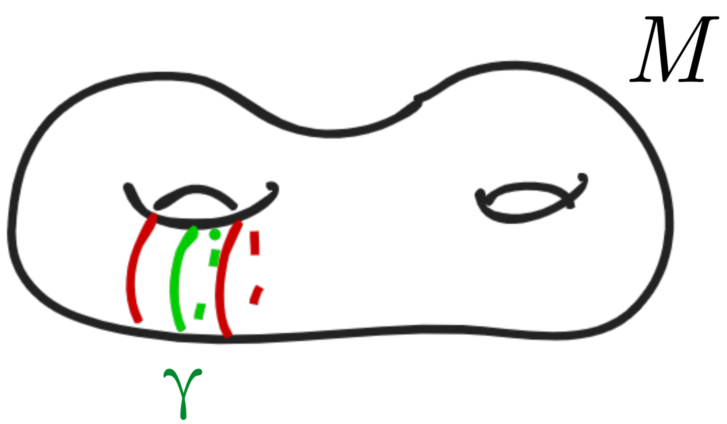

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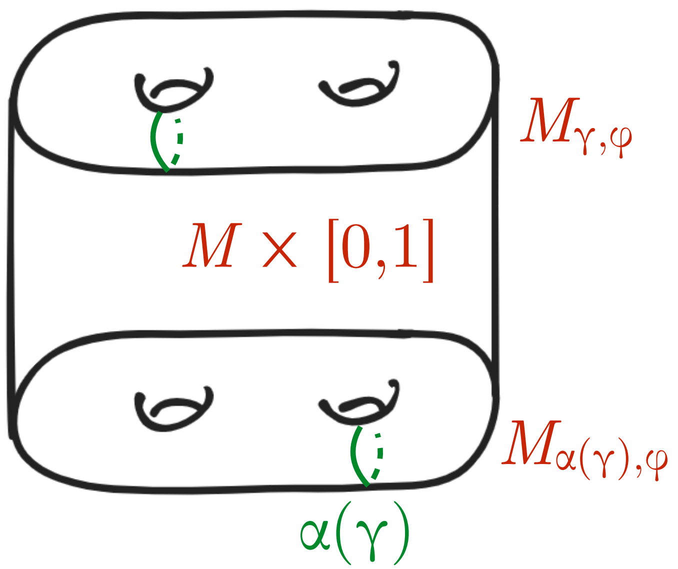

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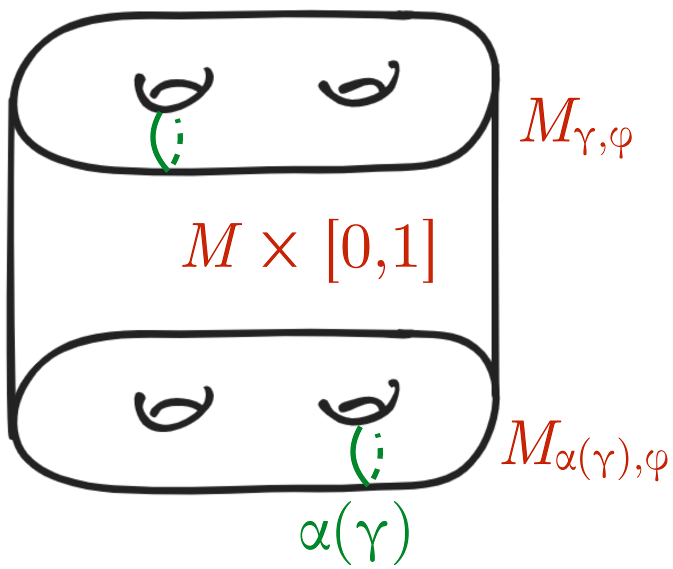

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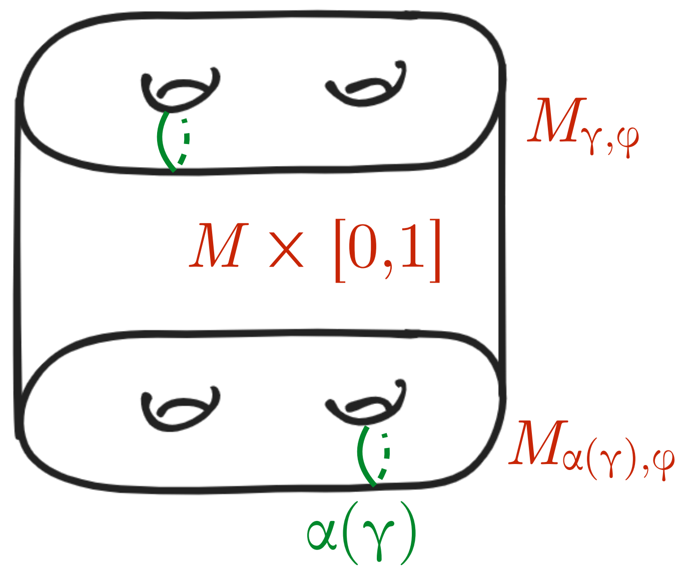

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  Assume there is homomorphism $\pi_1(M) \to \mathbb{Z}^d$ with $\alpha^i(\gamma) \longmapsto e_i$

  Then $M_{\alpha^i(\gamma),\varphi}$ and $M_{\alpha^j(\gamma),\varphi}$ are not concordant $\forall\; i,j$.

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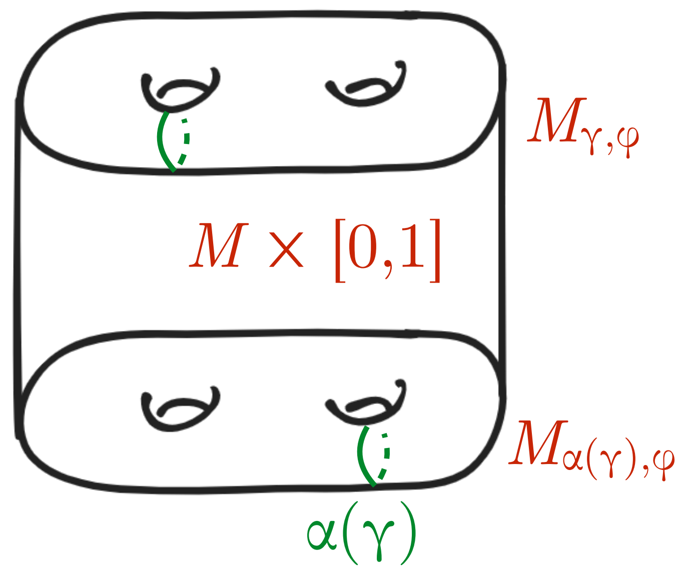

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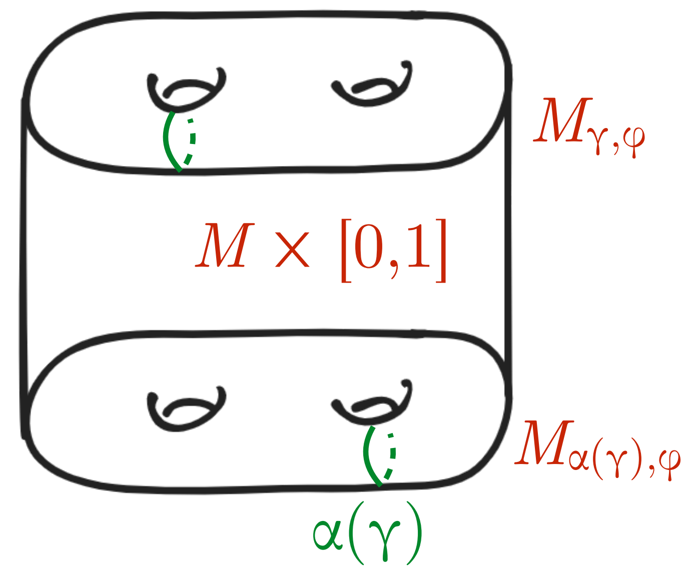
Assume there is homomorphism $\pi_1(M) \to \mathbb{Z}^d$ with $\alpha^i(\gamma) \longmapsto e_i$

Then $M_{\alpha^i(\gamma),\varphi}$ and $M_{\alpha^j(\gamma),\varphi}$ are not concordant $\forall\, i,j$.

$$\Longrightarrow \langle\alpha\rangle \cap \mathrm{Im}(p) = \{1\}$$

Proof Sketch

$N = M_{\gamma,\varphi}$



- Assume M has an order-d isometry α.

  WTS: $\langle\alpha\rangle \cap \mathrm{Im}(p) = \{1\}$.

  $p : \mathrm{Diff}(N) \to \mathrm{Out}(\pi_1(N)) \cong \mathrm{Isom}(M)$

- Step 1 If $\exists f \in \mathrm{Diff}(N)$ inducing $\alpha \in \mathrm{Out}(\pi_1(N))$

then $M_{\gamma,\varphi}$ and $M_{\alpha(\gamma),\varphi}$ are *concordant* smooth structures.

- Step 2 Assume M stably parallelizable.

$$\left\{ \begin{matrix} \text{smooth structures} \\ \text{on } M \end{matrix} \right\} \Big/ \text{concordance} \quad \overset{1\text{-}1}{\longleftrightarrow} \quad \left\{ \begin{matrix} \text{maps} \\ M \to \mathrm{Top}/\mathrm{O} \end{matrix} \right\} \Big/ \text{homotopy}$$

Assume there is homomorphism $\pi_1(M) \to \mathbb{Z}^d$ with $\alpha^i(\gamma) \longmapsto e_i$

Then $M_{\alpha^i(\gamma),\varphi}$ and $M_{\alpha^j(\gamma),\varphi}$ are not concordant $\forall\ i,j$.

$$\Longrightarrow \langle\alpha\rangle \cap \mathrm{Im}(p) = \{1\}$$

- Step 3 Show examples satisfying the assumptions exist.

Question

Does there exist M with $|\text{Isom}(M)| \gg 1$ and N exotic smooth structure that is *asymmetic*, i.e. $\text{Diff}(N)$ has no nontrivial finite order element?

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Does there exist M with $|\mathrm{Isom}(M)| \gg 1$ and N exotic smooth structure that is *asymmetic*, i.e. $\mathrm{Diff}(N)$ has no nontrivial finite order element?

Equivalently, $p : \mathrm{Diff}(N) \to \mathrm{Out}(\pi)$ is trivial.

Thank you.