Columbia
2018

I. Symmetry constant

M closed hyperbolic, $\pi = \pi_1 M$

N <u>exotic</u> smooth structure: $N \simeq M$ homeomorphic, $N \not\cong M$ not diffeomorphic

<u>Ex</u> $M = \mathbb{H}^n/\Gamma$ $\quad \Gamma < \{A \in SL_{n+1}\mathbb{R} \mid A^t B A = B\} \cap SL_{n+1}(\mathbb{Z}[\sqrt{2}])$

finite index torsionfree. $\quad B = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 & \\ & & & -\sqrt{2} \end{pmatrix}$ $\quad SO(n,1) \simeq \mathrm{Isom}\,\mathbb{H}^n$

$N = M \# \Sigma \qquad \Sigma \in \Theta_n$ = exotic spheres $\left(\begin{array}{l}\text{sm. str on } S^n \\ \text{up to or-pres. diffeo}\end{array}\right)$

$(\Theta_7 \simeq \mathbb{Z}/28)$

<u>Defn</u> Symmetry constant $\quad s(N) = \sup\limits_{\substack{F < \mathrm{Diff}(N) \\ \text{finite}}} |F|$.

Main <u>Q</u>: What are possible values of $s(N)$? (varying over M, $N \simeq M$)

<u>Ex</u> For M, $s(M) = |\mathrm{Isom}(M)|$.

(Borel) M aspherical, $Z(\pi) = 1 \Rightarrow$ Any $F < \mathrm{Diff}(M)$ finite acts faithfully on π

i.e. $F \hookrightarrow \mathrm{Diff}(M) \longrightarrow \mathrm{Out}(\pi)$ injective

$\Rightarrow s(M) \leq |\mathrm{Out}(\pi)|$.

(Mostow) $n \geq 3 \Rightarrow \mathrm{Out}(\pi) \simeq \mathrm{Isom}(M)$ (h.e. homotopic to isometry)

for $N \simeq M$, $\quad 1 \leq s(N) \leq |\mathrm{Isom}(M)|$ (so what's possible??)

<u>Ex</u> (positive curvature) For $\Sigma \in \Theta_n$ define $s(\Sigma) = \sup\limits_{\substack{G < \mathrm{Diff}(\Sigma) \\ \text{Lie gp}}} \dim G$.

$1 \leq s(\Sigma) \leq \dim SO(n+1)$

(Hsiang–Hsiang 1966) $\Sigma \not\cong S^n$, $n \geq 40 \Rightarrow s(\Sigma) < \frac{n^2}{8} + 1 < \frac{1}{4} \cdot \dim SO(n+1)$.

(sharp: $\exists\ \Sigma^{8k+1}$ w/ $s(\Sigma) = \frac{n^2}{8} + \frac{7}{8}$)

OTOH (Schultz) $\exists\ \Sigma^{10}$ with $1 \leq s(\Sigma) \leq 2$.

Not much known about $s(N)$

<u>Ex</u> (Farrell–Jones, 1989) If $\Theta_n \neq 0$ $\exists$ $N = M \# \Sigma$ w/ $s(N) \leq \frac{1}{2} |\mathrm{Isom}\, M|$.

<u>Theorem</u> (Bustamante–T)

- Fix n w/ $\Theta_{n-1} \neq 0$ $\forall d > 0$ $\exists$ M, $N \simeq M$ s.t. $s(N) \leq \frac{1}{d} |\mathrm{Isom}(M)|$.
- Fix n w/ $\Theta_n \neq 0$ $\forall d > 0$ $\exists$ M, $N \simeq M$ s.t. $s(N) = |\mathrm{Isom}\, M| \geq d$.

II. Nielsen realization

<u>Problem</u> ~~Given $F < \mathrm{Out}(\pi)$~~

$$\begin{tikzcd} & \mathrm{Diff}(N) \arrow[d, "\Psi_N"] \\ F \arrow[ur, dashed, "\exists ?"] \arrow[r, hook, "\text{Given}"'] & \mathrm{Out}(\pi) \end{tikzcd}$$

If yes, say F <u>realized by diffeos</u> of N

Borel $\Rightarrow$ $s(N) = \sup\limits_{\substack{F < \mathrm{Out}(\pi) \\ \text{realized}}} |F|$.   eg $s(N) = |\mathrm{Isom}(M)|$ $\Leftrightarrow$ $\mathrm{Out}(\pi)$ realized by diffeos of N.

<u>Ex</u>. $\alpha \in \mathrm{Out}(\pi)$, $F = \langle \alpha \rangle \simeq \mathbb{Z}/\ell$

- first obstruction to realizing F: $\alpha \notin \mathrm{im}\, \Psi_N$. (this happens! FJ)
- $\alpha \in \mathrm{im}\, \Psi_N \Rightarrow \exists$ $f: N \to N$, $\Psi_N(f) = \alpha$, f^ℓ homotopic to id.

  want f s.t. $f^\ell = id$.

(Nielsen, early 1900s) studied for surfaces $S_g =$ 

$$\begin{tikzcd} & \mathrm{Diff}(S_g) \arrow[d] \\ F \arrow[ur, dashed] \arrow[r, hook] & \mathrm{Out}(\pi_1(S_g)) \simeq \pi_0 \mathrm{Diff}(S_g) \end{tikzcd}$$

(Kerckhoff, 1983) every $F < \mathrm{Mod}_g$ finite realized by isometries wrt some hyp. metric on S_g.

<u>Warmup</u> (understand $\mathrm{im}\, \Psi_N$) Assume M stably parallelizable $(TM \oplus \underline{\mathbb{R}} \simeq \underline{\mathbb{R}}^{n+1})$ always true in cover
$N = M \# \Sigma$.

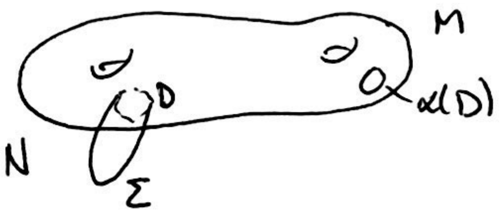
(a) For any Σ $\mathrm{im}\, \Psi_N \supset \mathrm{Out}^+(\pi)$

(b) $2\Sigma \neq 0 \Leftrightarrow \forall \alpha \in \mathrm{Out}(\pi) \setminus \mathrm{Out}^+(\pi)$ $\alpha \notin \mathrm{im}\, \Psi_N$. $(\Rightarrow s(N) \leq \frac{1}{2} |\mathrm{Isom}\, M|.)$

(c) ~~if~~ $|\mathrm{Isom}^+ M|$ & $|\Sigma|$ rel prime $\Rightarrow$ $\mathrm{Out}^+(\pi)$ realized by diffeos. of N.
$(\Rightarrow s(N) \geq \frac{1}{2} |\mathrm{Isom}\, M|)$

Proof of warmup:

(a) $\alpha \in \mathrm{Out}^+(\pi) \simeq \mathrm{Isom}^+(M)$



up to isotopy $\alpha(D) = D$

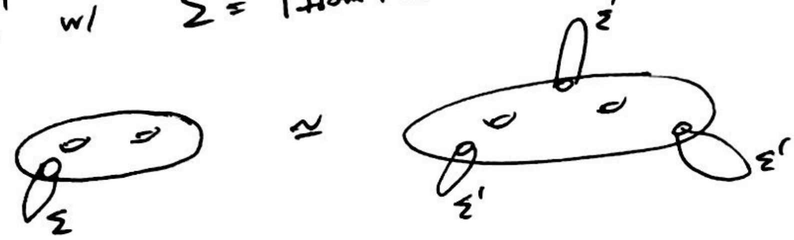
$\alpha|_D = id \in SO(n) \quad \rightsquigarrow \quad f : N \to N \quad$ w/ $\Psi_N(f) = \alpha$.

(b) $\alpha \in \mathrm{Out}(\pi) \setminus \mathrm{Out}^+(\pi) \quad \rightsquigarrow$ or. rev. diffeo $g : N \to M \# \overline{\Sigma}$.

suppose $\exists \; f : N \to N$ w/ $\Psi_N(f) = \alpha$. Then $g \circ f : N \to M \# \overline{\Sigma}$ or pres diffeo

$\underset{FJ}{\Longrightarrow} \quad \Sigma = \overline{\Sigma}$ in Θ_n ie $2\Sigma = 0$.

(c) $\exists \, \Sigma'$ w/ $\Sigma = |\mathrm{Isom}^+ M| \cdot \Sigma'$ in Θ_n.

N =  □.

Examples (Belolipetsky–Lubotzky) $\forall n \geq 2 \quad \forall \; G$ finite group $\exists \; M^n$ hyperbolic w/ $\mathrm{Isom}^+ M \simeq \mathrm{Isom}(M) \simeq G$ ~~(or fact~~ $\cancel{\mathrm{Isom}(M)}$~~)~~

This gives examples w/ $s(N) = 1$ or $1/2$.

Can even arrange ~~for~~ (in special cases) for $\mathrm{Out}^+(\pi)$ to act by isoms of N wrt neg. curved metric.

Q: what other values can $s(N)$ take for $N = M \# \Sigma$?

can you ever realize $\mathrm{Out}^+(\pi)$ when there's no obvious symmetry?! (embarrassing ...)

III. Obstructions to realization.

Thm Fix n w/ $\Theta_{n-1} \neq 0$. For $d > 0 \; \exists \; M, N \simeq M$ s.t. $s(N) \leq \frac{1}{d} |\mathrm{Isom}(M)|$.

Strategy: Find $M, N, \quad F < \mathrm{Out}(\pi) \quad F \simeq \mathbb{Z}/d\mathbb{Z} \;$ & $\; \mathrm{im}\Psi_N \cap F = \{1\}$.

Then $s(N) \leq |\mathrm{Im}\,\Psi_N| \leq \frac{1}{d} |\mathrm{Out}(\pi)|$.

<u>What's N?</u> Fix $j: S^1 \times D^{n-1} \hookrightarrow M^n$ framing of geodesic γ

$$M_{\gamma,\phi} = \left[S^1 \times D^{n-1} \sqcup \left(M \setminus j(S^1 \times \mathrm{int}(D^{n-1})) \right) \right] / \sim$$

$$\underset{(x,v)}{\overset{S^1\times S^{n-2}}{\cup}} \longleftrightarrow j(x, \phi(x)) \qquad \phi \in \mathrm{Diff}(S^{n-2})$$

$[\phi] \neq 1$ in $\pi_0 \mathrm{Diff}(S^{n-2}) \simeq \Theta_{n-1}$

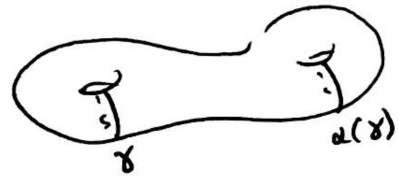
(Farrell-Jones) M stably parallelizable

$\exists \ \pi_1(M) \to \mathbb{Z}$, $[\gamma] \mapsto 1$ $\Rightarrow$ $M_{\gamma,\phi_*} \not\cong M$ not diffeo. if $[\phi] \neq 1$.

<u>What's the obstruction?</u> Fix $N = M_{\gamma,\phi}$, $\alpha \in \mathrm{Out}(\pi) \simeq \mathrm{Isom}(M)$

suppose $\exists$ $f \in \mathrm{Diff}(N)$ $\Psi_N(f) = \alpha$. (try to find obstruction...)

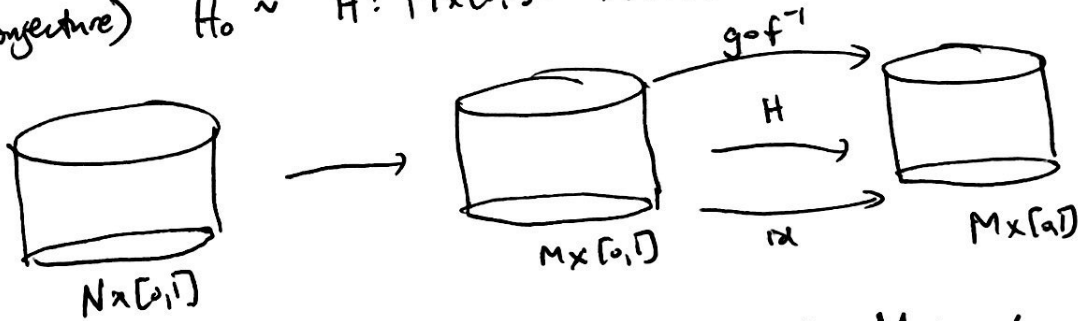
α induces diffeo $g: N \longrightarrow M_{\alpha(\gamma),\phi}$



$g \circ f^{-1}: \underset{M}{\underset{\approx}{N}} \longrightarrow \underset{M}{\underset{\approx}{M_{\alpha(\gamma),\phi}}}$ homeo of M homotopic to id. (induces id on π)

$\rightsquigarrow$ $H_0: M \times [0,1] \longrightarrow M \times [0,1]$ w/ $H_0|_{M\times 0} = \mathrm{id}$ $H_0|_{M \times 1} = g \circ f^{-1}$

(FJ, Borel conjecture) $H_0 \sim H: M\times[0,1] \longrightarrow M \times [0,1]$ homeo



$\rightsquigarrow$ sm. str on $M \times [0,1]$ st. $M \times 0 \simeq N$ $M \times 1 \simeq M_{\alpha(\gamma),\phi}$.

i.e. $\underset{\shortparallel\, M_{\gamma,\phi}}{N}$ & $M_{\alpha(\gamma),\phi}$ are <u>concordant</u> smooth structures.

This is the obstruction:

~~Pop/~~ <u>Claim</u>: if $\exists$ $\pi_1(M) \longrightarrow \mathbb{Z}^2$ then M_{γ,ϕ_*} & $M_{\alpha(\gamma),\phi}$ not
$\gamma \longmapsto (1,0)$ concordant for any $[\phi] \neq 1$. (this is new)
$\alpha(\gamma) \longmapsto (0,1)$

idea:

- smoothing theory: $\left\{ \begin{matrix} \text{sm. str} \\ \text{on } M \end{matrix} \right\} / \text{concordance} \simeq [M, Top/O]$.

$S^1 \times D^{n-1} \sqcup S^1 \times D^{n-1} \xrightarrow{\gamma \sqcup \alpha(\gamma)} M$ induces $M \longrightarrow \Sigma^{n-1}(S^1_+) \vee \Sigma^{n-1}(S^1_+)$.

show $[\Sigma^{n-1}(S^1_+) \vee \Sigma^{n-1}(S^1_+), Top/O] \longrightarrow [M, Top/O]$ injective

$Top/O \simeq \Omega^{n+2} Y$ $\quad [\Sigma^{2n+1}(S^1_+) \vee \Sigma^{2n+1}(S^1_+), Y] \cong$ $\quad [\Sigma^{n+2} M, Y] \cong \quad \to [\Sigma^{2n}(T^2_+), Y]$

show this composition injective directly.

$M^n \times D^{n+2} \hookrightarrow T^2 \times D^{2n}$ open embedding. $\square$

Summary: Start w/ M w/ $\pi_1(M) \twoheadrightarrow F_r$ (Lubotzky)
$r \geq 2$

pass to $\mathbb{Z}/d = \langle \alpha \rangle$ cover to find good γ ~~and deck transformation α~~

s.t. none of $M_{\gamma,\phi}, \ldots, M_{\alpha^{d-1}(\gamma),\phi}$ concordant

$\Rightarrow$ $\text{im}\, \Psi_N \cap \langle \alpha \rangle = \{1\}$ as desired.

<u>Q</u>: $\exists ?$ $M, N \simeq M$ w/ $s(N) = 1$ and $|\text{Isom}(M)| \gg 0$?