Arithmetic groups and characteristic classes of manifold bundles

Bena Tshishiku

June 18, 2018

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and use to study M bundles.

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Characteristic classes of manifold bundles and arithmetic groups

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Question. Is (unstable) cohomology of $\mathrm{Sp}_{2g}(\mathbb{Z})$ a source for characteristic classes of S_g bundles?

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Plan: give characteristic class construction for bundles with structure group in $\mathrm{SL}_n(\mathbb{Z})$.

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Specifically, construct nonzero $c \in \mathrm{H}^{n-1}(\mathrm{B}\Gamma;\mathbb{Q})$ for certain (congruence) subgroups $\Gamma < \mathrm{SL}_n(\mathbb{Z})$.

Characteristic class construction

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Definition: (P,L) is <u>q-orthogonal at $b \in B$</u> if $\exists$ iso $\varphi : (\mathbb{R}^n, \mathbb{Z}^n) \to (W_b, \Lambda_b)$ s.t. $W_b = \varphi(P) \oplus \varphi(L)$ is orthogonal wrt q_b.

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Given $(W,\Lambda) \to B$, $\exists$? q such that (P,L) nowhere q-orthogonal?

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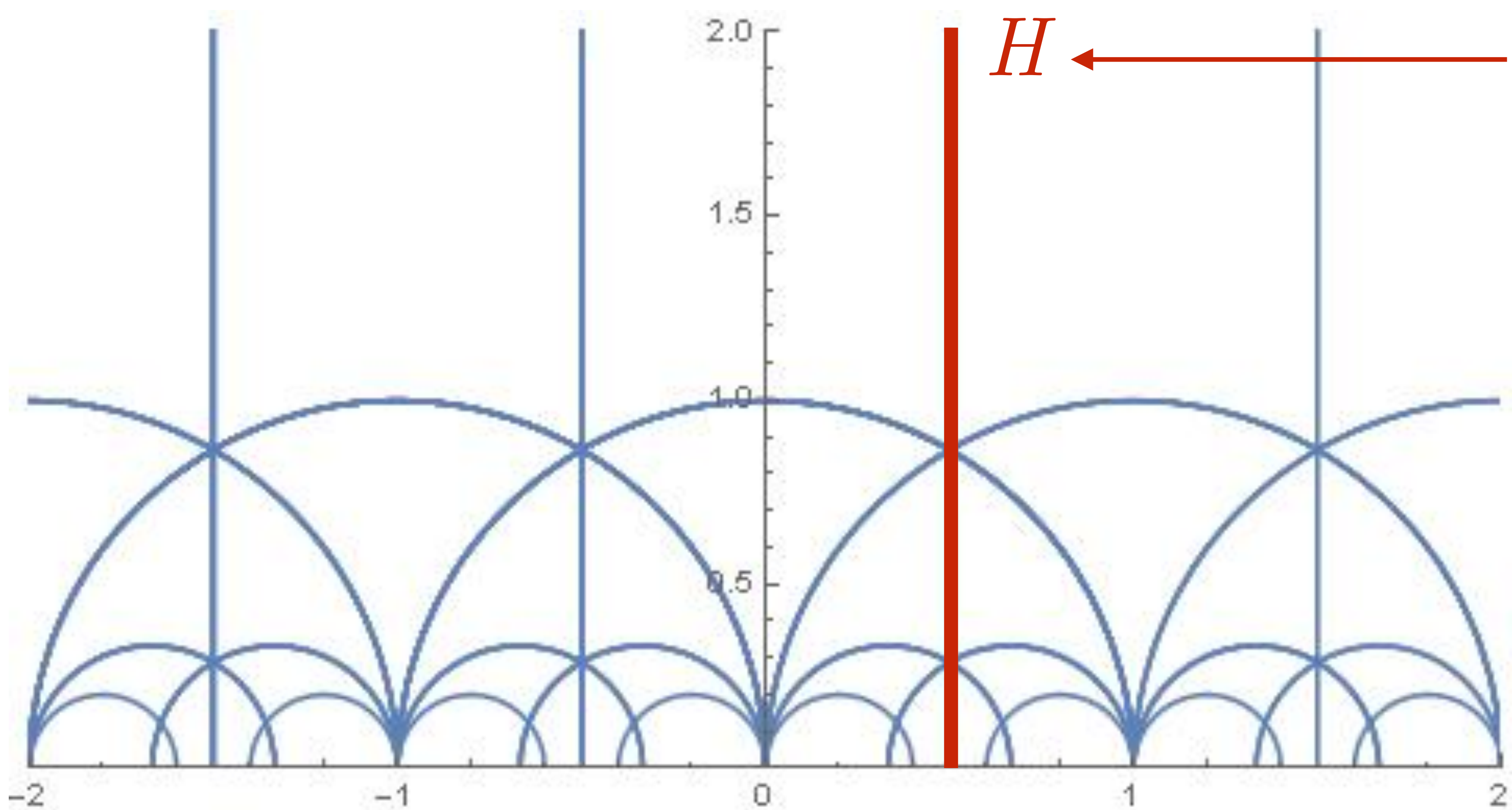

inner prod. with $\langle e_1 \rangle \oplus \langle \frac{1}{2}e_1 + e_2 \rangle$ orthogonal

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<u>**Main Theorem**</u> (T, 2017). $1 \leq p \leq q$, $p+q \geq 3$, p odd.

$\forall\ N>0$, $\exists\ \Gamma < \mathrm{SO}_{p,q}(\mathbb{Z})$ finite index so that

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Hard part: showing these characteristic classes are *nonzero/independent*.

Application to manifold bundles

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is an isomorphism for $i \leq g-1$ and injective for $i = g$.

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Theorem. $g \geq 3$ odd. $\forall\ N{>}0$, $\exists\ \Gamma < \mathrm{SO}_{g,g}(\mathbb{Z})$ finite index with

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(Berglund-Madsen). $g \geq 4$ and $k \geq (g+1)/2$.

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is an isomorphism for $i \leq g-1$ and injective for $i = g$.

Theorem. $g \geq 3$ odd. $\forall\ N>0$, $\exists\ \Gamma < \mathrm{SO}_{g,g}(\mathbb{Z})$ finite index with

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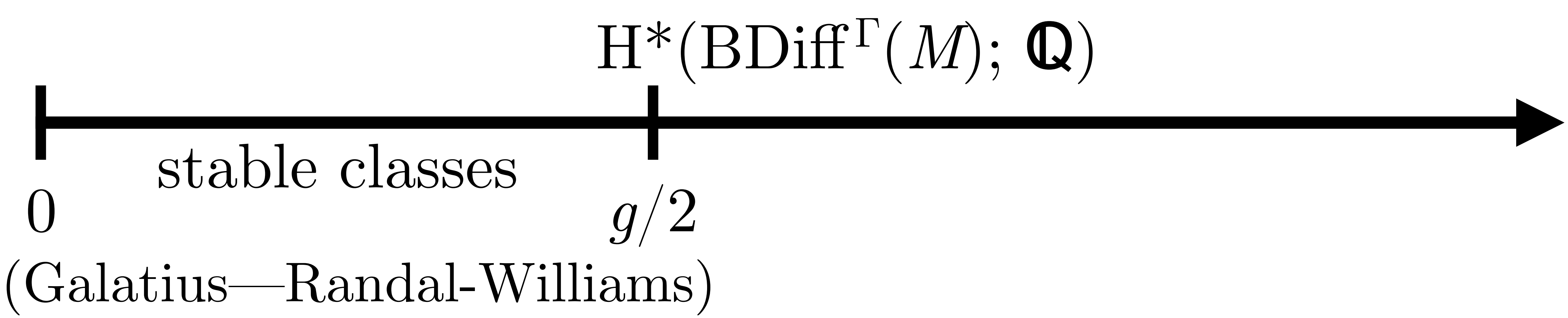

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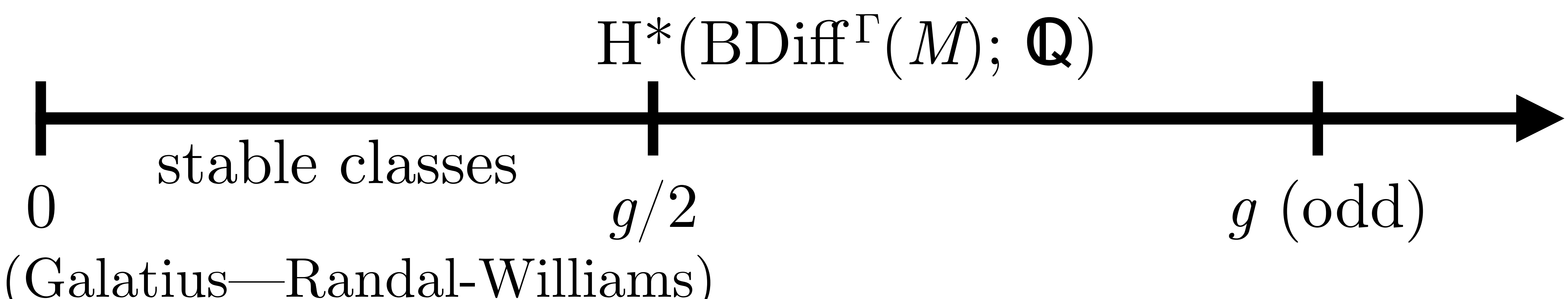

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- if z lifts to $\mathrm{BDiff}^\Gamma(\mathrm{M})$, $\exists$ K3 bundle over 3-manifold with no fiberwise Einstein metric;
- if z doesn't lift, then $\mathrm{Diff}(M) \to \pi_0\mathrm{Diff}(M)$ does not split (Nielsen realization problem).

Thank you.