Convex cocompact subgroups of the Goeritz group

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UC-Riverside Topology Seminar

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Convex cocompactness in mapping class groups

Hyperbolicity of surface group extensions

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in particular every $g \neq \mathrm{id} \in G$ is pseudo-Anosov, but this is (potentially) weaker than being convex cocompact

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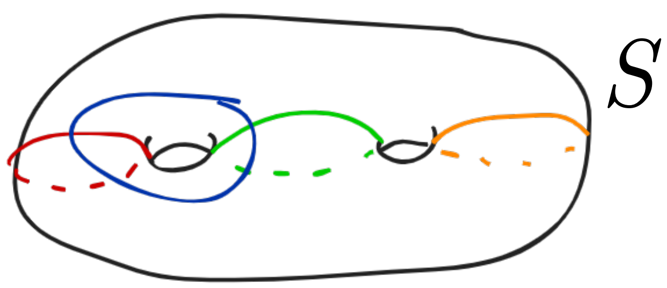

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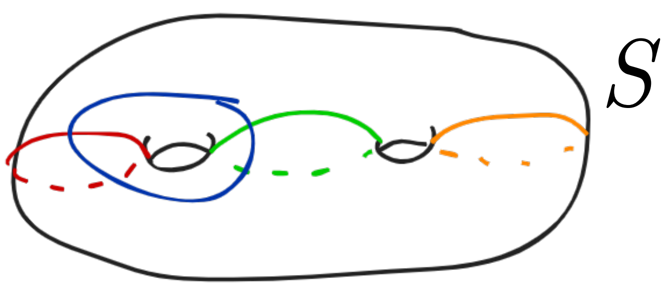

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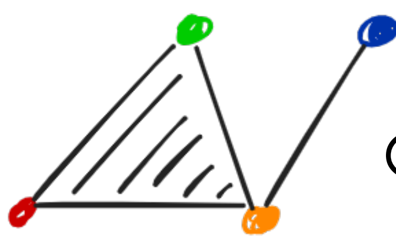

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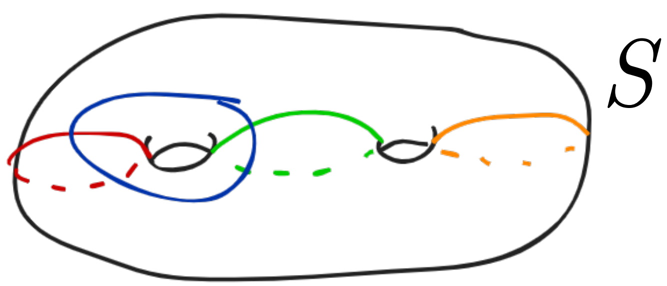

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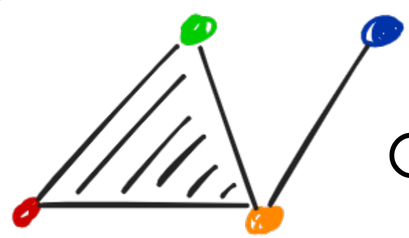

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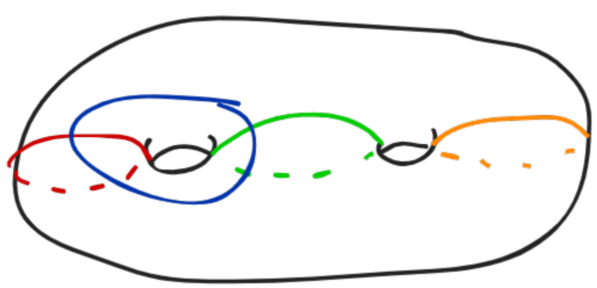

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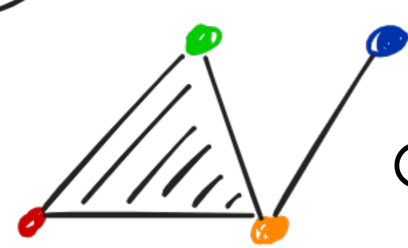

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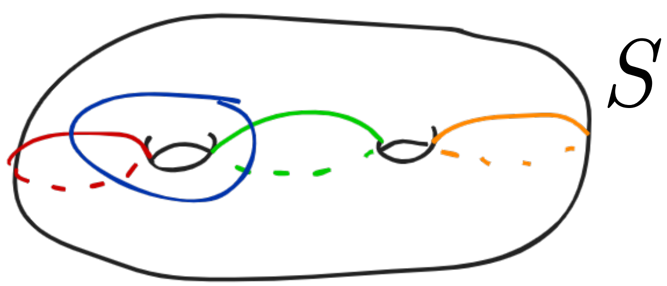

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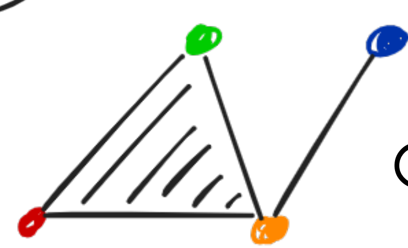

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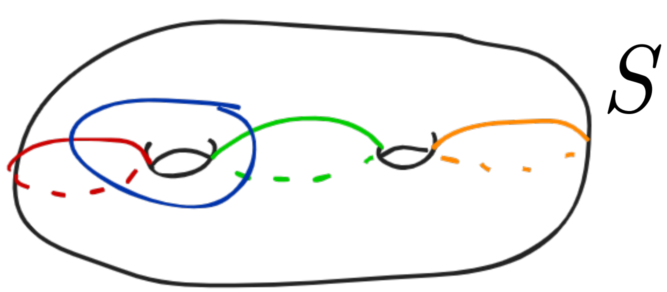

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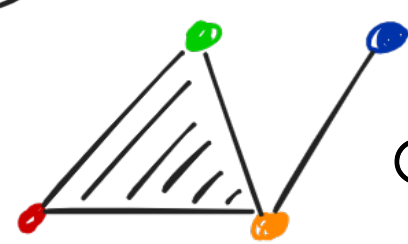

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Summary so far

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Γ_G is hyperbolic $\iff$ $G < \mathrm{Mod}(S)$ is *convex cocompact*

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All known examples of convex co-cpt $G < \mathrm{Mod}(S)$ are virtually free.

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If $G < \mathrm{Mod}(S)$ is purely pseudo-Anosov, then G is convex cocompact.

Some results

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- certain right-angled Artin subgroups (Koberda-Mangahas-Taylor)

Some results

$G < \mathrm{Mod}(S)$ $\quad\quad$ Γ_G hyperbolic $\iff$ G is convex cocompact

$$\iff G \to \mathscr{C}(S) \text{ q.i. embedding}$$

Problem (Farb-Mosher). Prove/disprove

G purely pseudo-Anosov $\implies$ *G convex cocompact.*

Known cases. This is true if $G < H$, for H

- a Veech group $\mathrm{Aff}(X,\omega)$ (Kent-Leininger-Schleimer)
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This talk: genus-2 Goeritz group

The Goeritz group
and convex cocompact subgroups

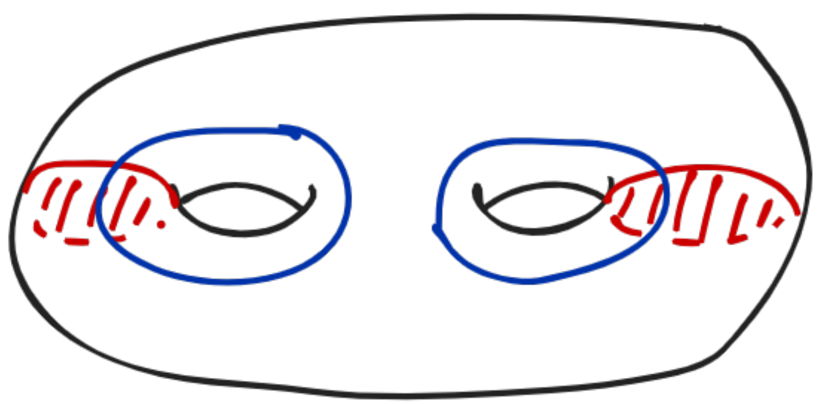
Goeritz group

Goeritz group

$S^3 = V \cup_{S_g} W$ genus-g Heegaard splitting

Goeritz group

$$S^3 = V \underset{S_g}{\cup} W \quad \text{genus-}g \text{ Heegaard splitting}$$



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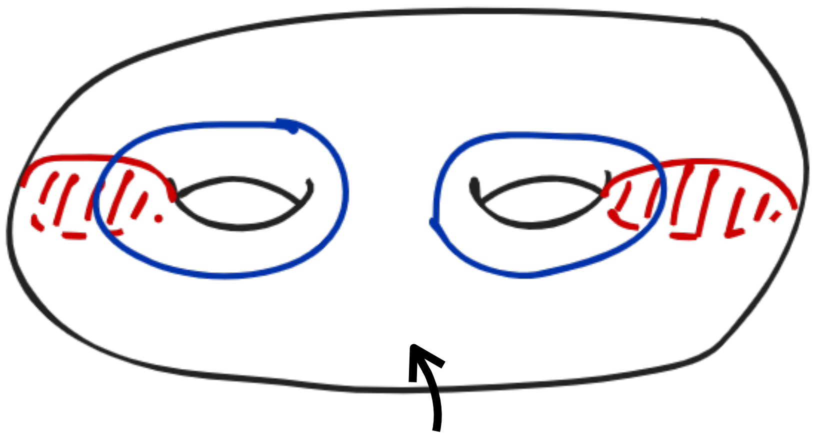

V (inside handlebody)

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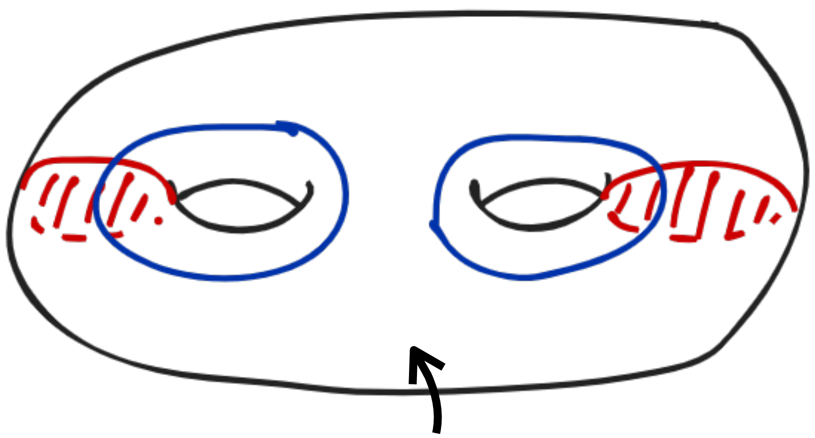
$$S^3 = V \underset{S_g}{\cup} W \quad \text{genus-}g \text{ Heegaard splitting}$$

W (outside handlebody)



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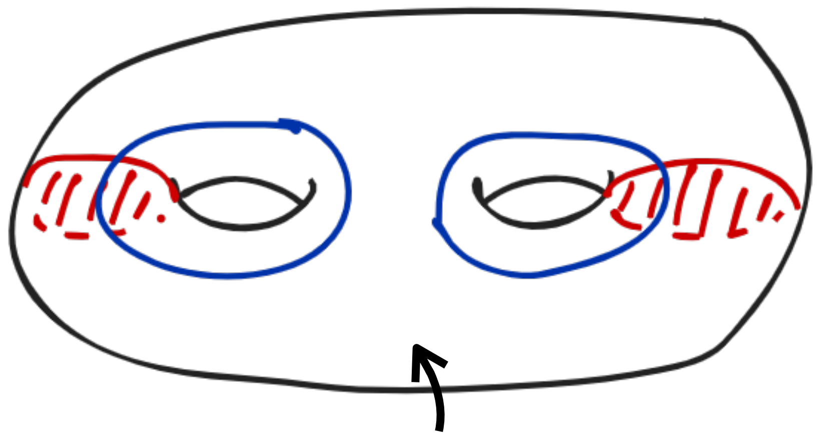

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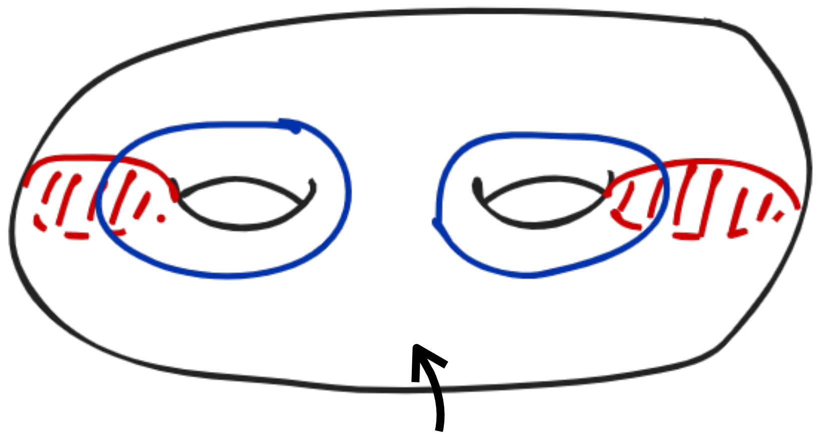

V (inside handlebody)

genus-g Goeritz group

$$\mathrm{Homeo}(S^3, V)$$

Goeritz group

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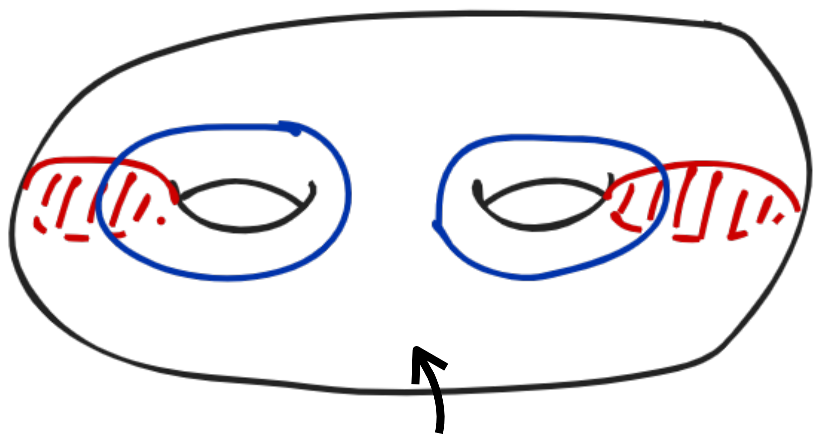

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genus-g Goeritz group

$$\pi_0(\mathrm{Homeo}(S^3, V))$$

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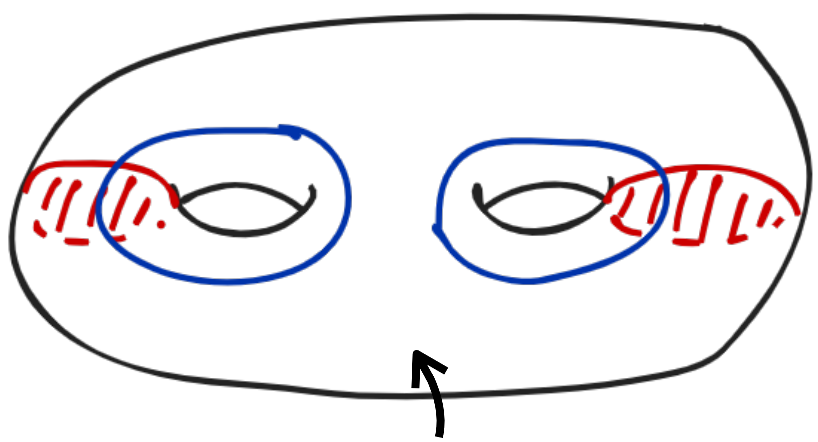

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genus-g Goeritz group

$$\pi_0(\mathrm{Homeo}(S^3, V)) \to \pi_0(\mathrm{Homeo}(S_g)) = \mathrm{Mod}(S_g)$$

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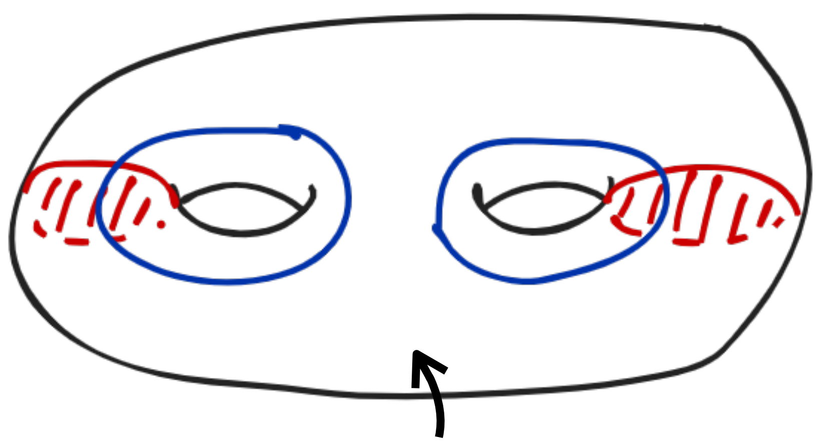

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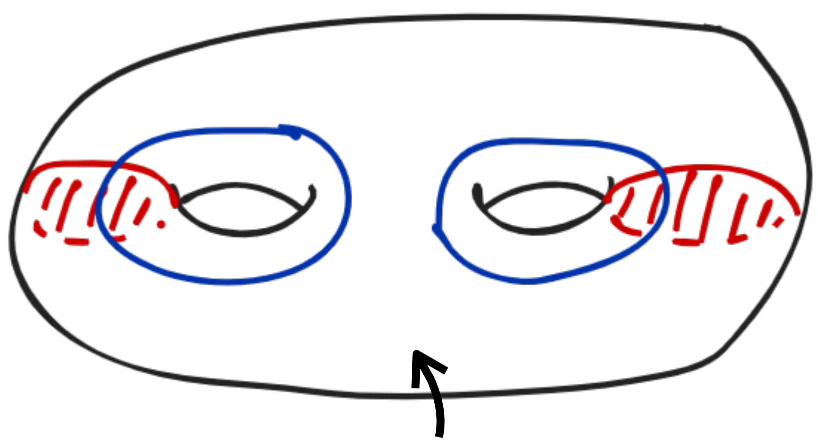

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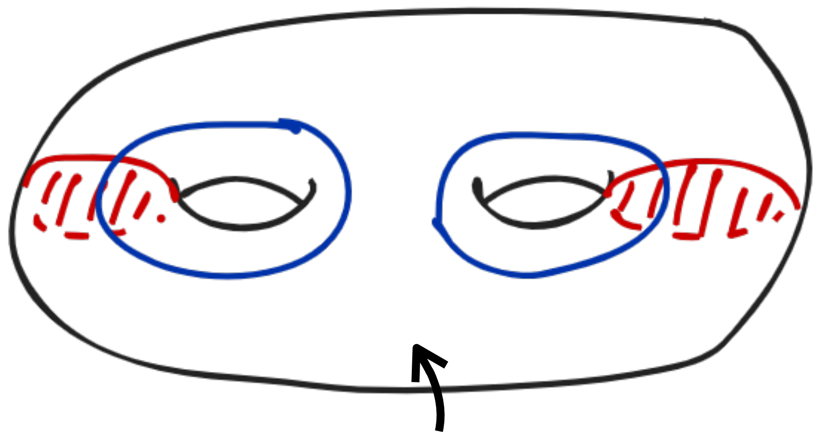

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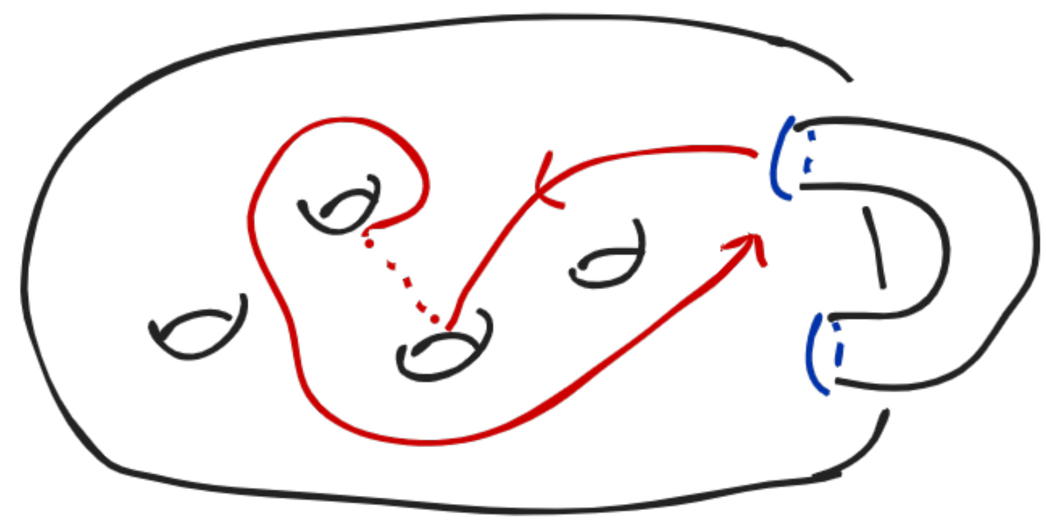

handle drag

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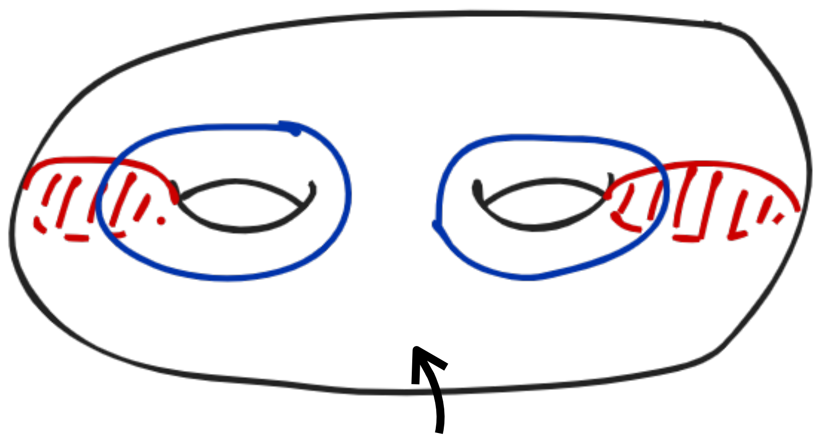

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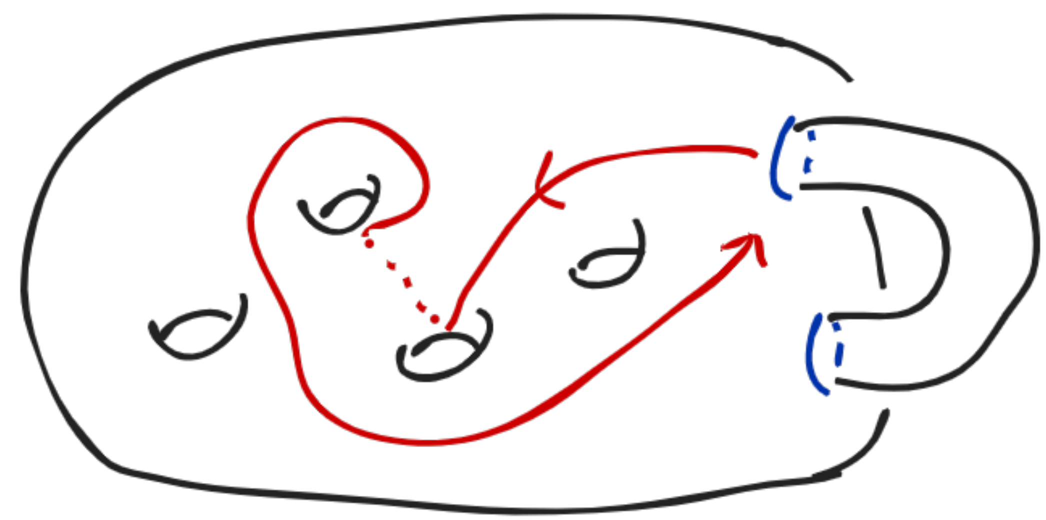

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Conjecture (Powell). $\mathcal{G}_g$ is finitely generated $\forall\ g$.

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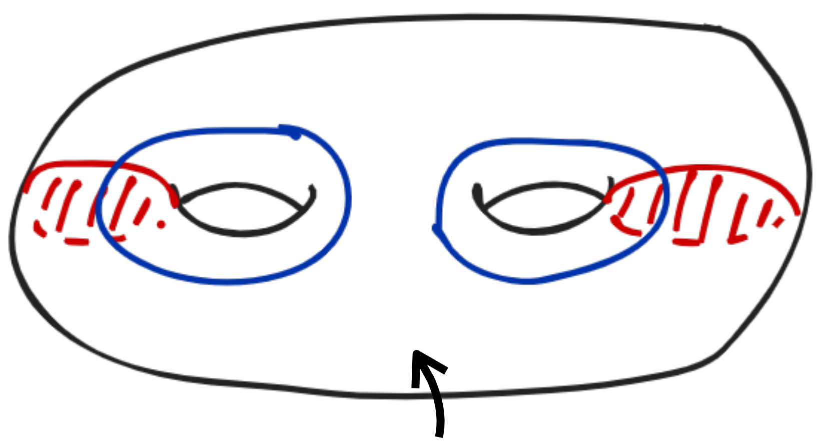

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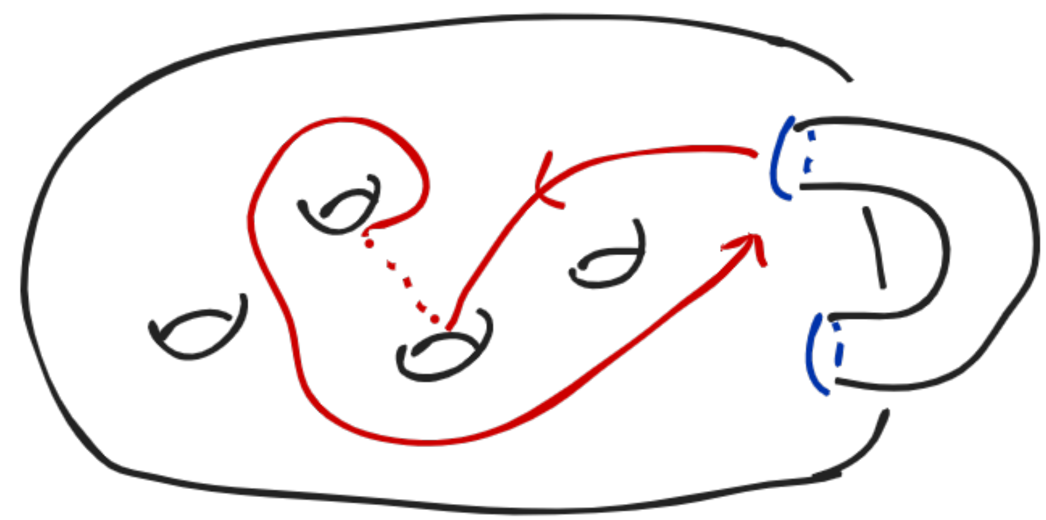

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Known for $g \leq 3$ (Goeritz, Scharlemann-Freedman)

Goeritz group in genus 2

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Heegaard splitting

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<u>Generators of $\mathcal{G}_2$</u>

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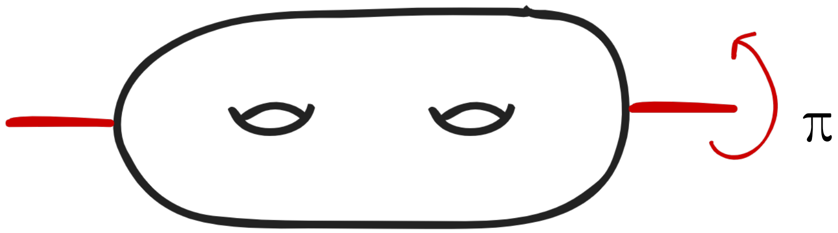

α *hyperelliptic*

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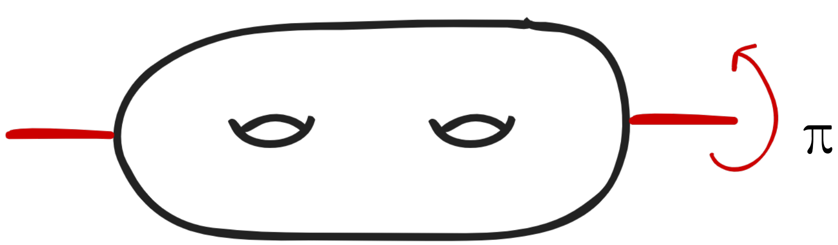

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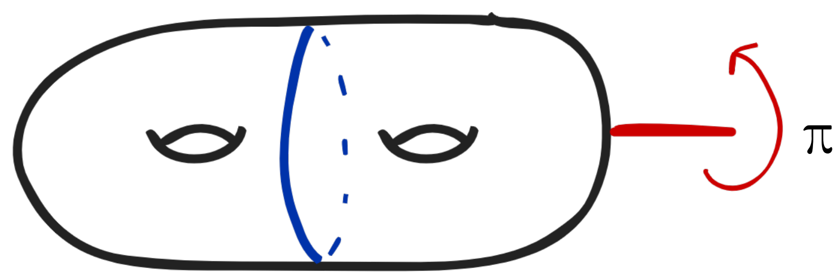

β *half-twist*

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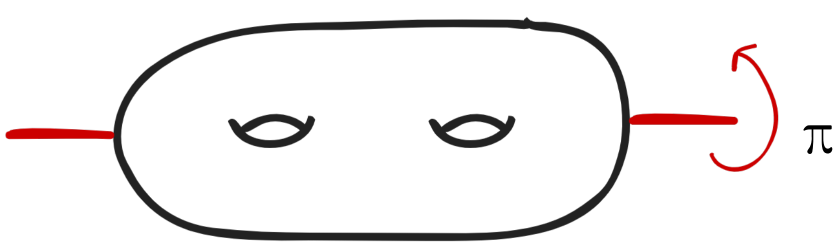

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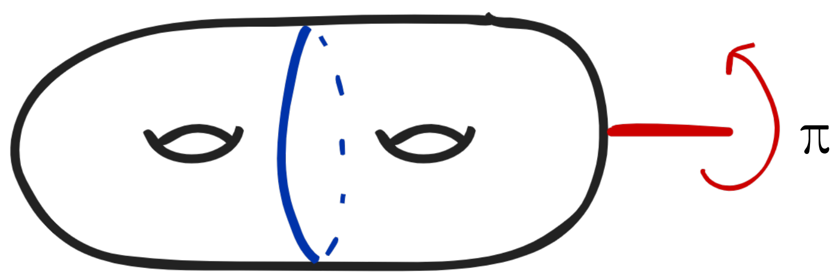

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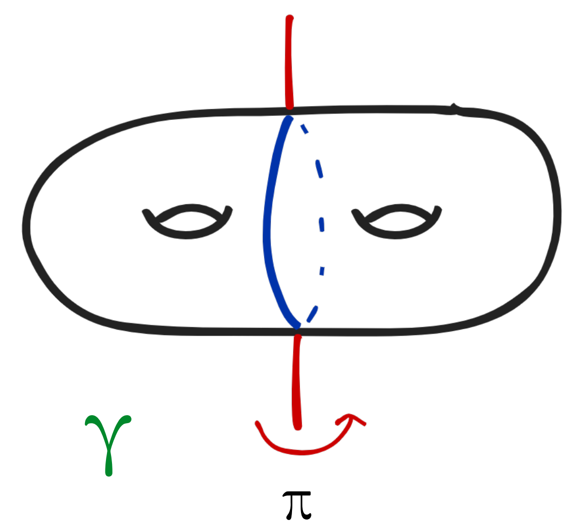

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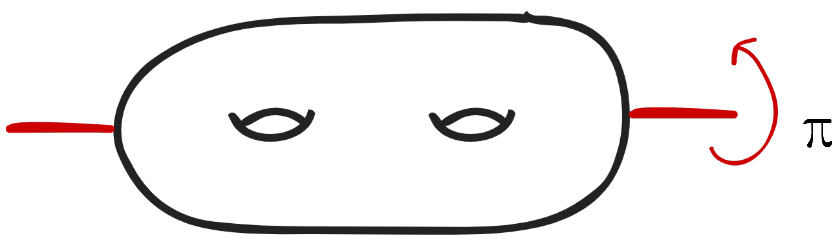

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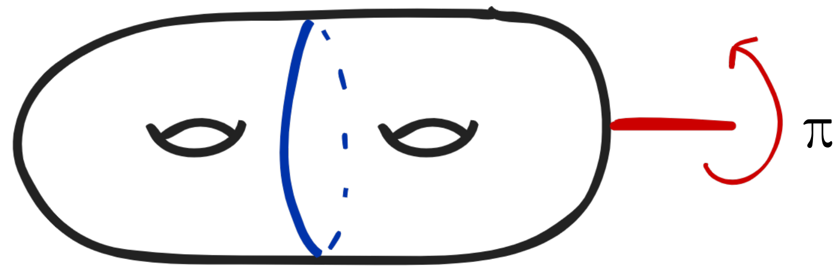

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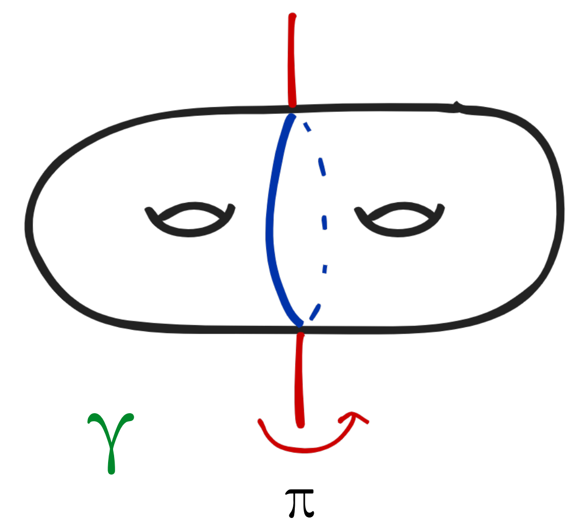
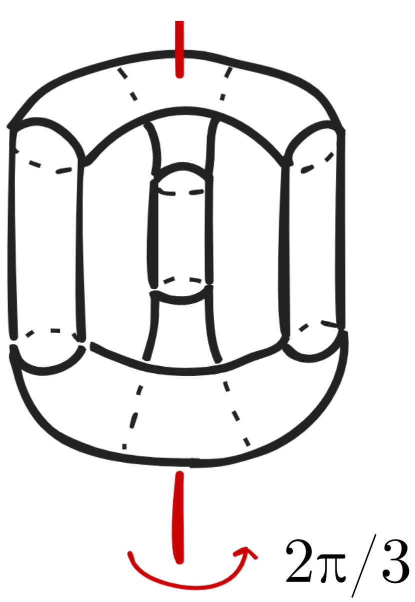


γ



δ

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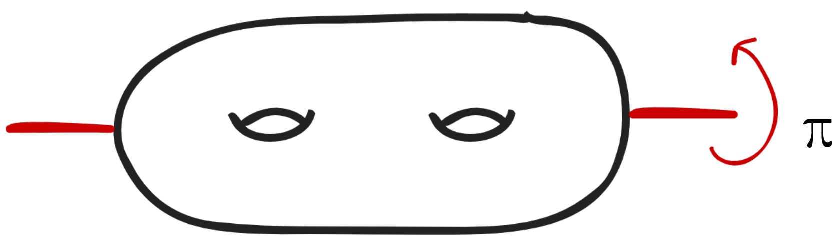

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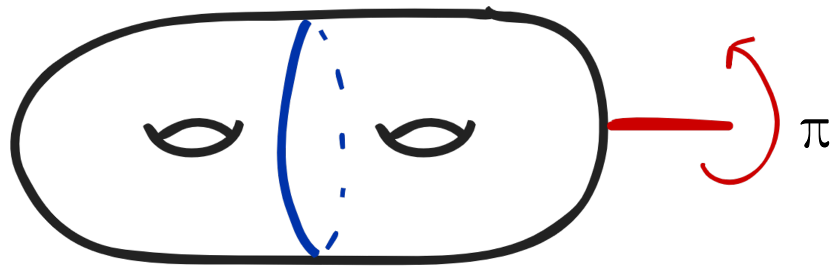

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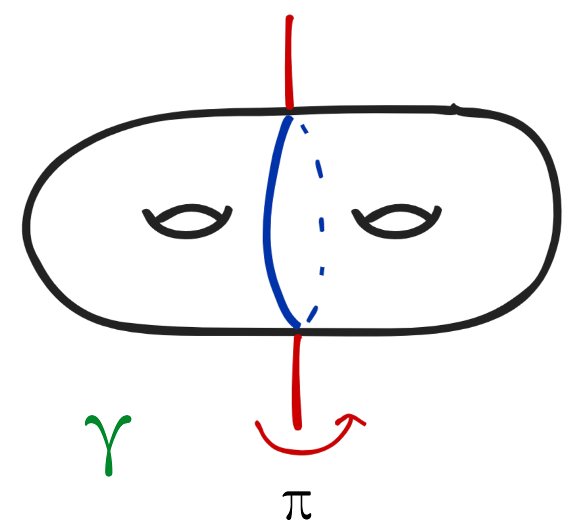
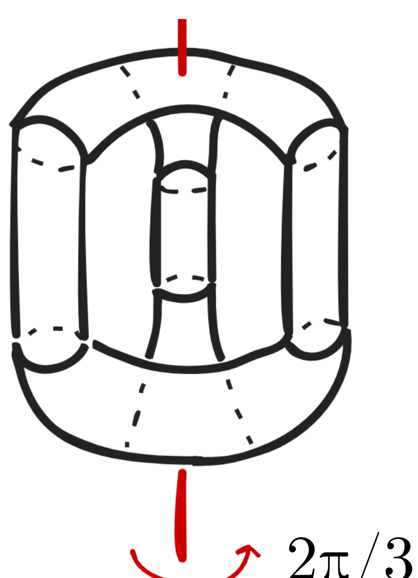


γ



δ

(Scharlemann, Akbas, Cho)

$$\boldsymbol{\mathcal{G}}_2 \cong \underset{\mathbb{Z}_2\times\mathbb{Z}_2}{\left[(\mathbb{Z}_2 \times \mathbb{Z})\rtimes \mathbb{Z}_2\right]} * (S_3\times\mathbb{Z}_2)$$

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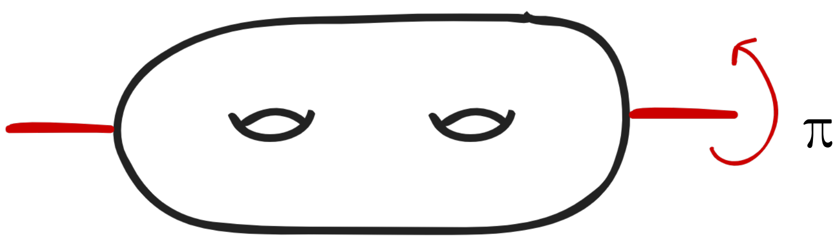

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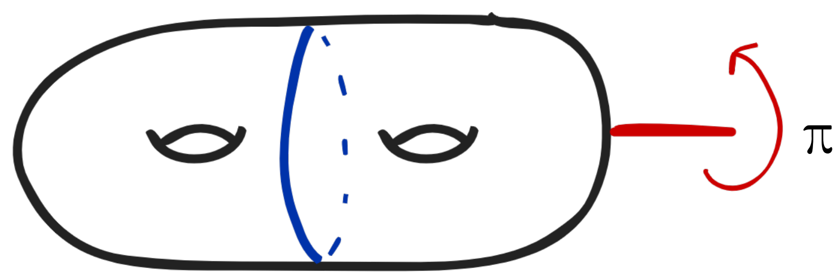

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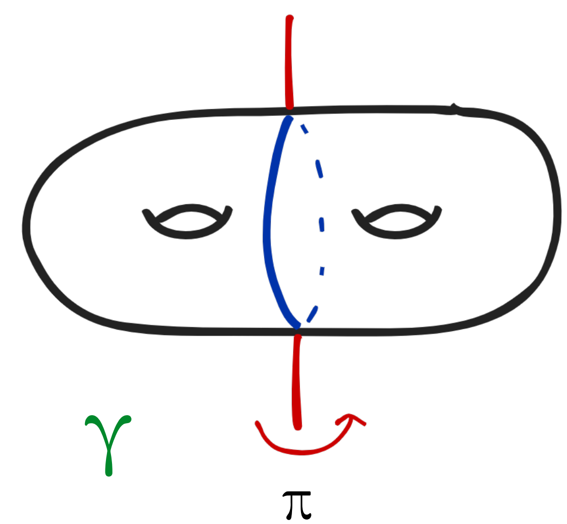

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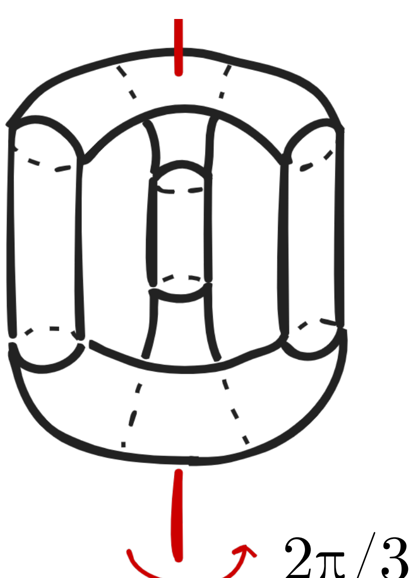

δ

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$$\mathcal{G}_2 \cong \overset{(\langle\alpha\rangle\times\langle\beta\rangle)\rtimes\langle\gamma\rangle}{\left[(\mathbb{Z}_2 \times \mathbb{Z})\rtimes \mathbb{Z}_2\right]} \underset{\mathbb{Z}_2\times\mathbb{Z}_2}{*} \overset{\langle\gamma,\delta\rangle\times\langle\alpha\rangle}{(S_3\times\mathbb{Z}_2)}$$

Main results (in progress)

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(sample) Corollary. For each $n \geq 2$,

$G_n = \langle\ \beta^n\delta,\ \delta\beta^n\ \rangle$ is purely pseudo-Anosov, hence convex cocompact.

Key ingredient: primitive disk complex

$$S^3 = V \underset{S}{\cup} W$$

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orbit map $\boldsymbol{\mathcal{G}} \to \mathscr{C}(S)$ requires choice of basepoint

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D is called a <u>primitive disk</u>

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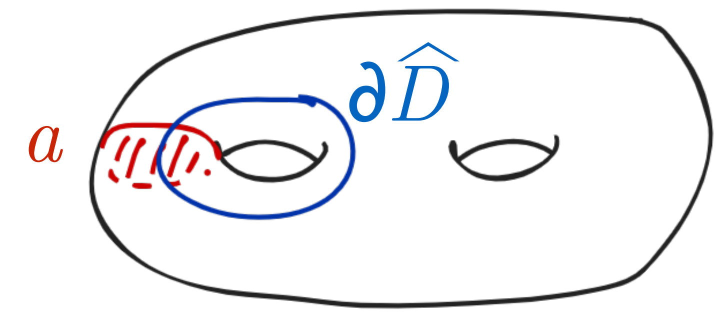

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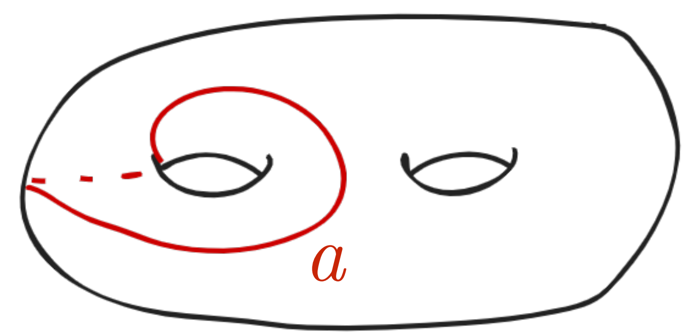

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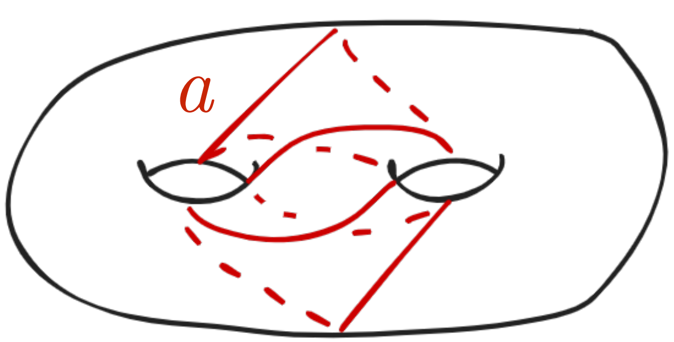
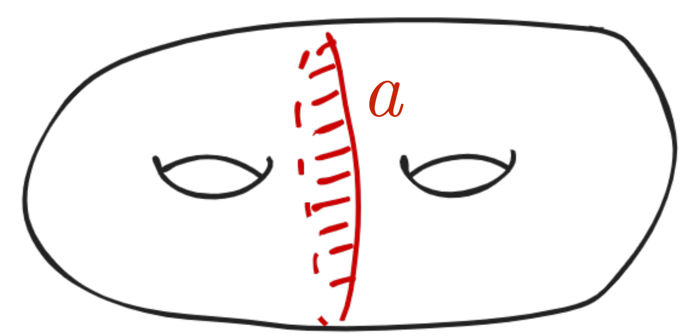


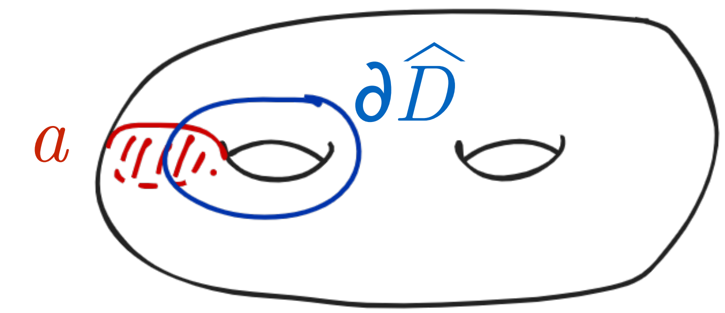


Key ingredient: primitive disk complex

orbit map $\mathscr{G} \to \mathscr{C}(S)$ requires choice of basepoint

$$S^3 = V \underset{S}{\cup} W$$

a geometrically meaningful orbit:



$a \in \mathscr{P}$

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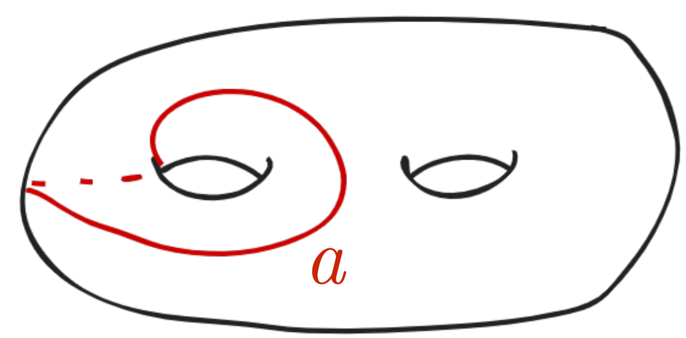

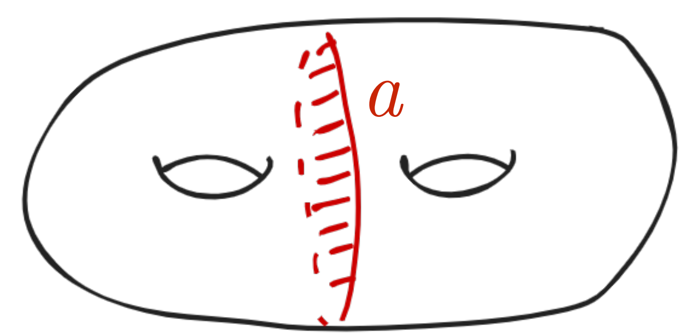

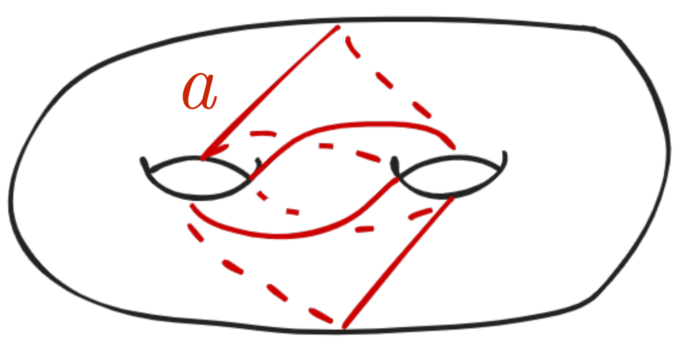

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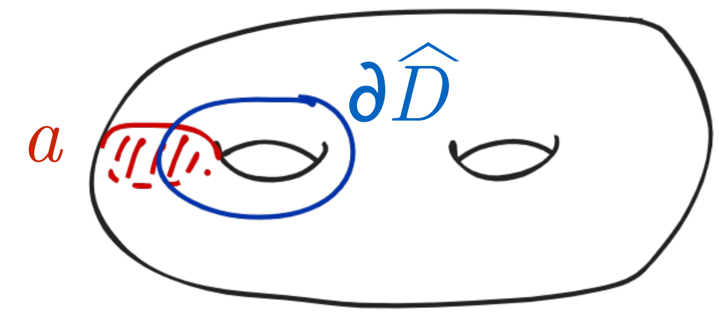

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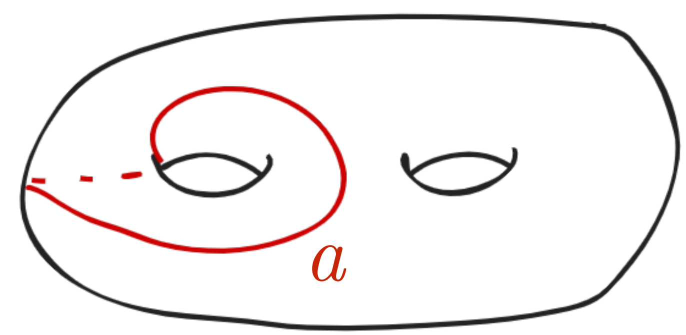

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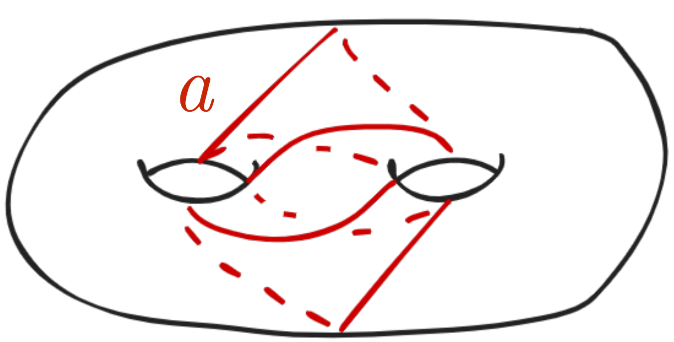

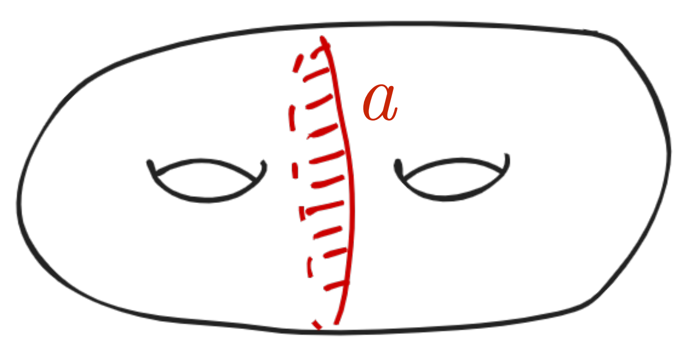

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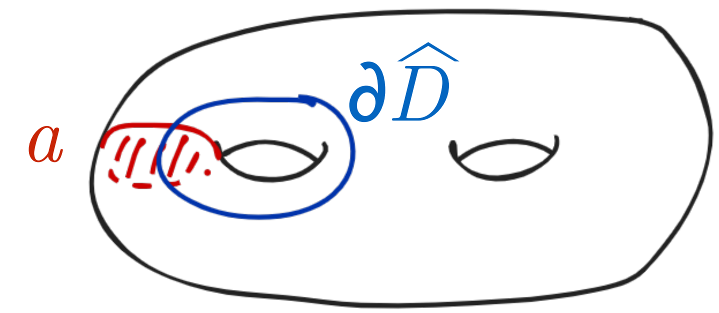

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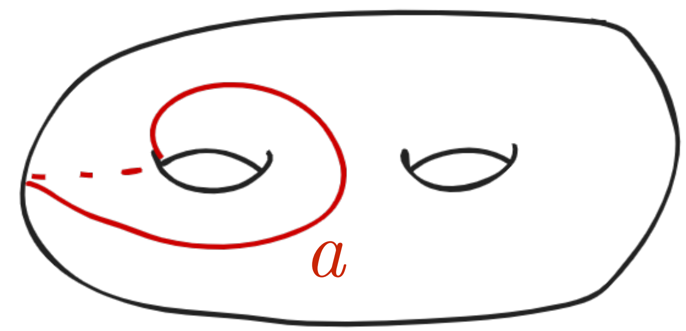

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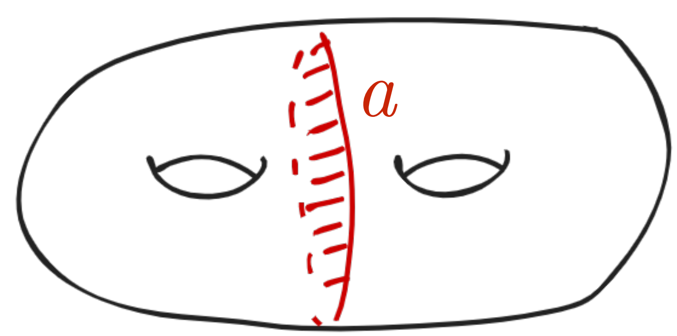

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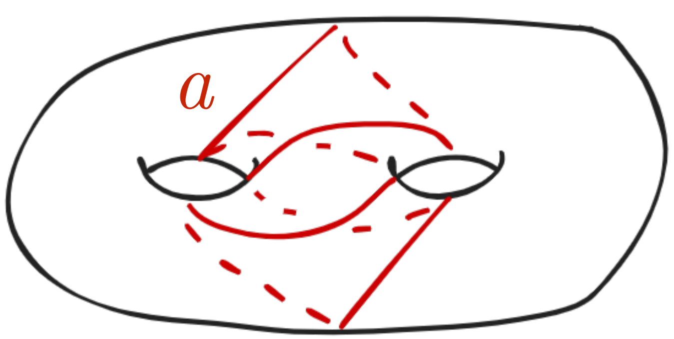

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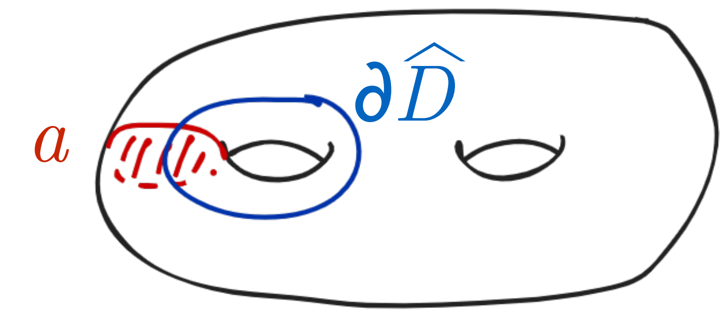

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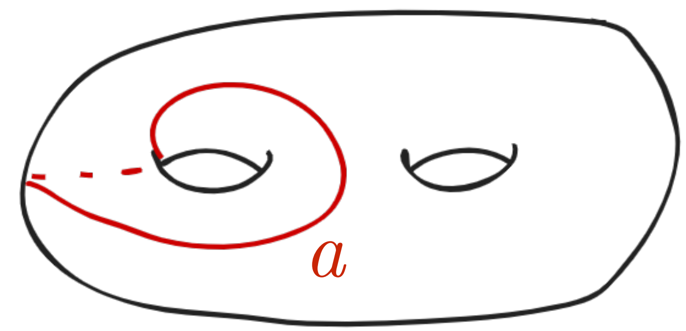

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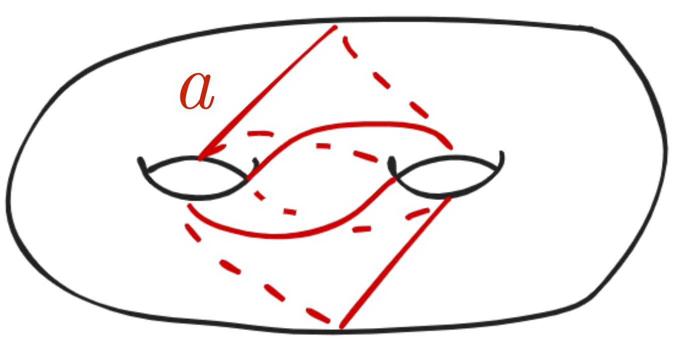

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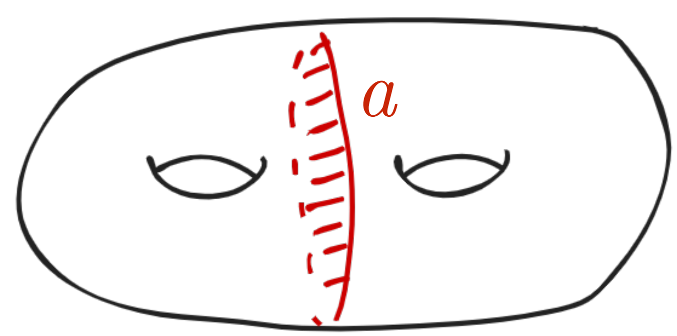

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Example:

the I-bundle subgroup of $\boldsymbol{\mathscr{G}}$

Theorem B (characterization of pseudo-Anosovs in $\mathcal{G}$)

$g \in \mathcal{G} < \mathrm{Mod}(S)$ is pseudo-Anosov $\Longleftrightarrow$ g is not conjugate into any of the following subgroups

- primitive disk stabilizer $\langle\ \alpha,\ \beta,\ \gamma\delta\ \rangle$
- reducing sphere stabilizer $\langle\ \alpha,\ \beta,\ \gamma\ \rangle$
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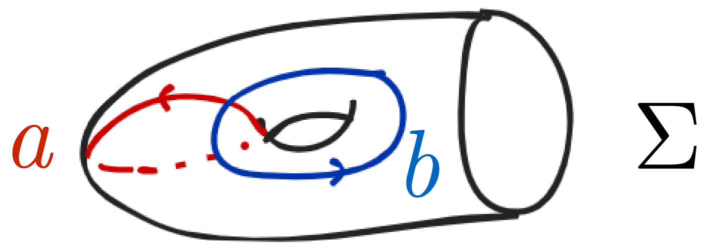

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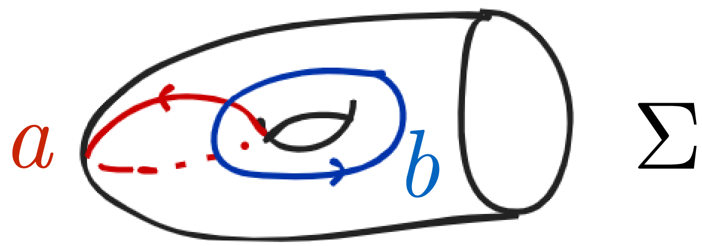

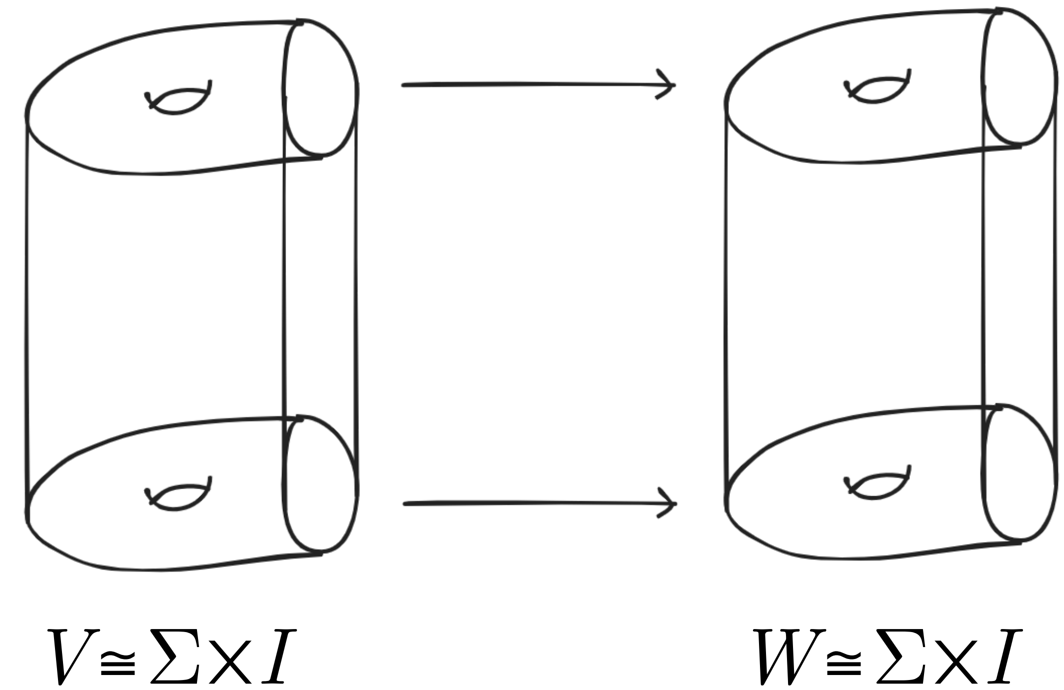

$$S^3 \cong (\Sigma \times I) \cup (\Sigma \times I)$$

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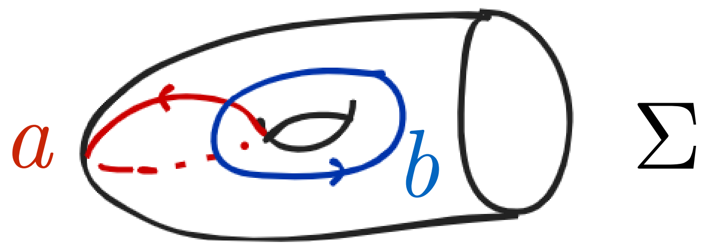

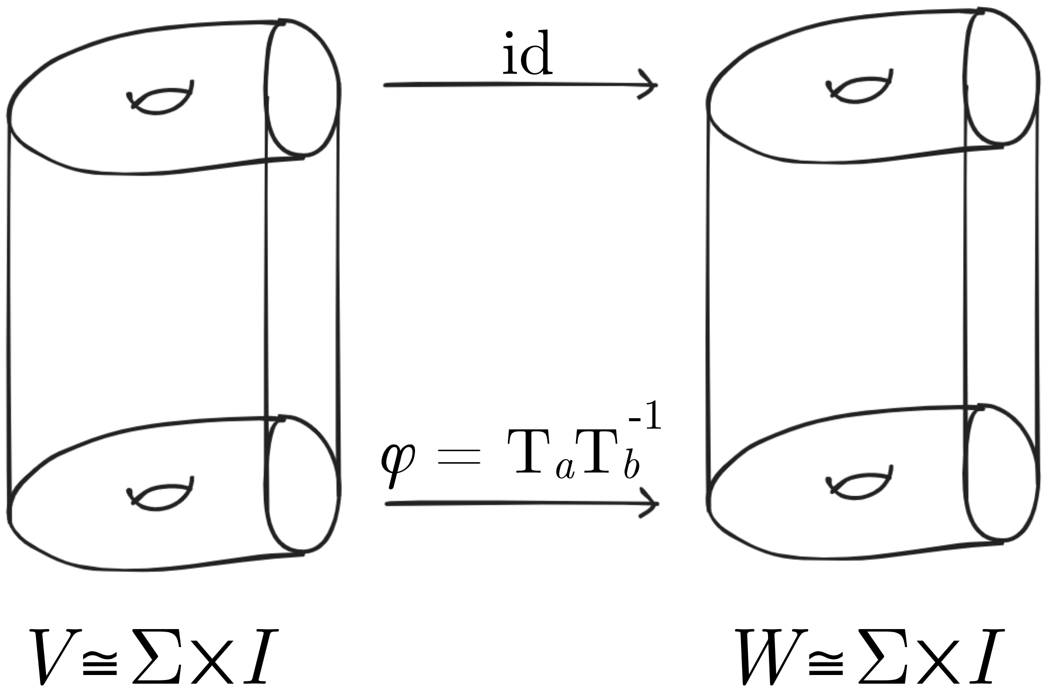

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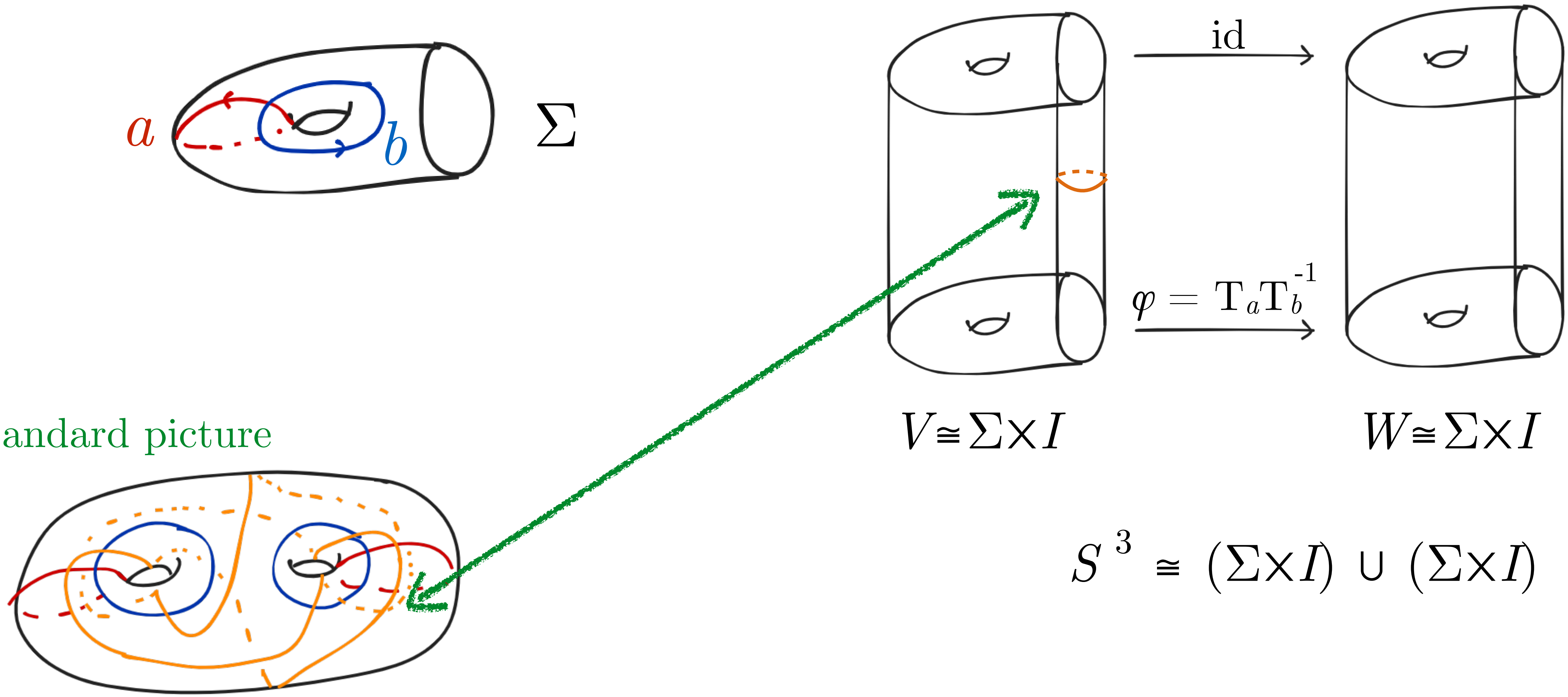

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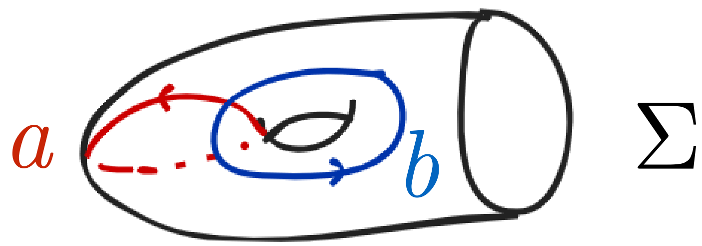

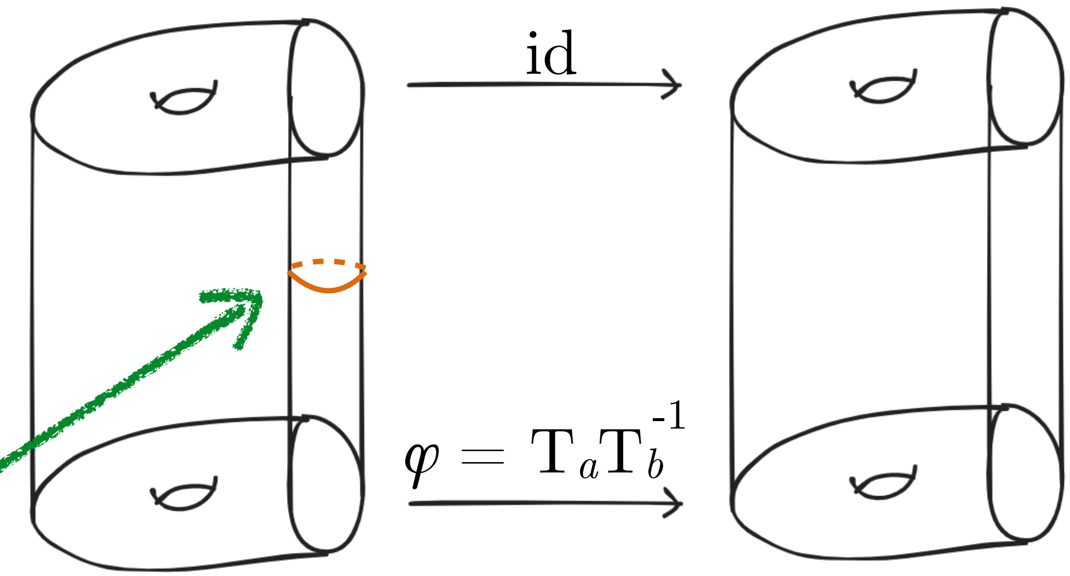

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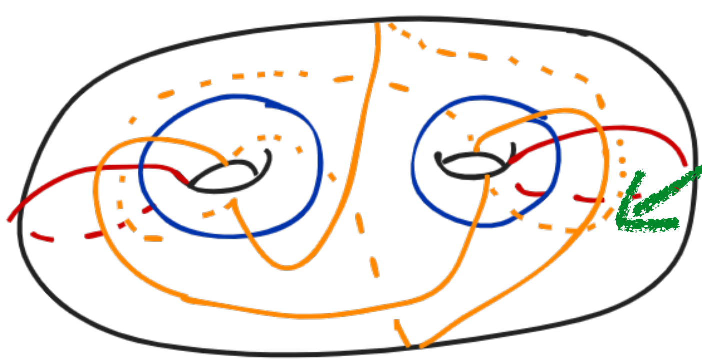
standard picture



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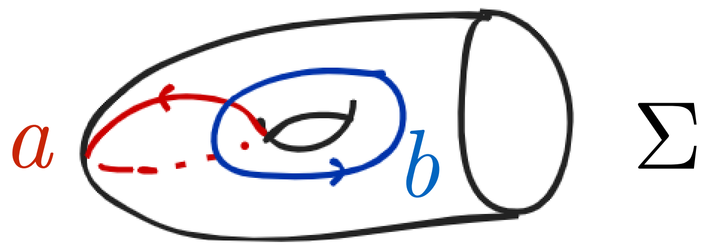

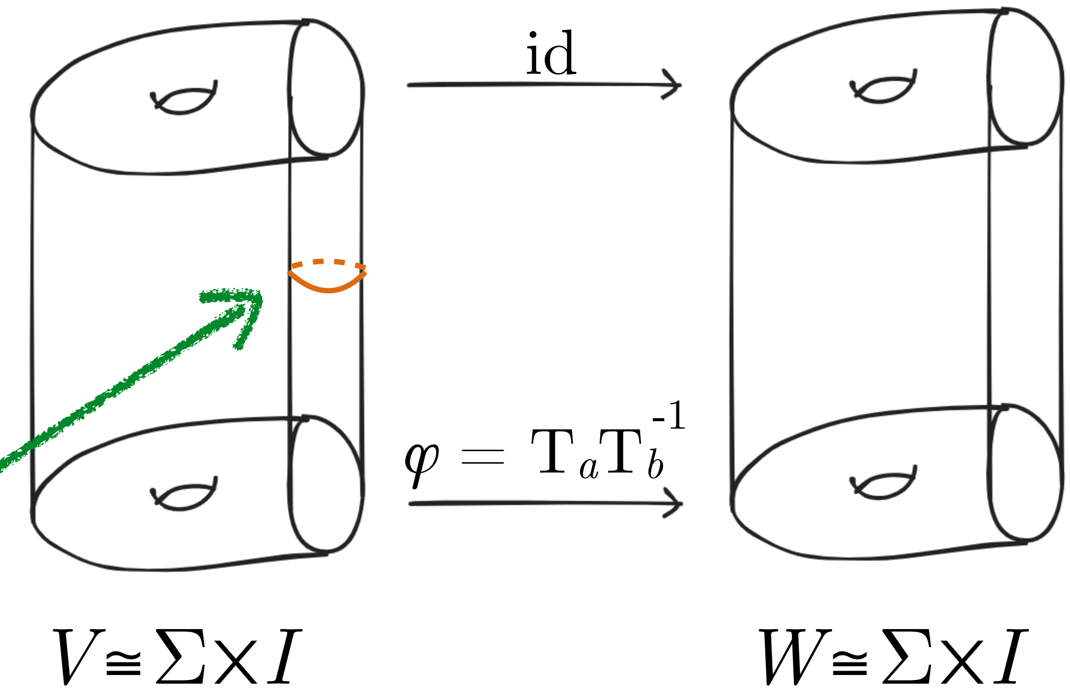

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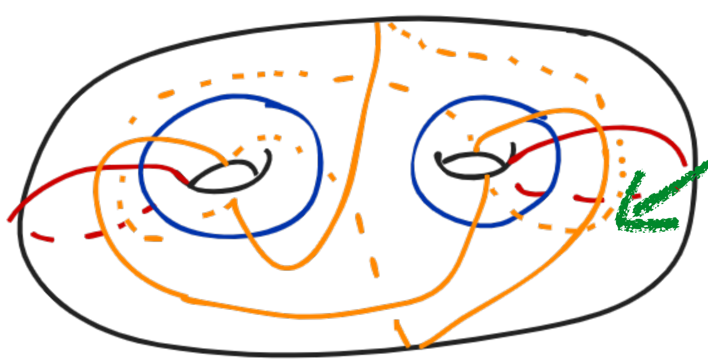

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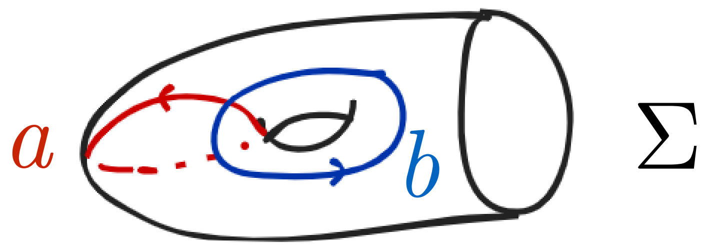

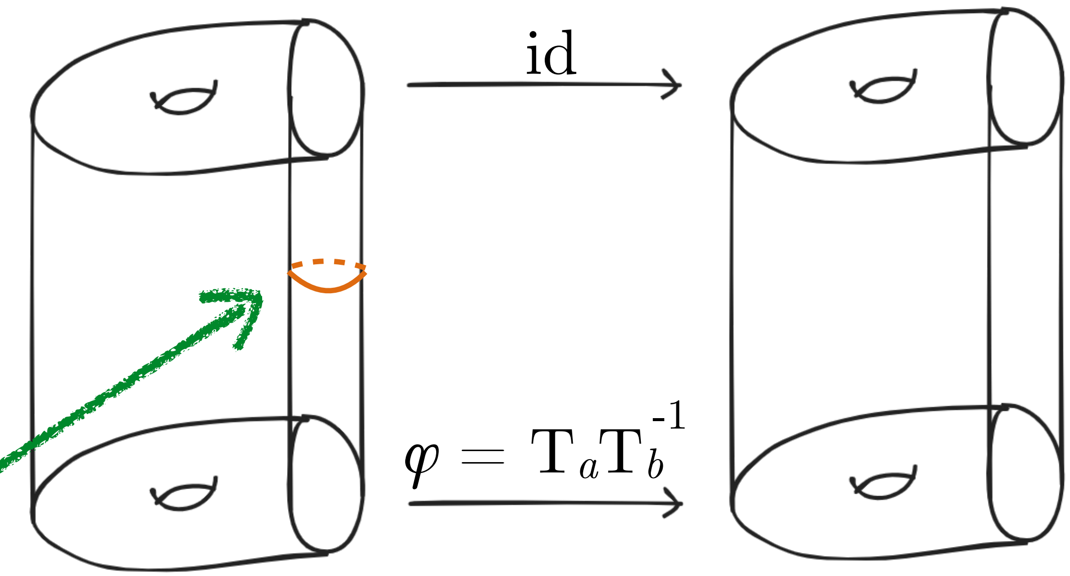

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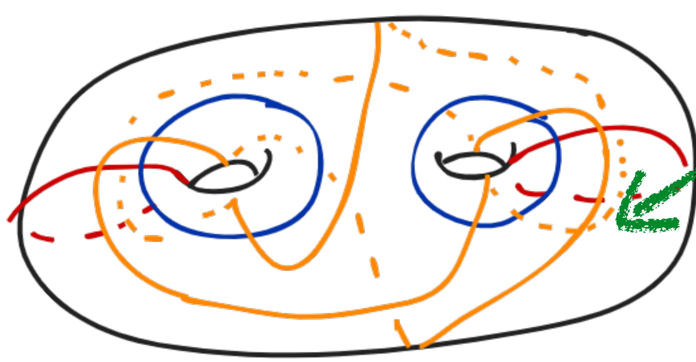

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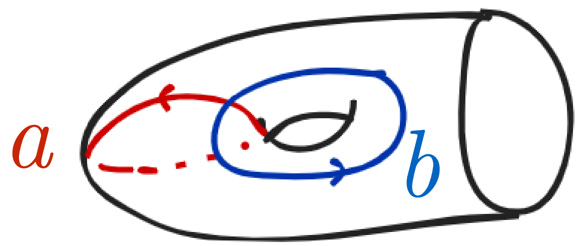

Σ

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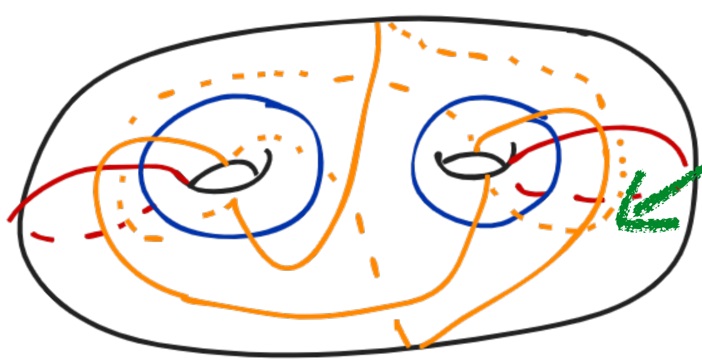

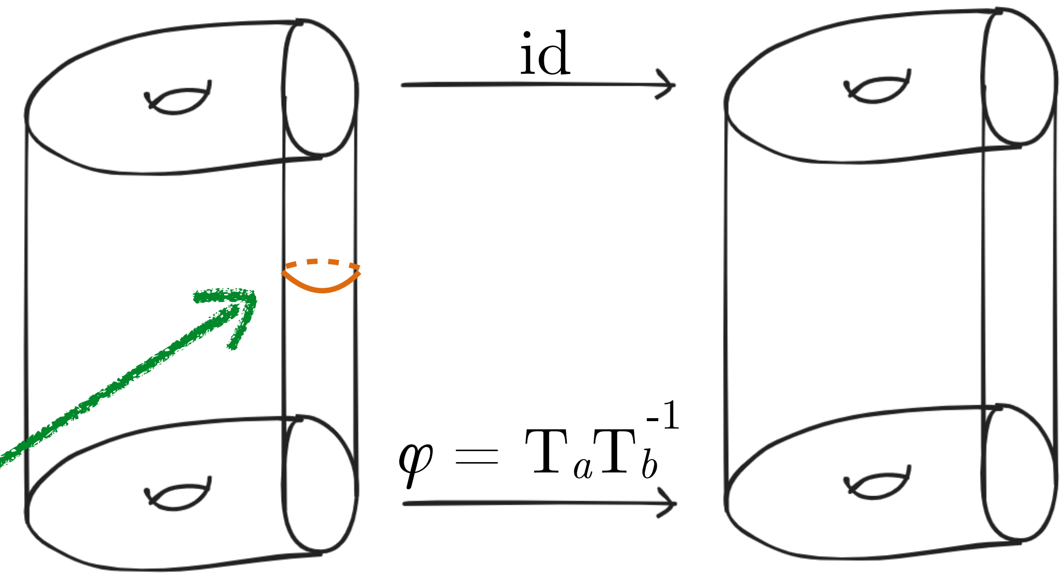

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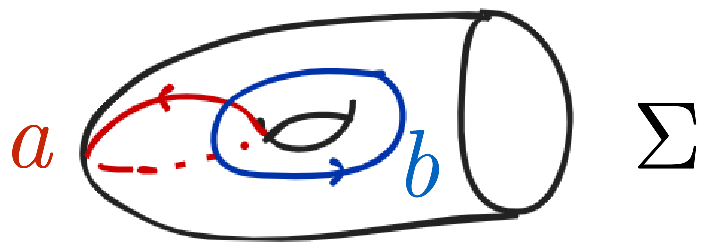

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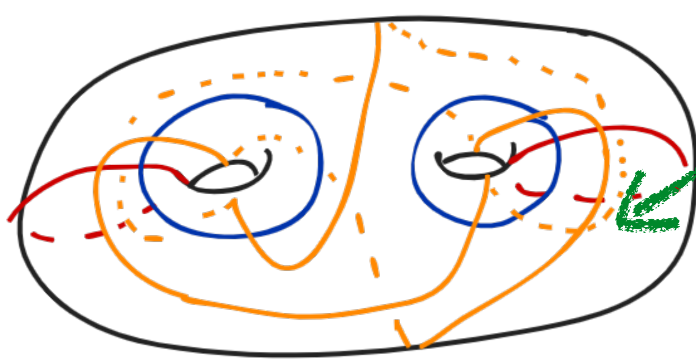

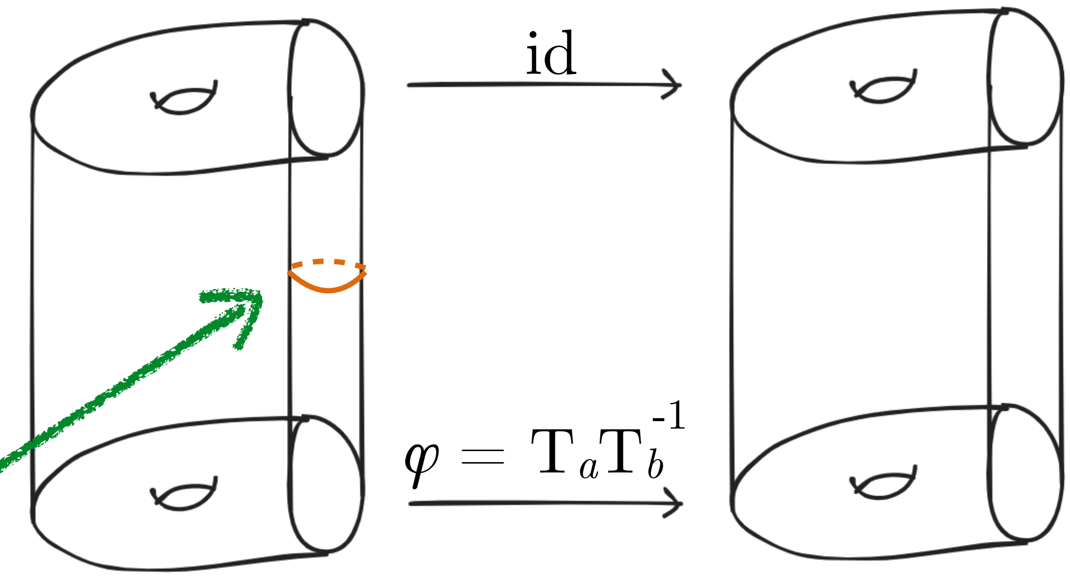

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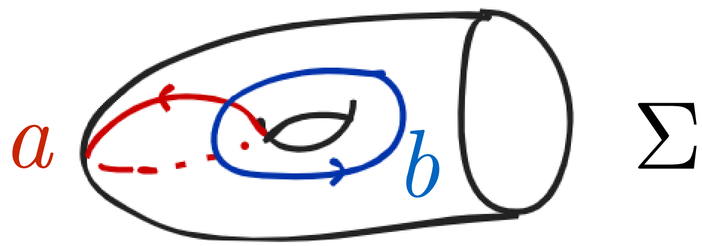
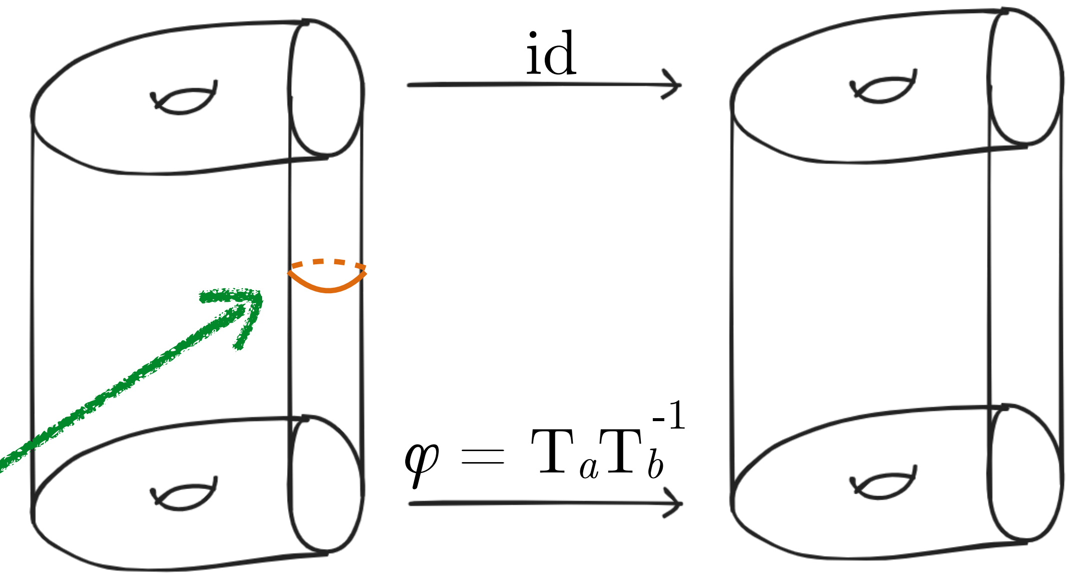
I-bundle subgroup of $\mathscr{G}$





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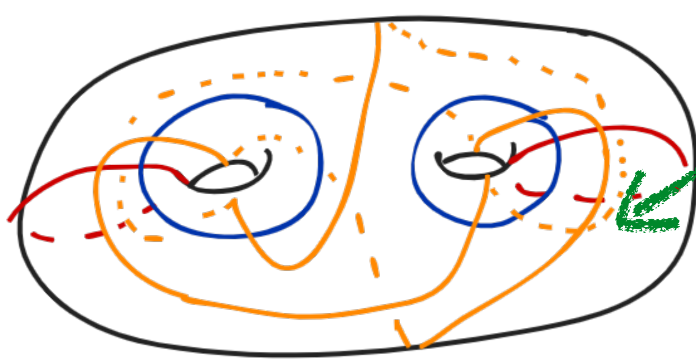

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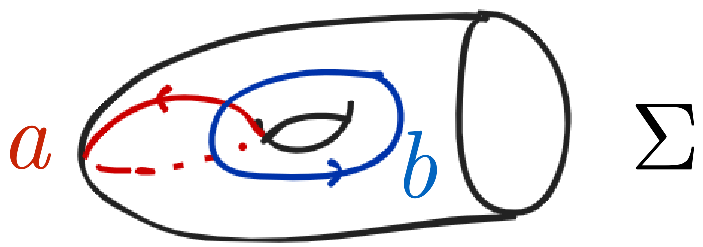
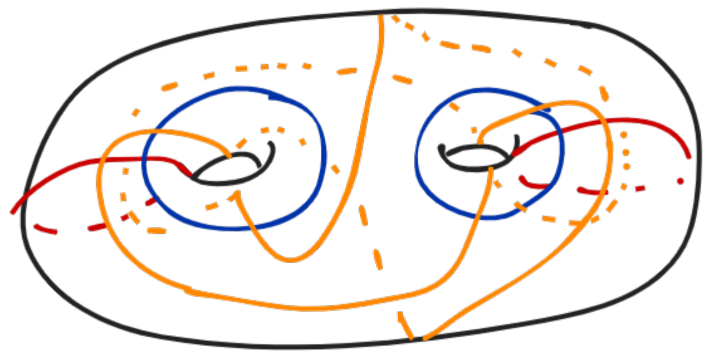
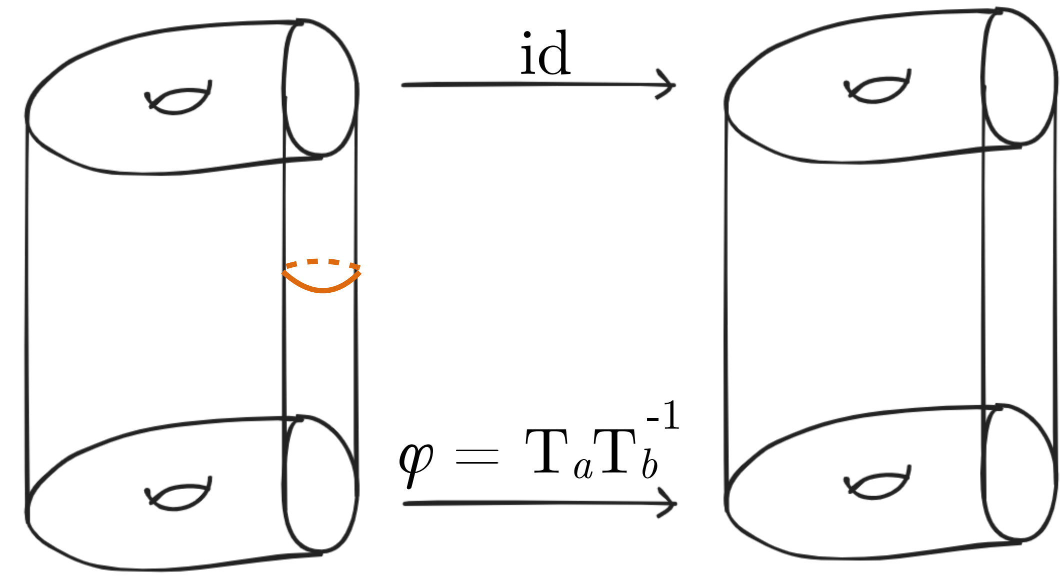
- replace φ with $\varphi \circ T^n_{\partial\Sigma} \rightsquigarrow M^3$ nontrivial homology sphere.

Significance of the I-bundle subgroup



Standard picture





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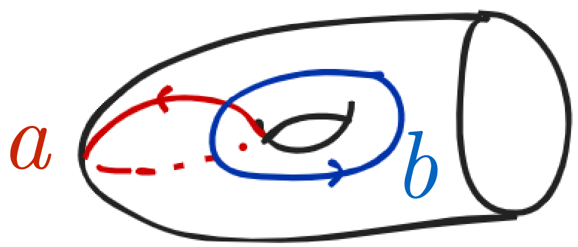

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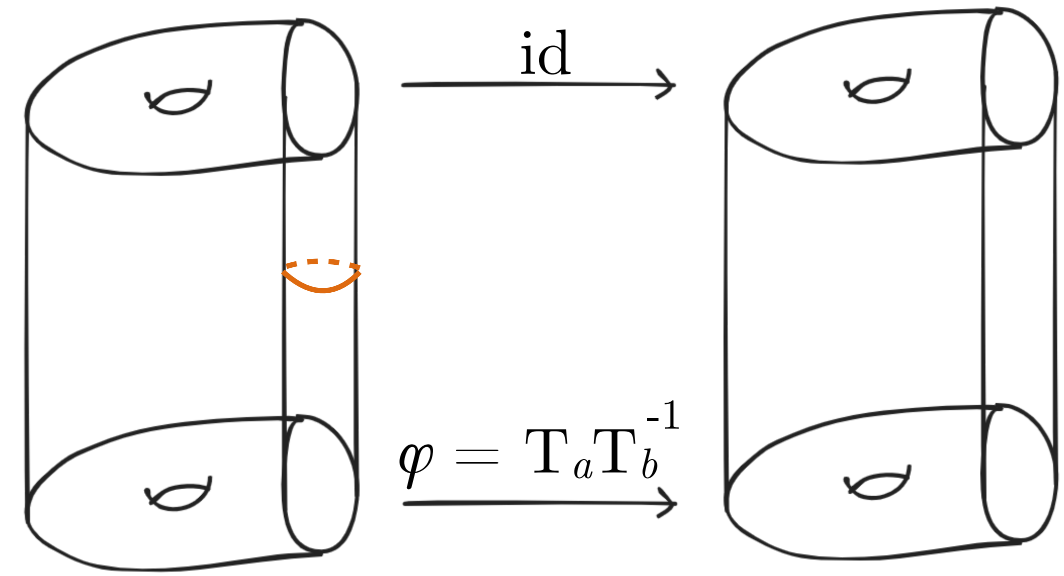

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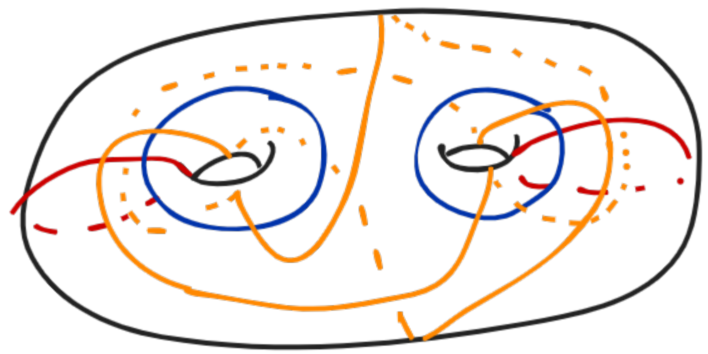

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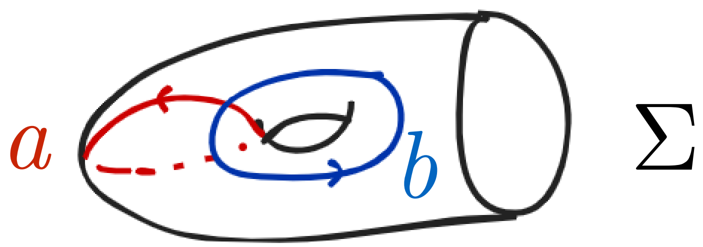

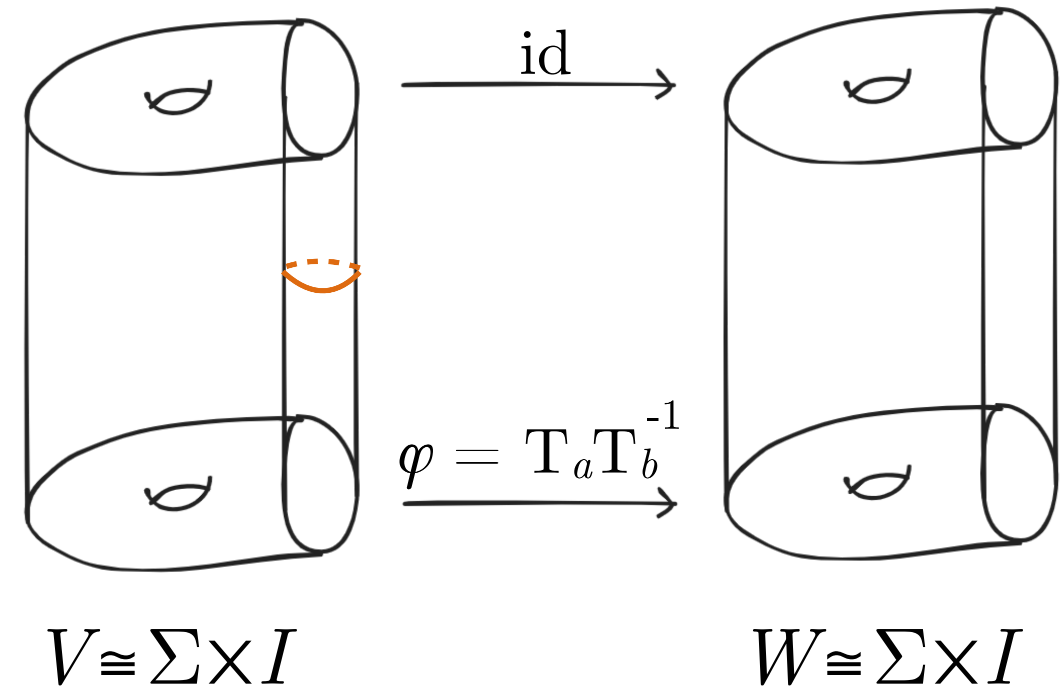

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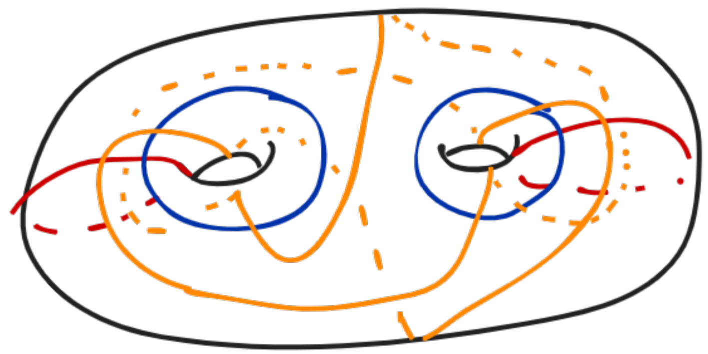

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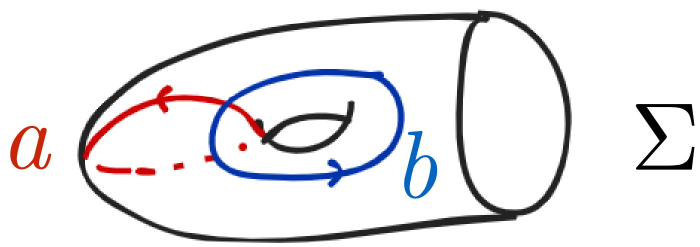
Significance of the I-bundle subgroup



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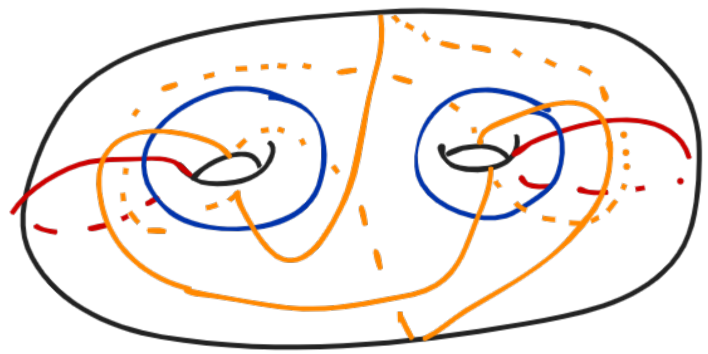

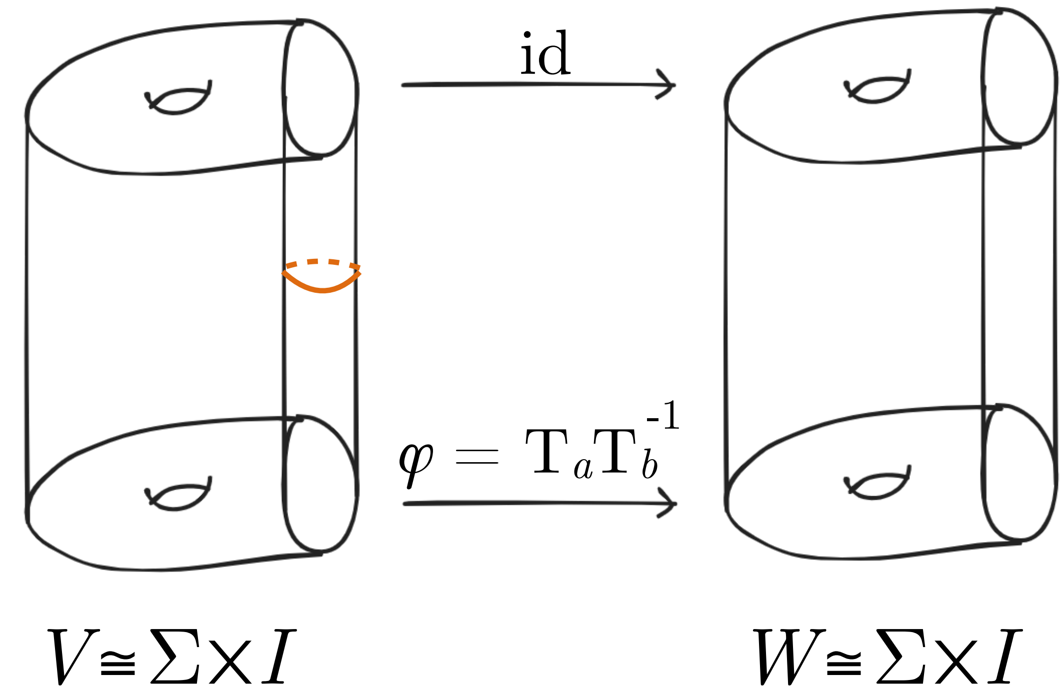

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- Responsible for summands $X \neq S$ in distance formula ($X = \Sigma \times 1$).
- Responsible for the fact that $\mathscr{P} \hookrightarrow \mathscr{C}(S)$ is not a q.i. emb.
- Classification of I-bundle subgroups key to Theorem B (characterizing p.A. elements in $\mathscr{G}$).

Thank you

Extra

About proof of Theorem A

Fin. gen. purely p.A. subgroups $G < \boldsymbol{\mathcal{G}}$ *are convex cocompact.*

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- $\boldsymbol{\mathscr{G}}$ is virtually free, so $G < \boldsymbol{\mathscr{G}}$ is q.i. embedded
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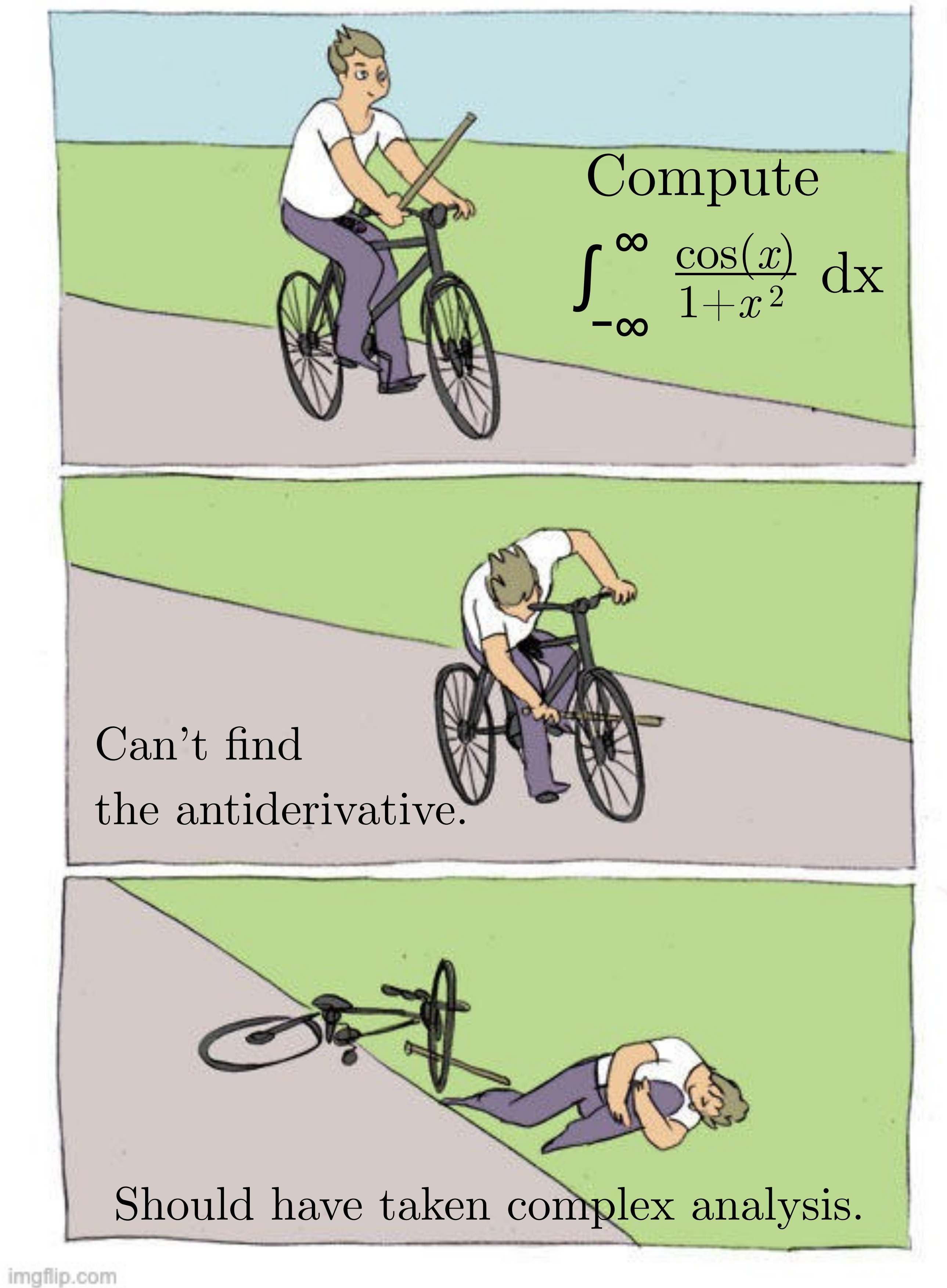
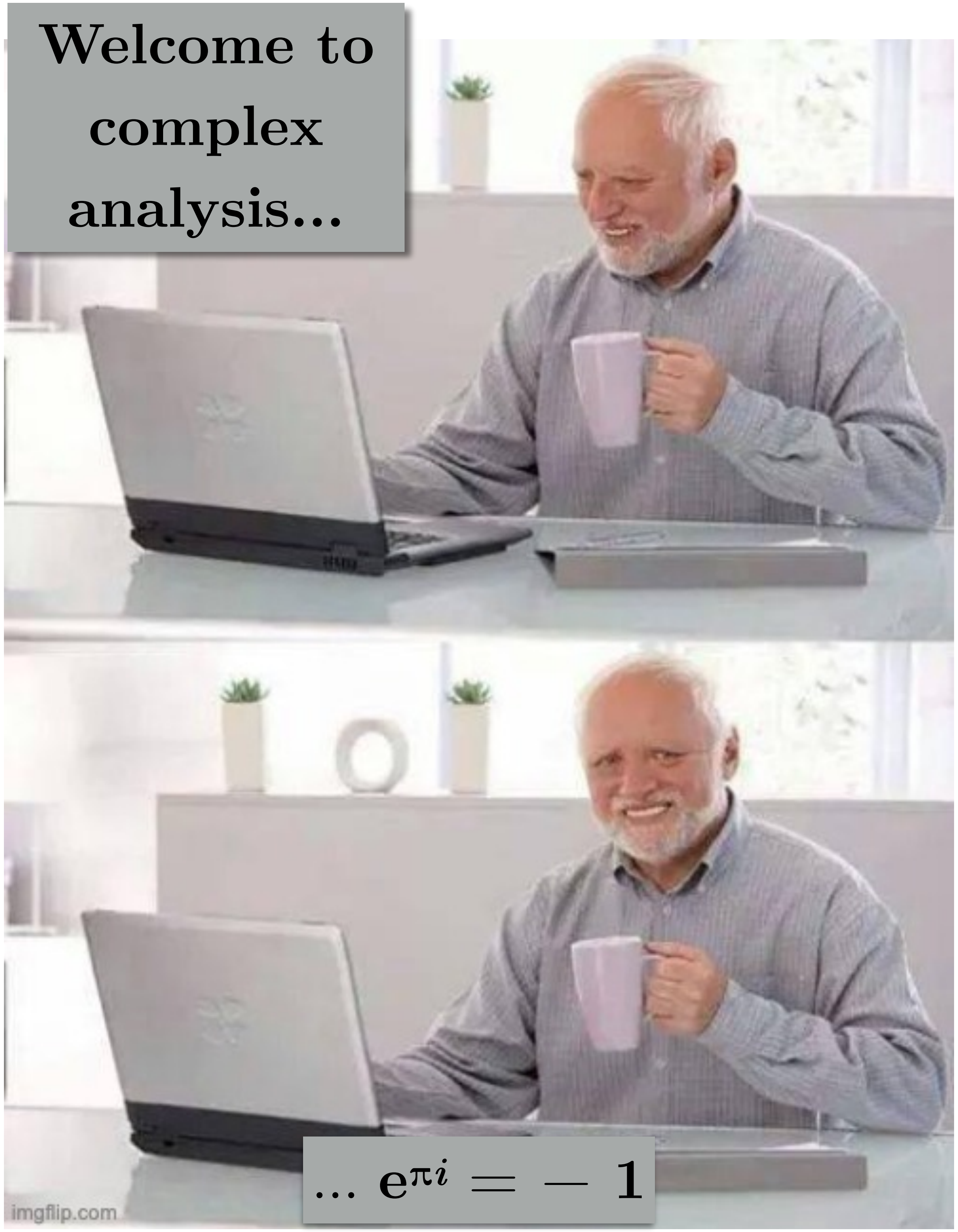
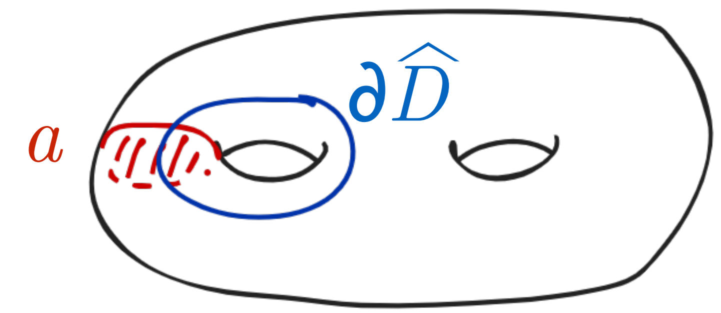
- (Manning-Abbott) limit-set criterion to determine if $G \to \boldsymbol{\mathscr{P}}$ is q.i. embedding
- Show if $G \to \boldsymbol{\mathscr{P}}$ is not q.i. embedding, then G contains an element that fixes a primitive disk (in particular G contains a reducible element).

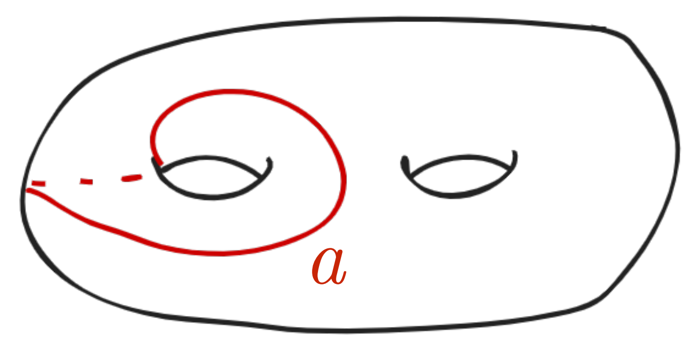

Compute
$\int_{-\infty}^{\infty} \frac{\cos(x)}{1+x^2}$ dx
Can't find
the antiderivative.
Should have taken complex analysis.
imgflip.com

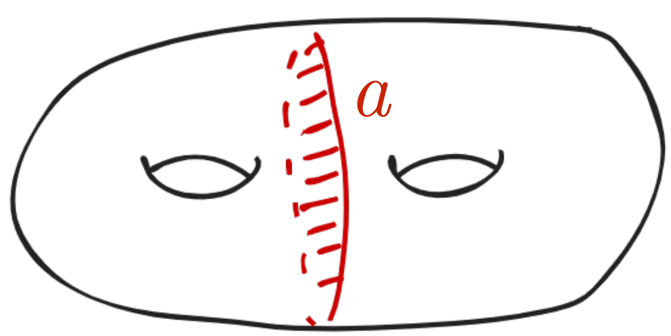

Welcome to complex analysis...
... $\mathbf{e}^{\pi i} = -1$
imgflip.com

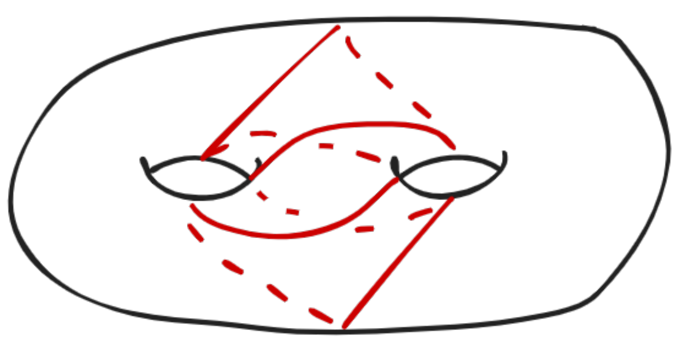
Key ingredient: primitive disk complex

$$S^3 = V \underset{S}{\cup} W$$









Key ingredient: primitive disk complex

orbit map $\mathcal{G} \to \mathscr{C}(S)$ requires choice of basepoint

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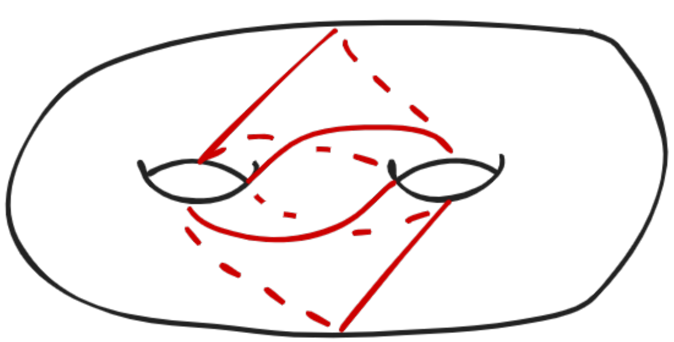

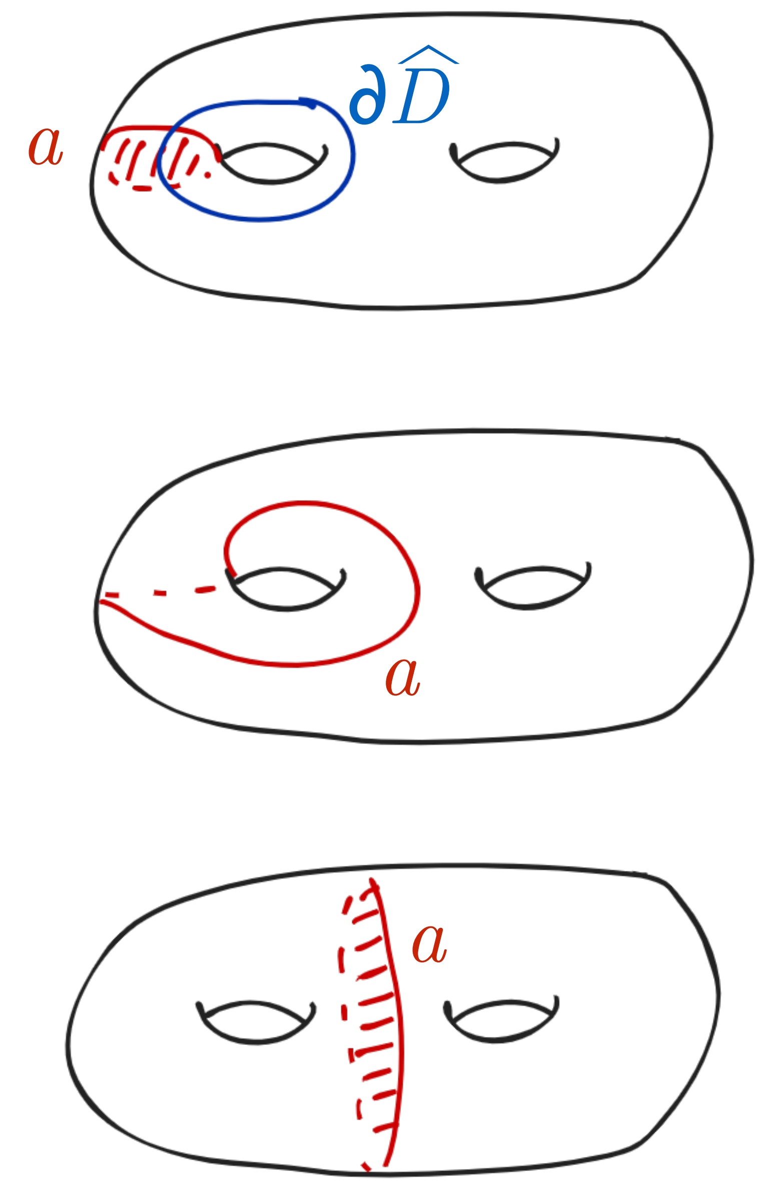

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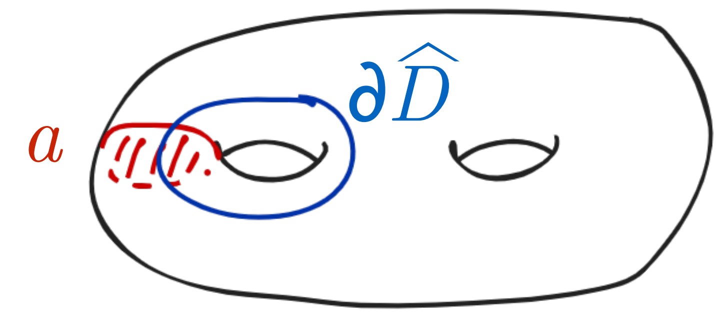
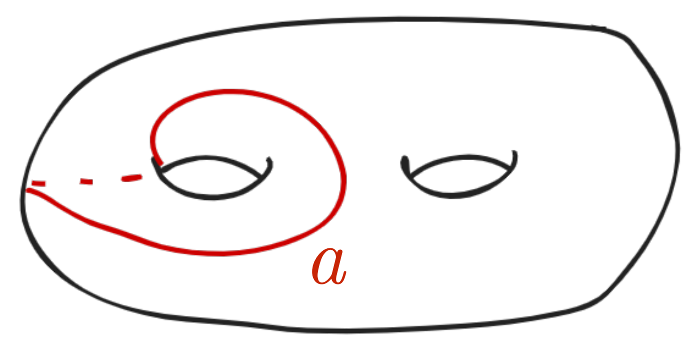
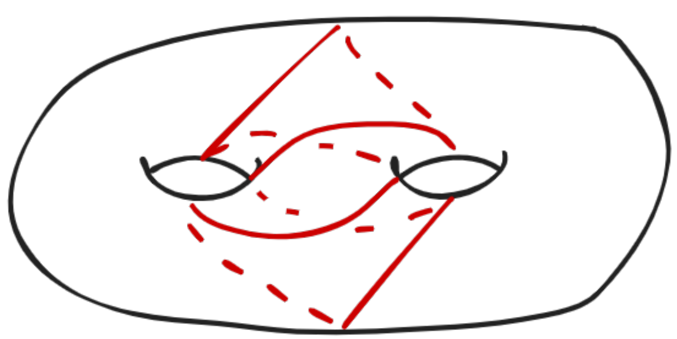
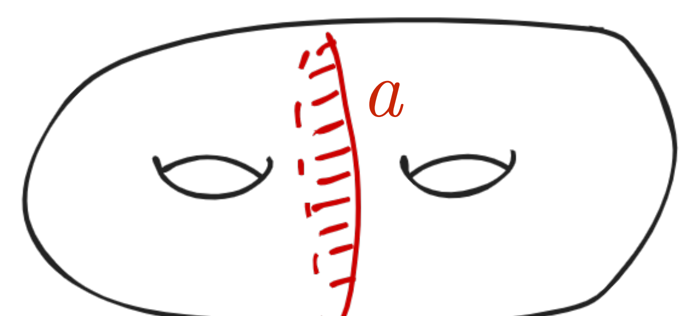
a geometrically meaningful orbit:









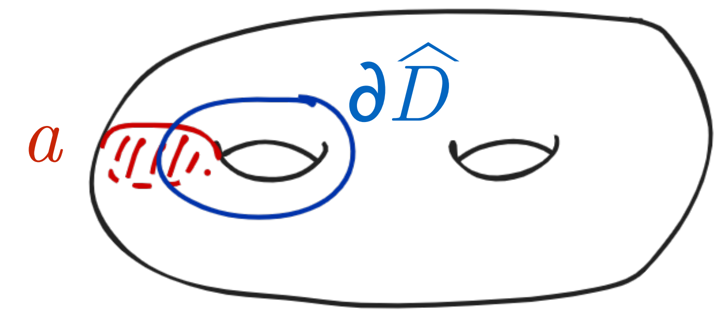
Key ingredient: primitive disk complex

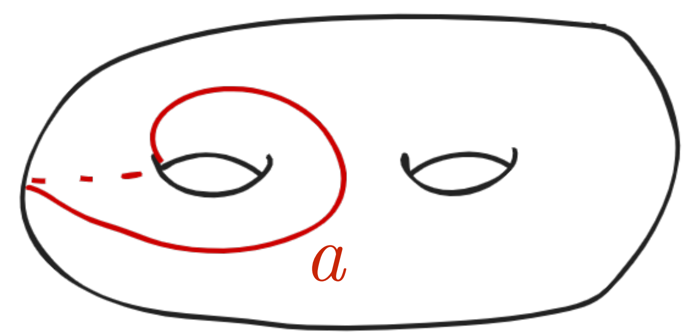
orbit map $\boldsymbol{\mathscr{G}} \to \mathscr{C}(S)$ requires choice of basepoint

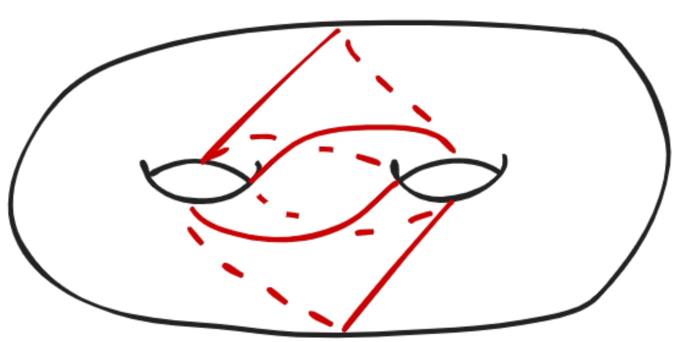
$$S^3 = V \underset{S}{\cup} W$$

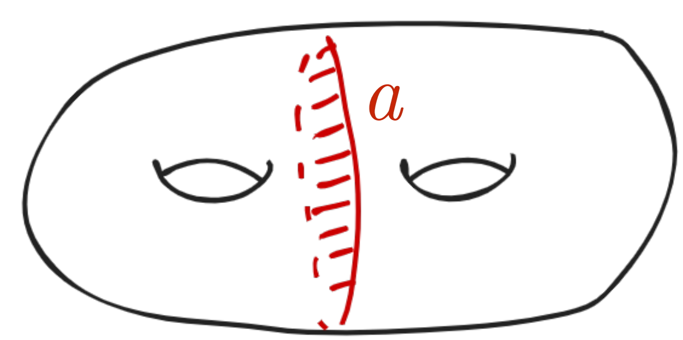
a geometrically meaningful orbit:

<u>Primitive disks complex</u> $\boldsymbol{\mathscr{P}} \subset \mathscr{C}(S)$









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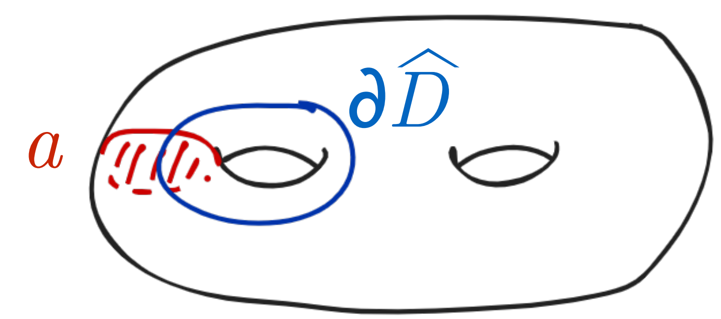
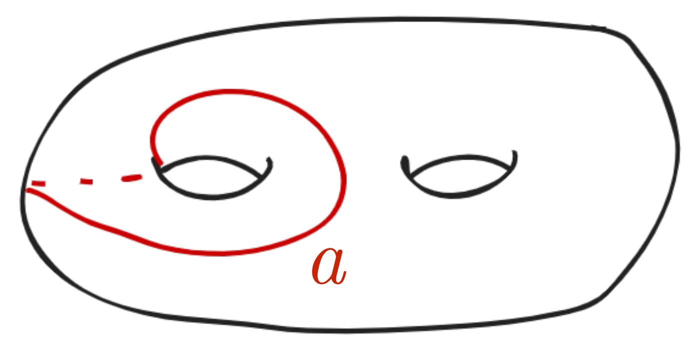
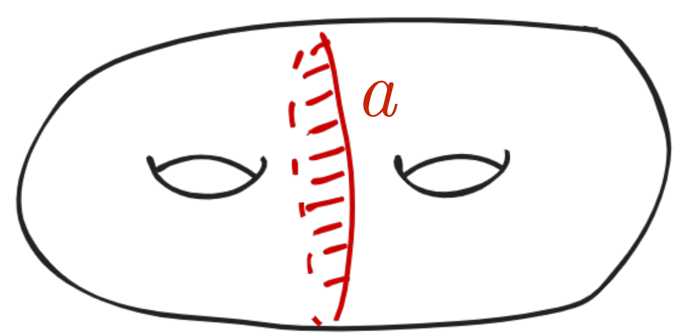
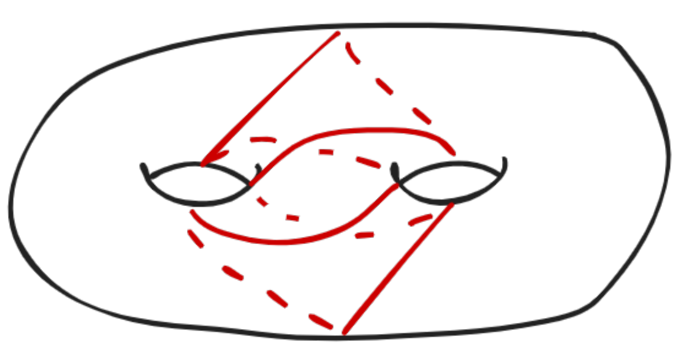


Primitive disks complex $\mathscr{P} \subset \mathscr{C}(S)$

spanned by vertices $a \in \mathscr{C}(S)$ where







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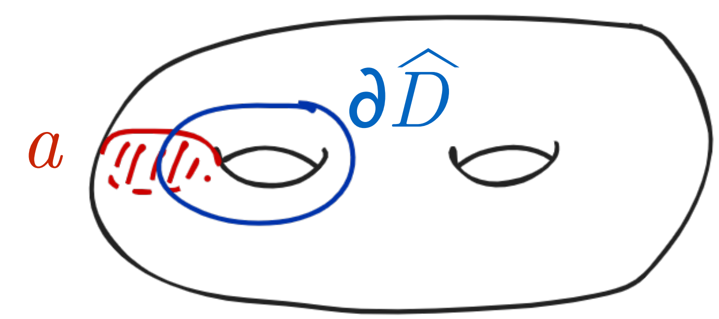

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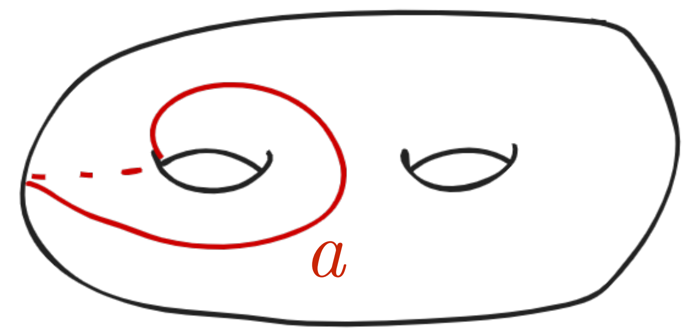

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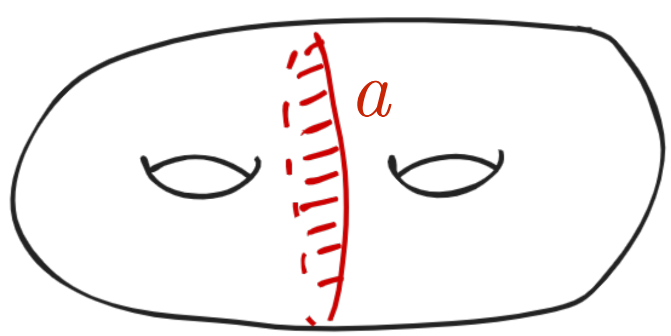

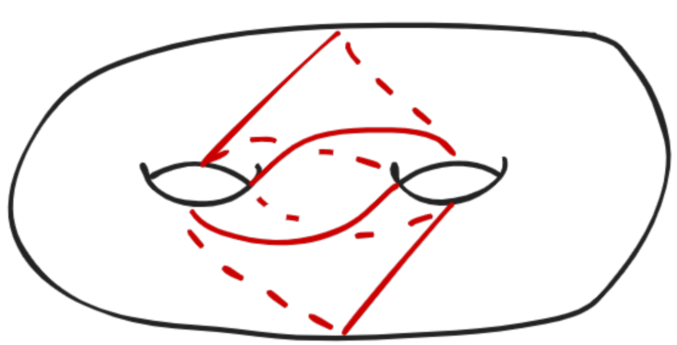

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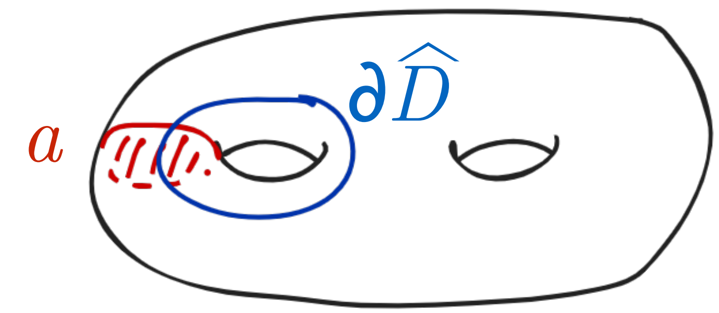

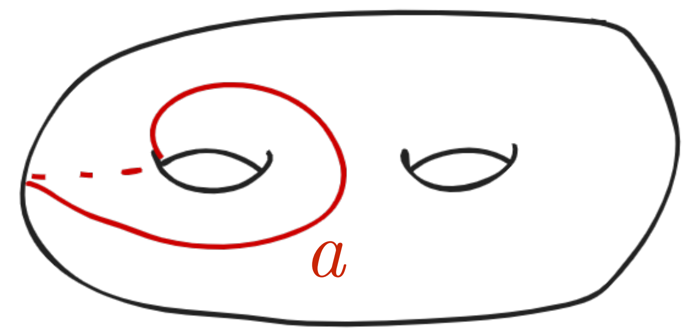

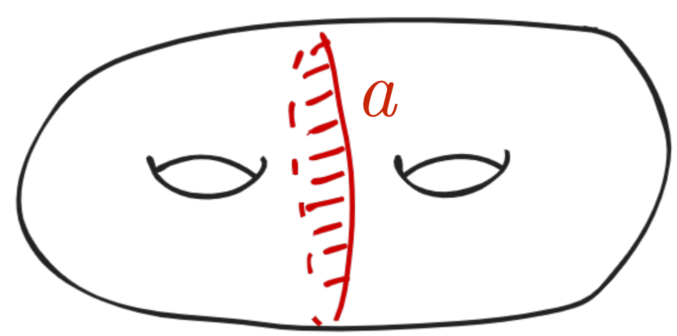

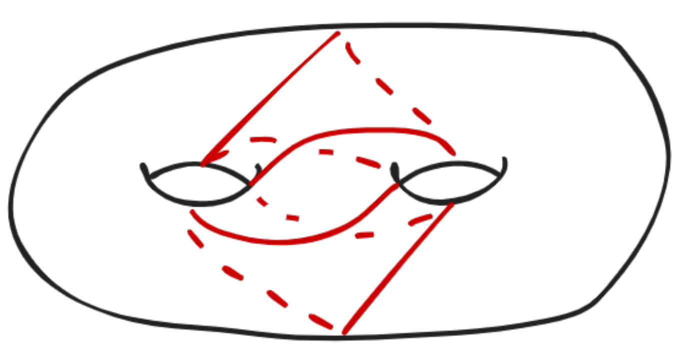

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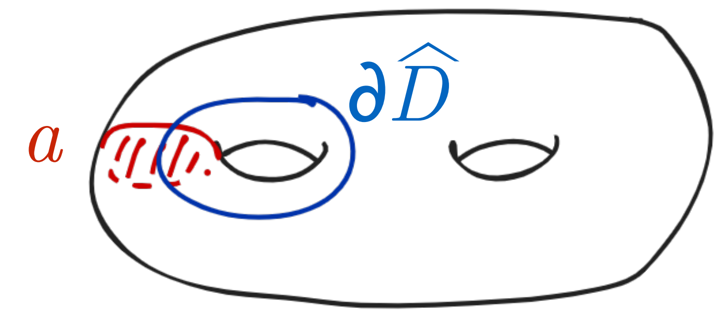

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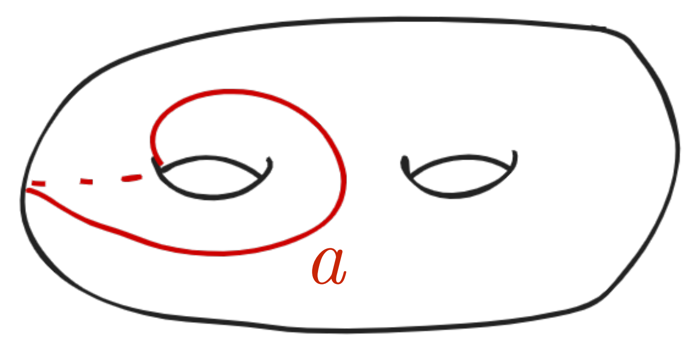

- $a = \partial D$ for some disk $D \subset V$
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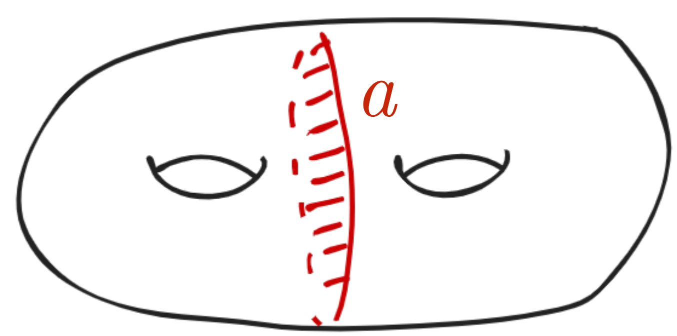
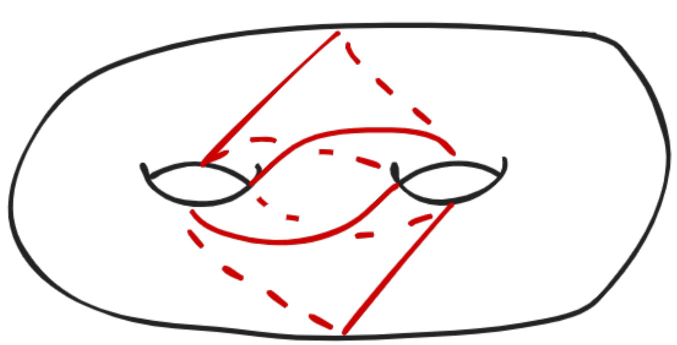
D is called a <u>primitive disk</u>





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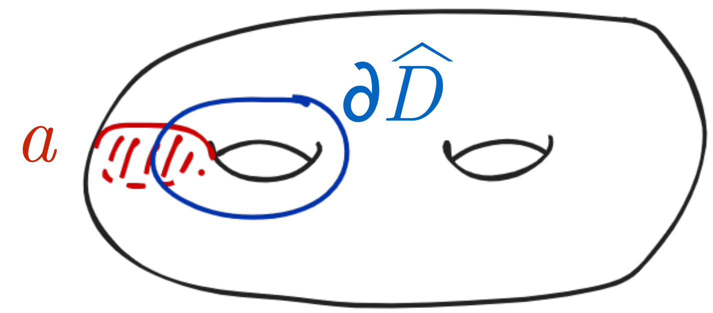

$a \in \mathscr{P}$

vertex

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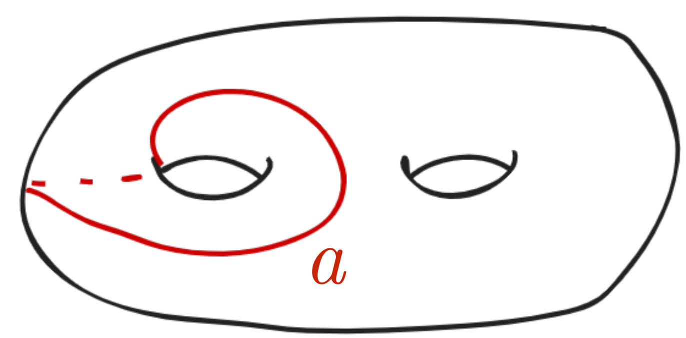

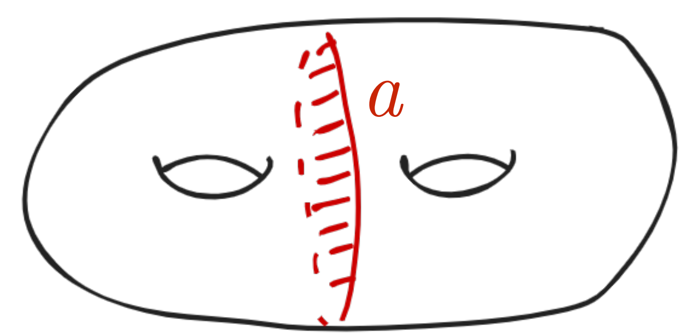

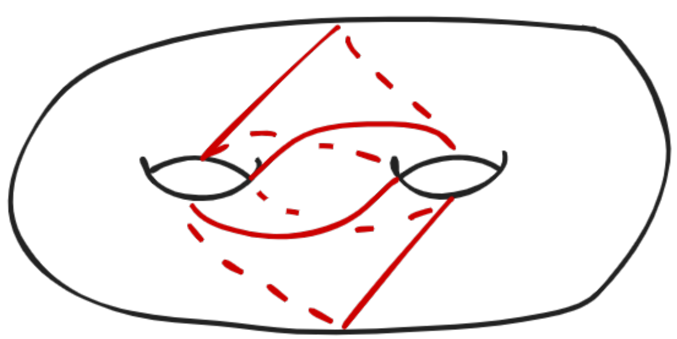

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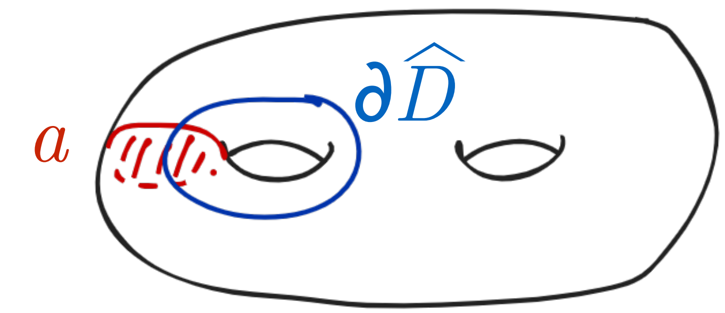

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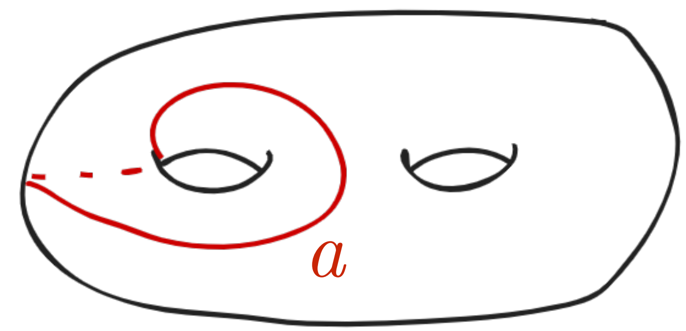
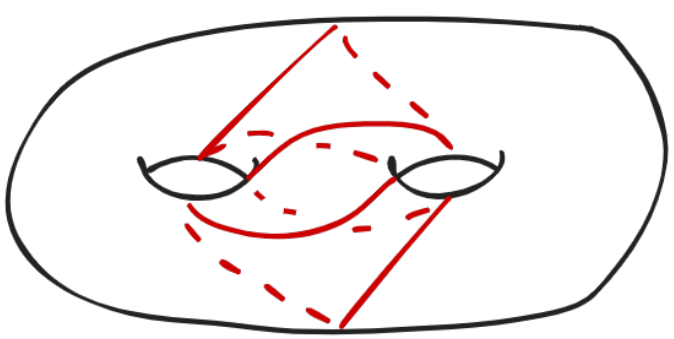
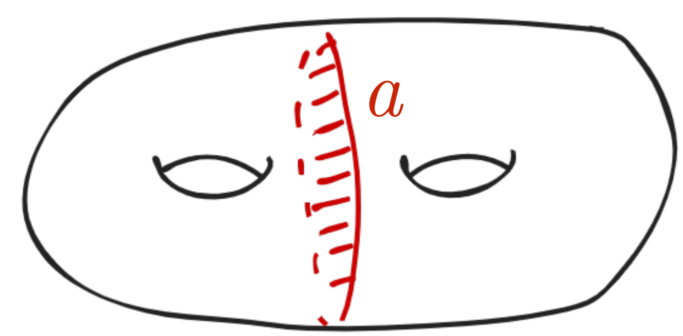


$a \notin \mathscr{P}$

doesn't bound disk in V





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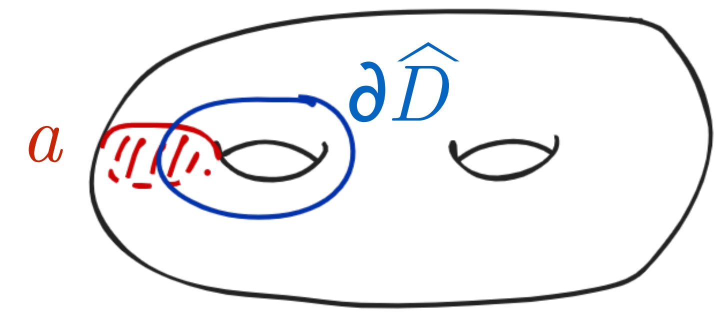

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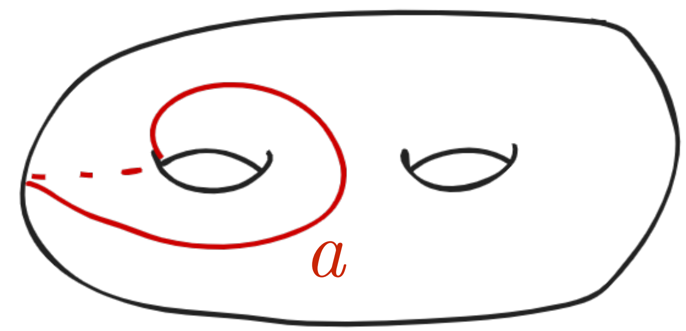
vertex

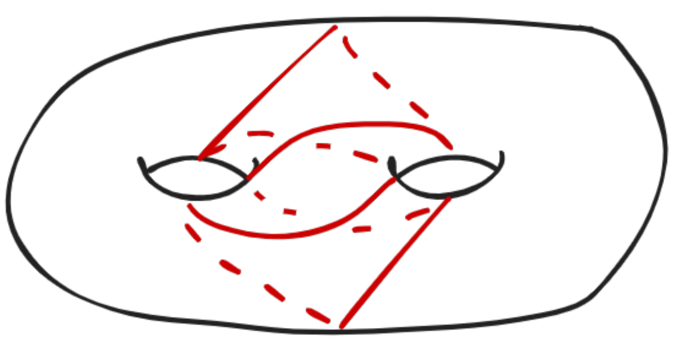


$a \notin \mathscr{P}$

doesn't bound disk in V

$a \notin \mathscr{P}$

bounds disk in V, but a is separating



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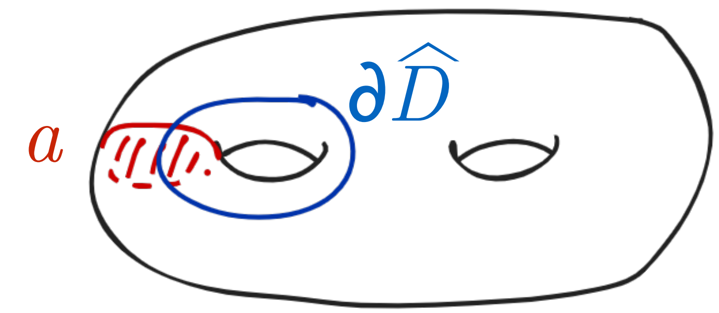

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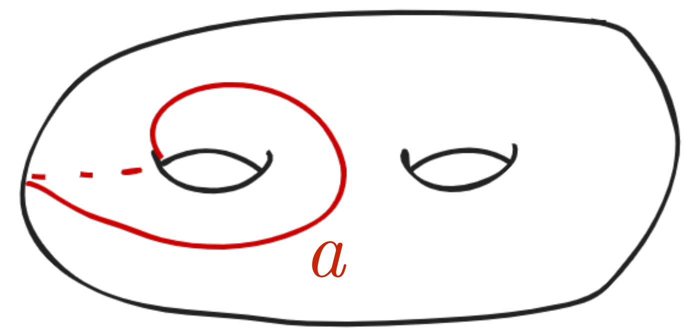
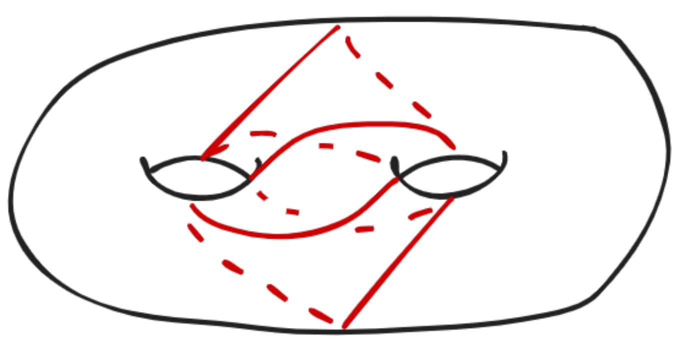


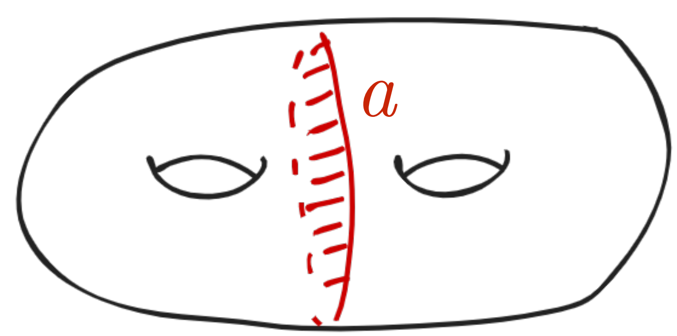
$a \notin \mathscr{P}$

doesn't bound disk in V



$a \notin \mathscr{P}$

bounds disk in V, is nonseparating, but $\nexists$ $\widehat{D}$



$a \notin \mathscr{P}$

bounds disk in V, but a is separating