Geometric Cycles | Question

- $G/\mathbb{Q}$ algebraic group
- $Y = \Gamma \backslash G(\mathbb{R})/K$ $\quad \Gamma < G(\mathbb{Z})$ $\quad [\Gamma = \Gamma(p^k)$ $\quad p$ prime & (p fixed, $k \gg 0$ variable$)]$

$\quad K < G(\mathbb{R})$ max cpt

$\uparrow$

$Y_1 = \Gamma_1 \backslash G_1(\mathbb{R})/K_1$ $\quad G_1 < G$ $\mathbb{Q}$-subgroup (reductive), $\Gamma_1 = \Gamma \cap G_1$ cocompact in $G_1(\mathbb{R})$

<u>Q</u>: Is $[Y_1] \in H_\cdot(\cancel{X};\mathbb{Q}) \cong H_\cdot(\Gamma;\mathbb{Q})$ nonzero? (for some $k \gg 0$)

(For $k \gg 0$, Y, Y_1 are oriented manifolds & $Y_1 \hookrightarrow Y$ embedding)

"geometric cycle"

<u>Examples</u> ① $G = SO(x_1^2 + \cdots + x_n^2 - \sqrt{2}\, x_{n+1}^2)$ $\quad$ Yes

$\quad \cancel{\&}\ G_1 = SO(y_1^2 + y_2^2 - \sqrt{2}\, y_3^2)$ $\quad$ Millson-Raghunathan 1981

② $G = SL_3$ $\quad \cancel{\&}\ G_1 = SO(x_1^2 + x_2^2 - d x_3^2)$ $\quad$ Yes

$d \equiv 3 (4)$ prime $\quad$ Lee-Schwermer 1986

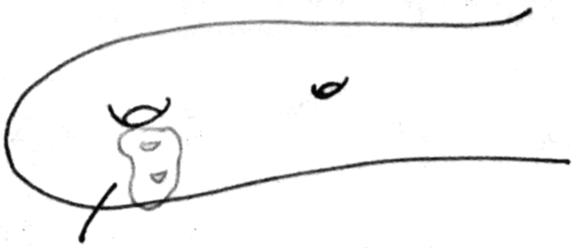
③ (a) $G = SL_3$ $\quad G_1 = \text{Centralizer}(A)$ $\quad A \in SL_3(\mathbb{Z})$ irreducible (char poly) real eigenvalues

Yes Avramidi-Nguyen-Phan 2015

(b) $G < Sp_4$ $\quad G_1 = \text{Centralizer} \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & -14 \\ 0 & 0 & 1 & 7 \end{pmatrix}$

unknown!

(next: more about flat cycles & how different from ...)



Y compact in (1)
noncompact in (2), (3)

Y_1 compact surface
hyperbolic in (1), (2)
flat in (3)

Results on $\boxed{\text{Flat cycles}}$ $\quad Y = \Gamma \backslash G(\mathbb{R}) / K \quad$ $\boxed{(\Gamma = \Gamma(p^k) \quad k \gg 0)}$

Y_1 compact, flat manifold

$\dim Y_1 = \operatorname{rank}_{\mathbb{R}} G.$

corresponding G_1 is max $\mathbb{Q}$-anisotropic alg. torus in G.

These always exist (Prasad-Raghunathan)

compare w/ Gunnells description of modular forms — take max $\mathbb{Q}$-split torus ...

$\mathbb{Q}$-anisotropic means $\operatorname{rank}_{\mathbb{Q}} G_1 = 0$

<u>Q</u>: Is $[Y_1] \in H_\bullet(Y; \mathbb{Q})$ nonzero?

<u>Yes</u> for $G = SL_n \qquad SO(n,m) \qquad R_{F/\mathbb{Q}}(SL_2)$

A–NP 2015 $\qquad$ T– 2017 $\qquad$ Zschumme 2019

$H_{n-1}(\Gamma; \mathbb{Q}) \neq 0$ (limit of stability)

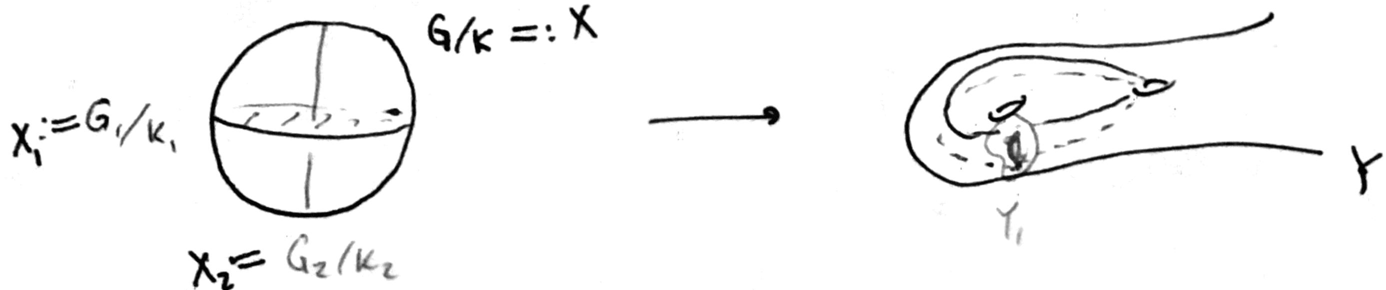
<u>Unknown</u> for $G = Sp_{2g}(\mathbb{Z})$, $g \geq 2$ $\quad$, $U(n,m)$, ...

↳ flat cycles live in deg $\min\{n,m\}$

Same degree studied in Mathilde's talk

Strategy for showing $[Y_1] \neq 0$ (Millson–Raghunathan)

Find $Y_2 \hookrightarrow Y$ s.t. $\dim Y_1 + \dim Y_2 = \dim Y$ & each intersection of Y_1 & Y_2 transverse w/ same sign.



- in MR $Y_1 = Y^{\tau_1}$ τ_1 involution (isometry)
  can choose $Y_2 = Y^{\tau_2}$ τ_2 involution & τ_1, τ_2 commute.
  "special cycle"

When Y_1 is compact, flat $Y_1 \neq Y^{\tau}$
and there may be no natural choice for Y_2 (eg $U(n,1)$)

- if can find Y_2,
  components of $Y_1 \cap Y_2$ $\longleftrightarrow$ subset of $\Gamma_1 \backslash \Gamma / \Gamma_2$
  Suffices to show for $\gamma \in \Omega$ can write $\gamma = g_1 g_2$
  where $g_i \in G_i(\mathbb{R})$ preserves or. on X_i & X.
  - for special cycles, can use Galois cohomology
  - for flat cycles: direct p-adic argument

$\boxed{\text{Application}}$ (reason I got interested in flat cycles).

<u>Thm</u> (T, 2017) $G = SO(n,m)$ $n \geq m$. Given $d > 0$ $\exists$

$\Gamma < G(\mathbb{Z})$ st. $\dim H_m(\Gamma;\mathbb{Q}) \geq d$.

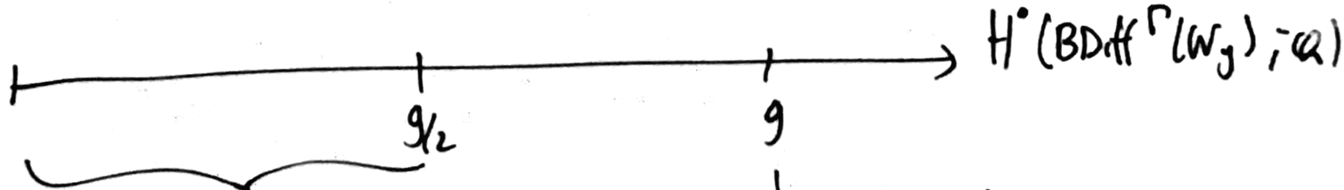
① $W_g^{4k} = \#_g (S^{2k} \times S^{2k})$

$$\begin{tikzcd} \mathrm{Diff}(W_g) \arrow[r, "\alpha"', "\text{action on } H_{2k}(W_g)"] & O(g,g;\mathbb{Z}) \\ \mathrm{Diff}^\Gamma(W_g) := \alpha^{-1}(\Gamma) \arrow[u, hook, "\text{f.i.}"] \arrow[r] & \Gamma \arrow[u, hook, "\text{f.i.}"'] \end{tikzcd}$$

(Berglund-Madsen, Krannich) $k \geq 2$, $g \geq 5$

$H^i(B\Gamma;\mathbb{Q}) \longrightarrow H^i(B\mathrm{Diff}^\Gamma(W_g);\mathbb{Q})$ inj for $i \leq 2k+4$.

<u>Cor</u> new characteristic classes



stable classes

Galatius, ~~Randal~~ Randal-Williams

poly alg gen in even deg

(some come from Γ...)

$\hookrightarrow$ dim $\geq d$.

not in alg gen by stable (eg if g odd)

② K3 surface : $H^3 \neq 0$ for f.i. subgp of Mod(K3)

coming from $H^3(\Gamma)$ $\Gamma < SO(3,19)$

Problems

- flat cycles for $U(n,m)$
- growth of flat cycles (compare w/ Mathilde's Thm)
- $\mathrm{Mod}(S_g) \longrightarrow Sp_{2g}(\mathbb{Z})$ (Melody)