Flat cycles and homology growth

Bena Tshishiku

Stability in Topology, Arithmetic, and Rep Theory

3/26/2021

Cohomology of congruence subgroups

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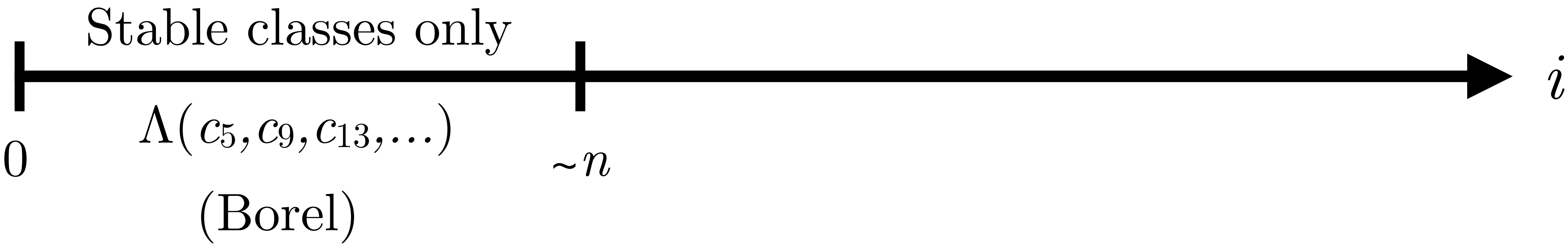

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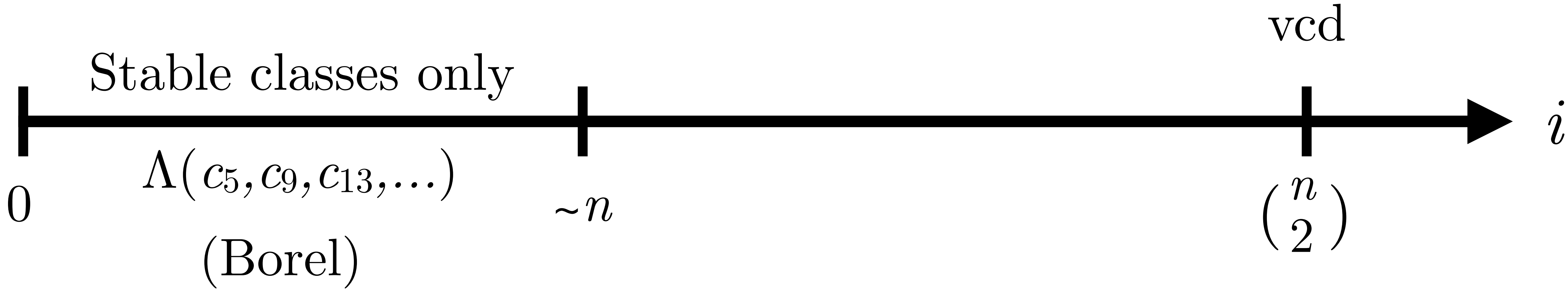

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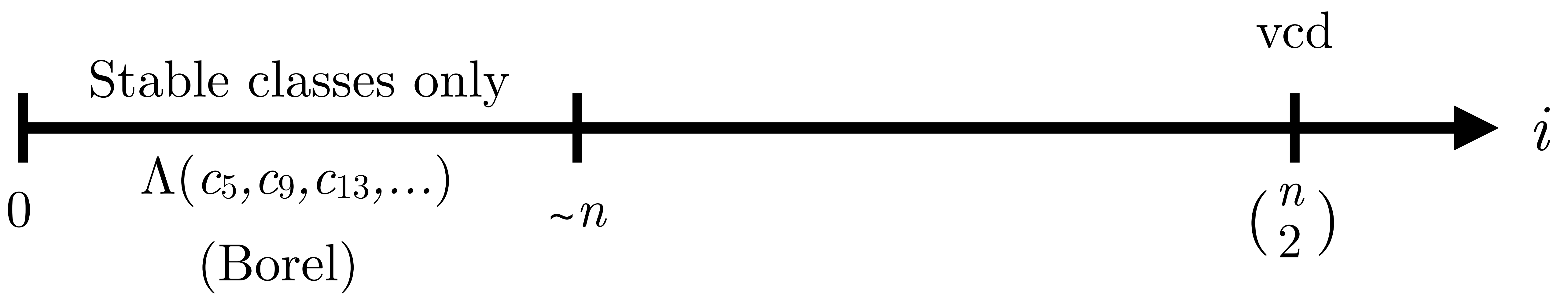

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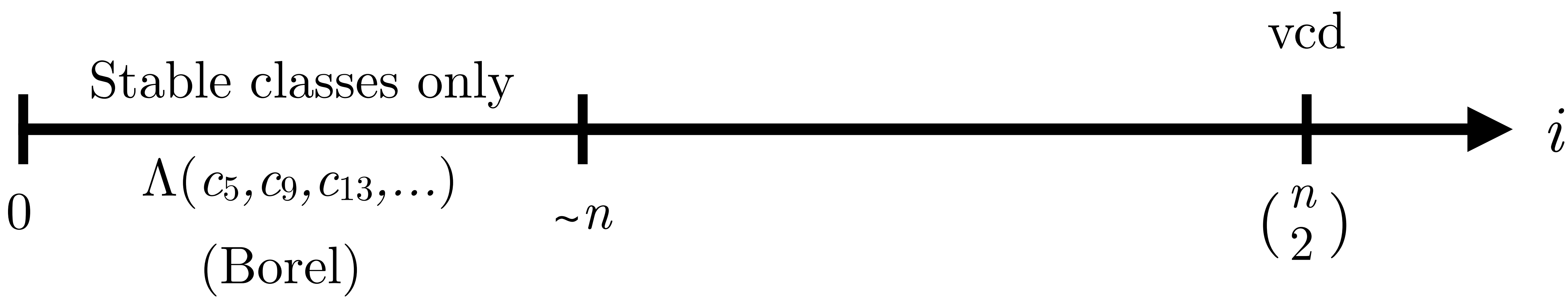

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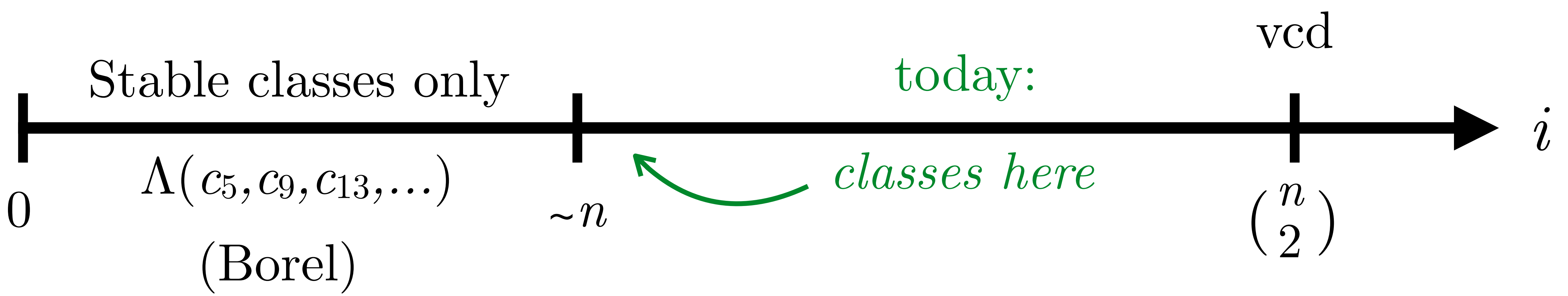

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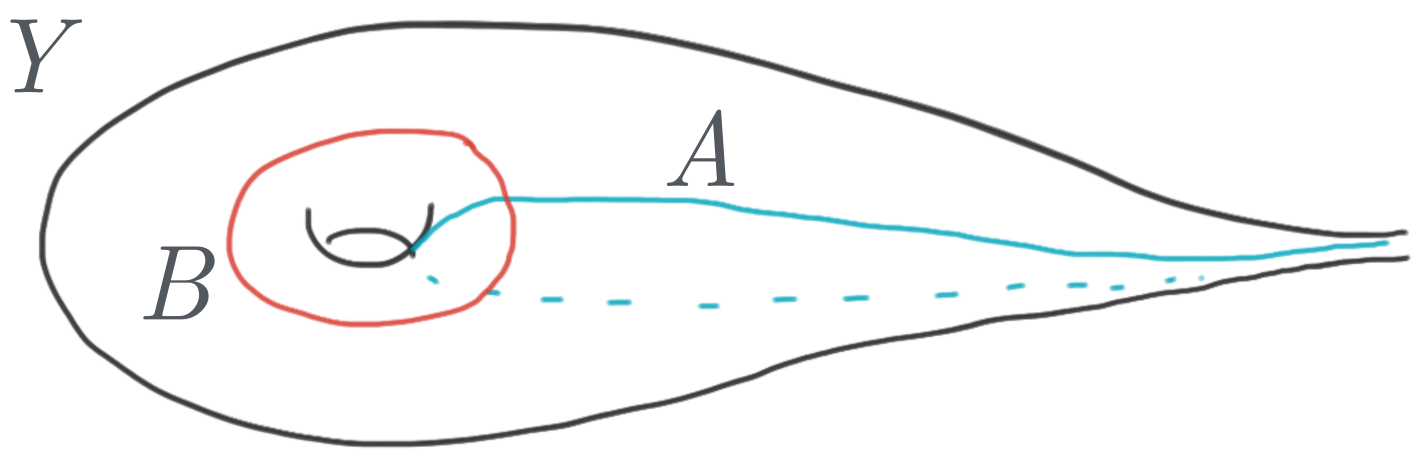

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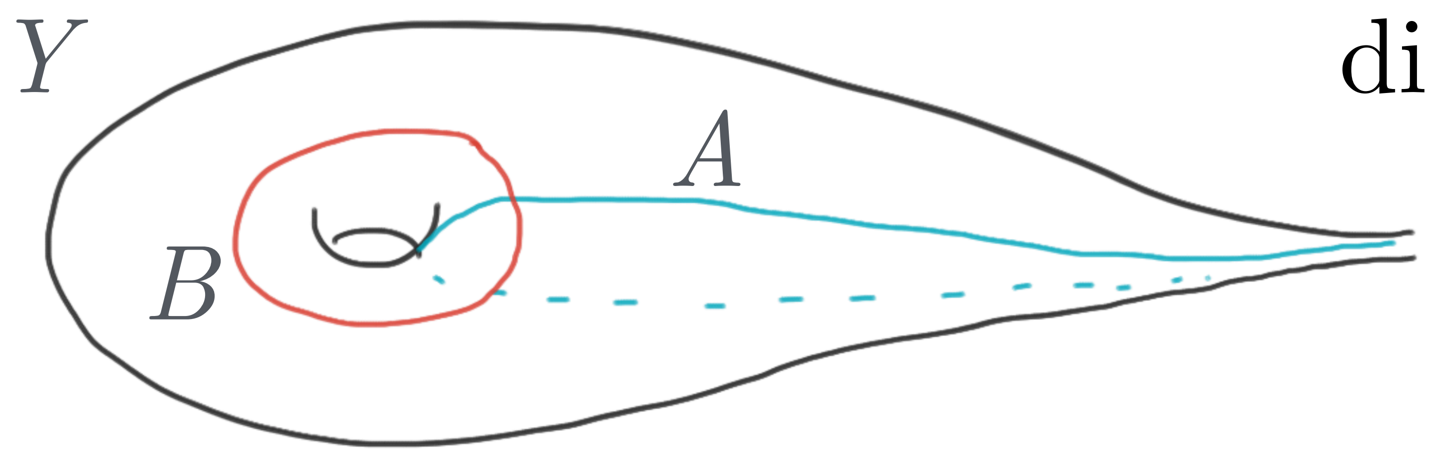

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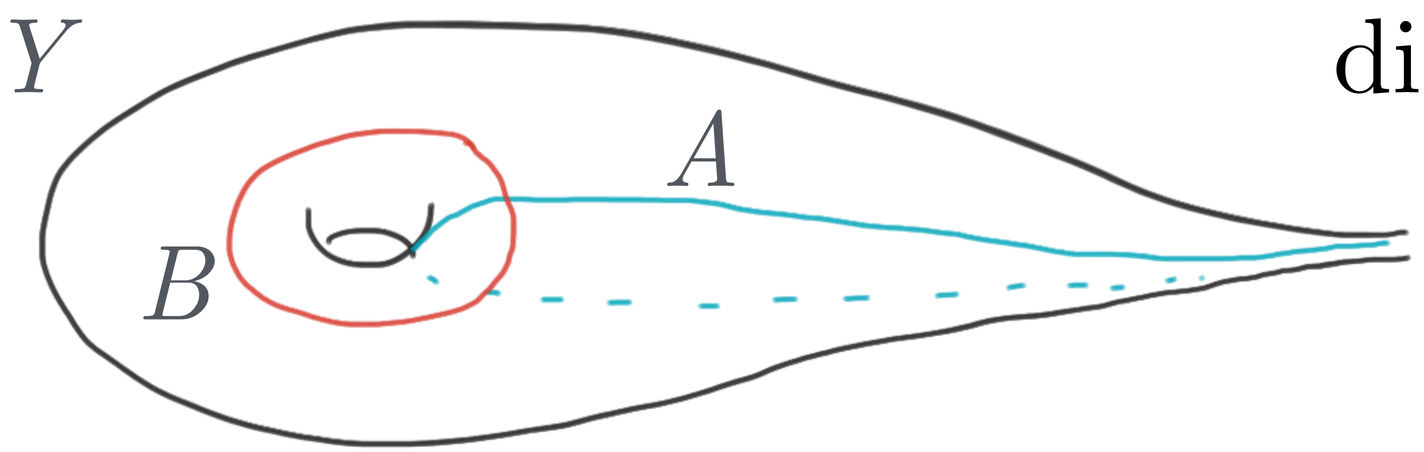

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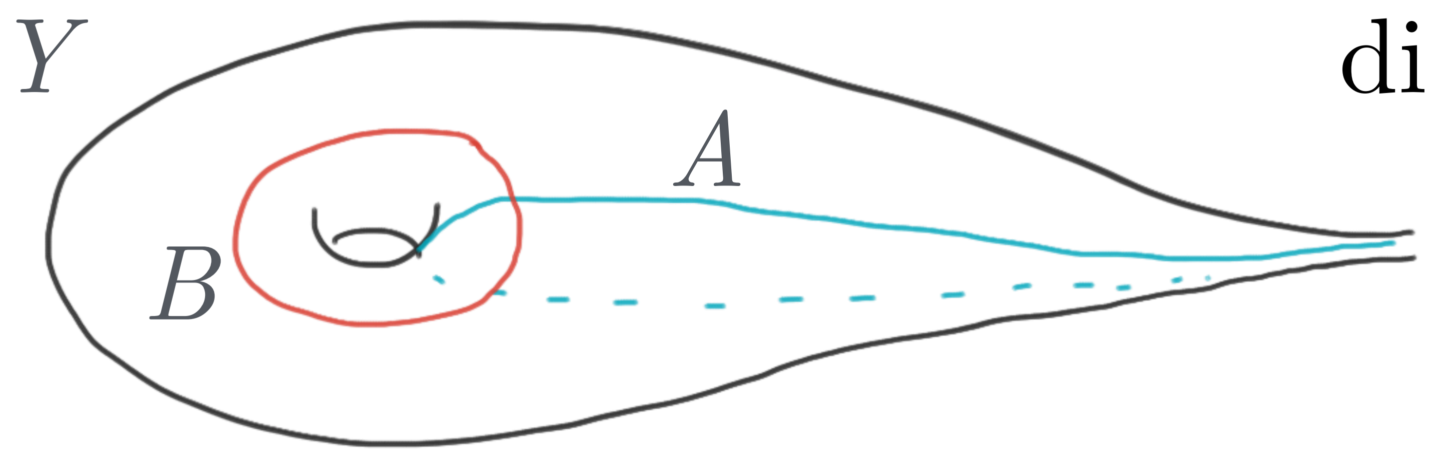

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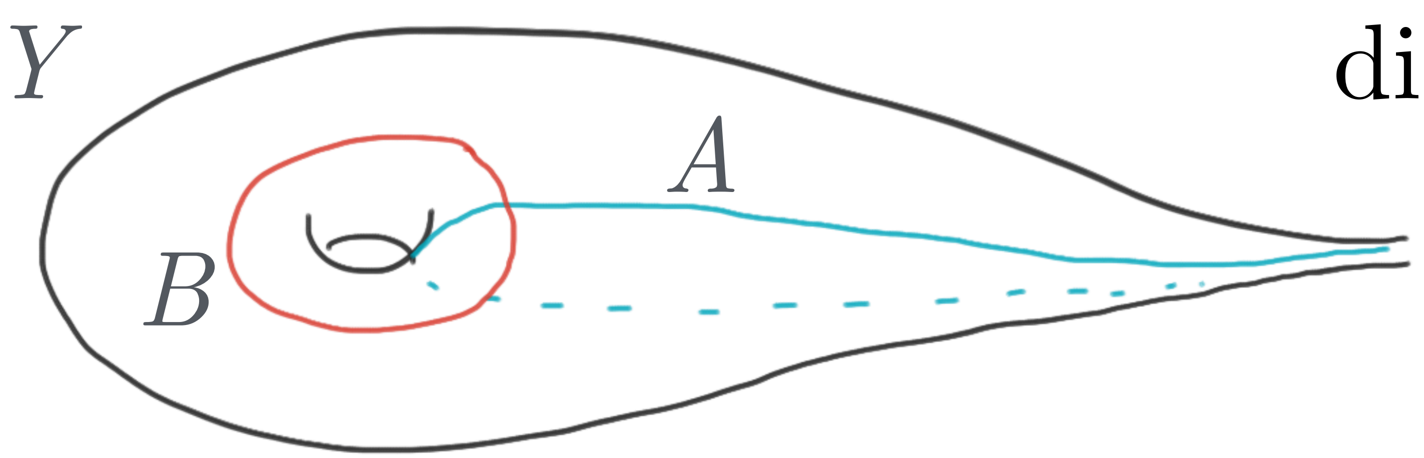

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$$\langle \mathrm{PD}(A), B \rangle = A \cdot B \neq 0 \implies \mathrm{PD}(A) \neq 0 \text{ and } [B] \neq 0$$

Low-dimensional example

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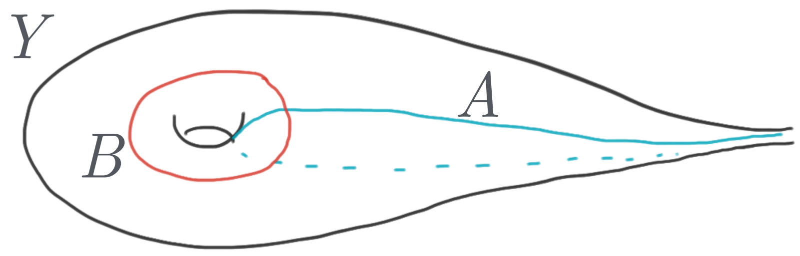

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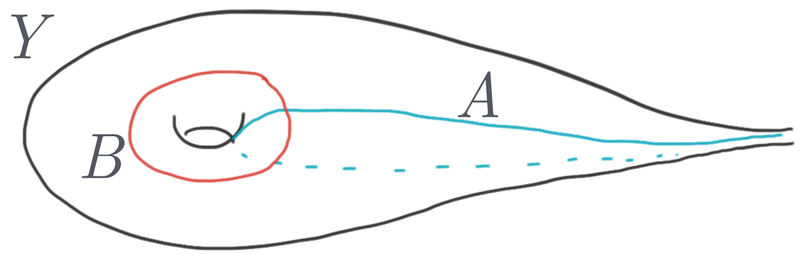

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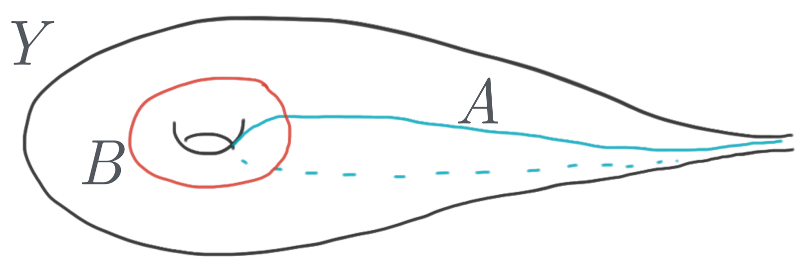

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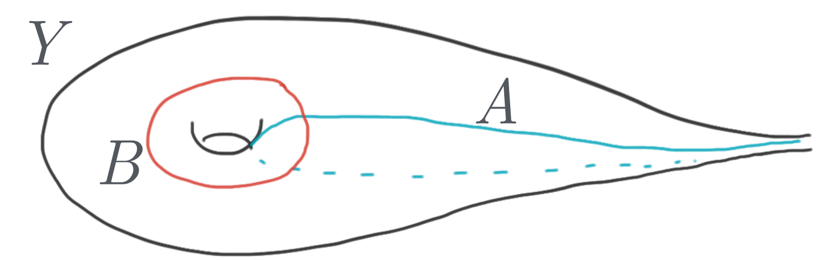

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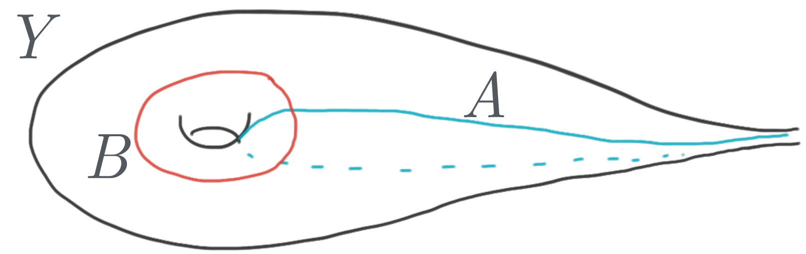

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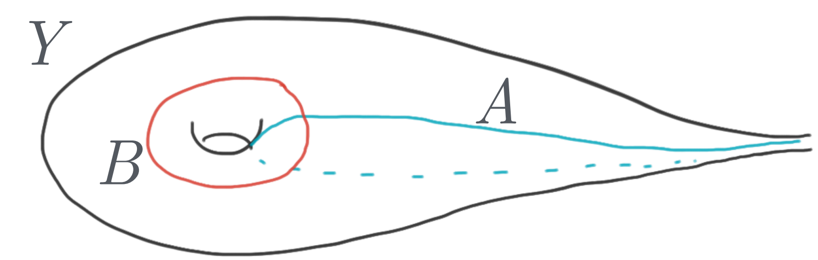

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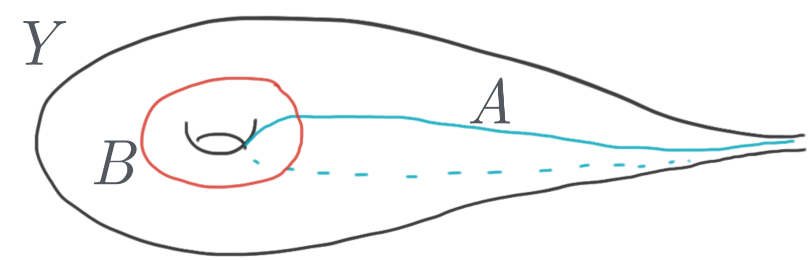

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$\mathrm{SO}(q;\mathbb{Z}) < \mathrm{SO}(q;\mathbb{R})$ *not cocompact if* $q(x_1..x_n)$ *integral, indefinite,* $n \gg 0$.

Flat cycles

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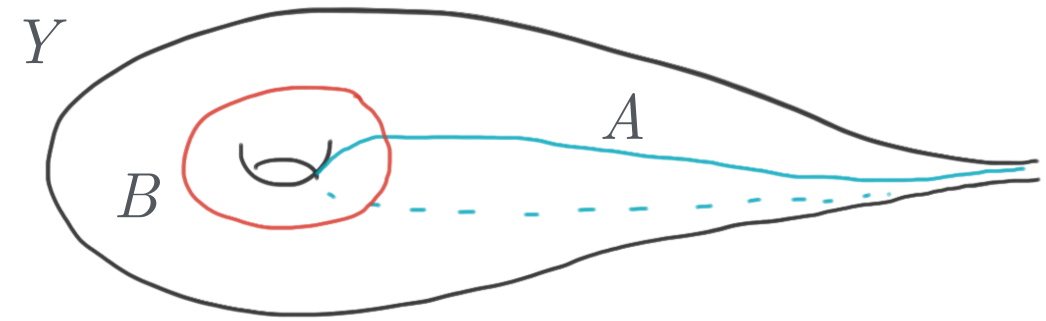

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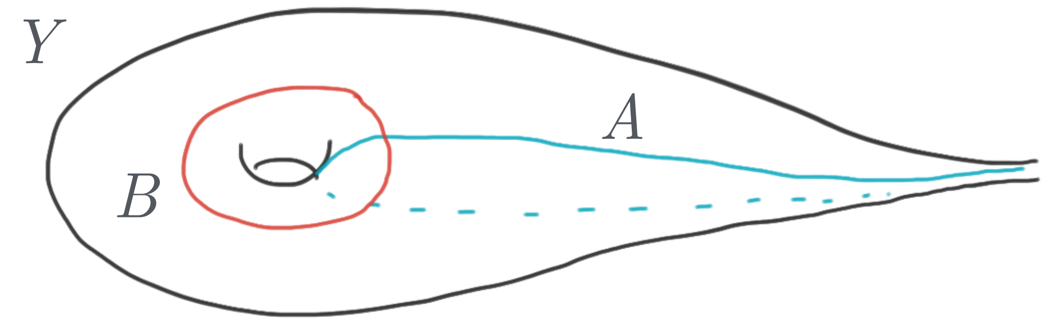

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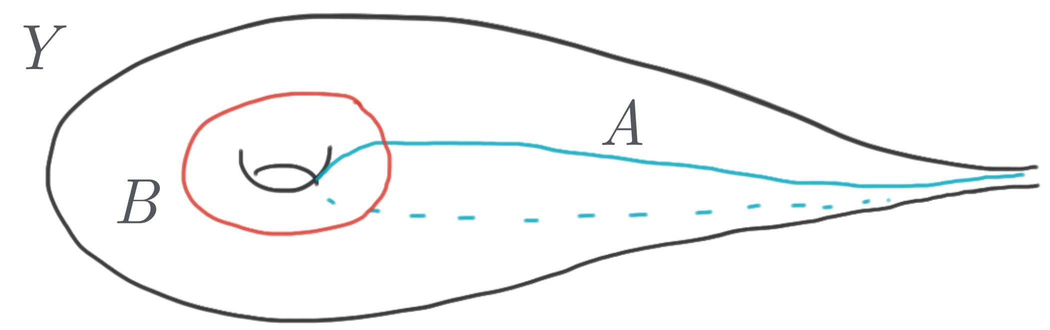

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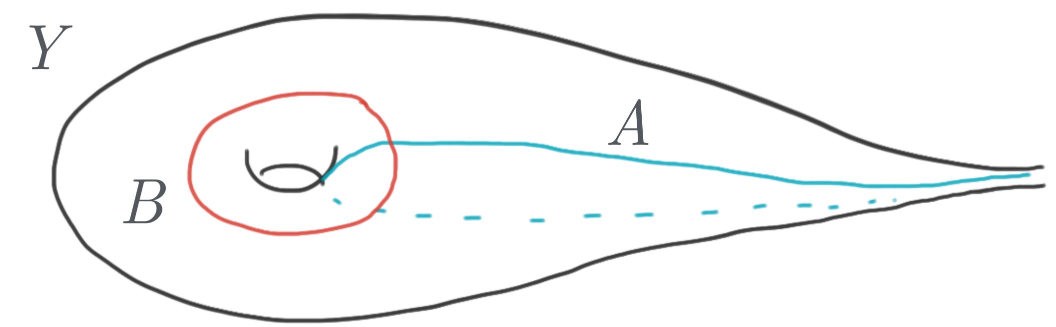

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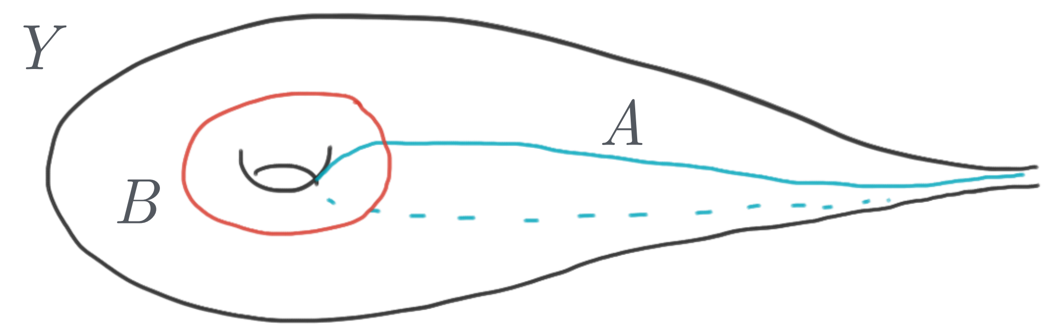

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$$\Longrightarrow \quad \mathrm{PD}(A) \neq 0 \text{ in } \mathrm{H}^{n-1}(\mathrm{Y}_n(\ell);\mathbb{Q})$$

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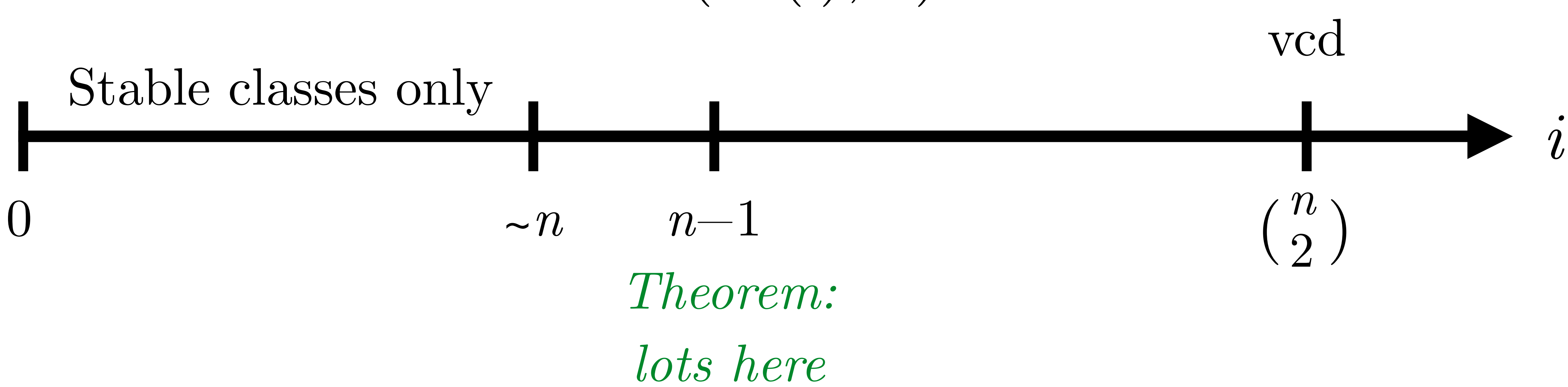

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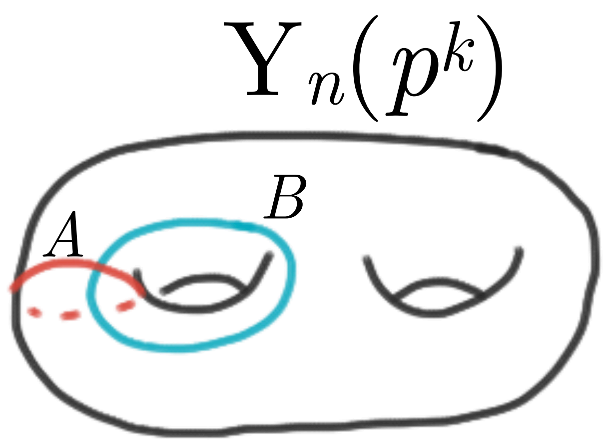

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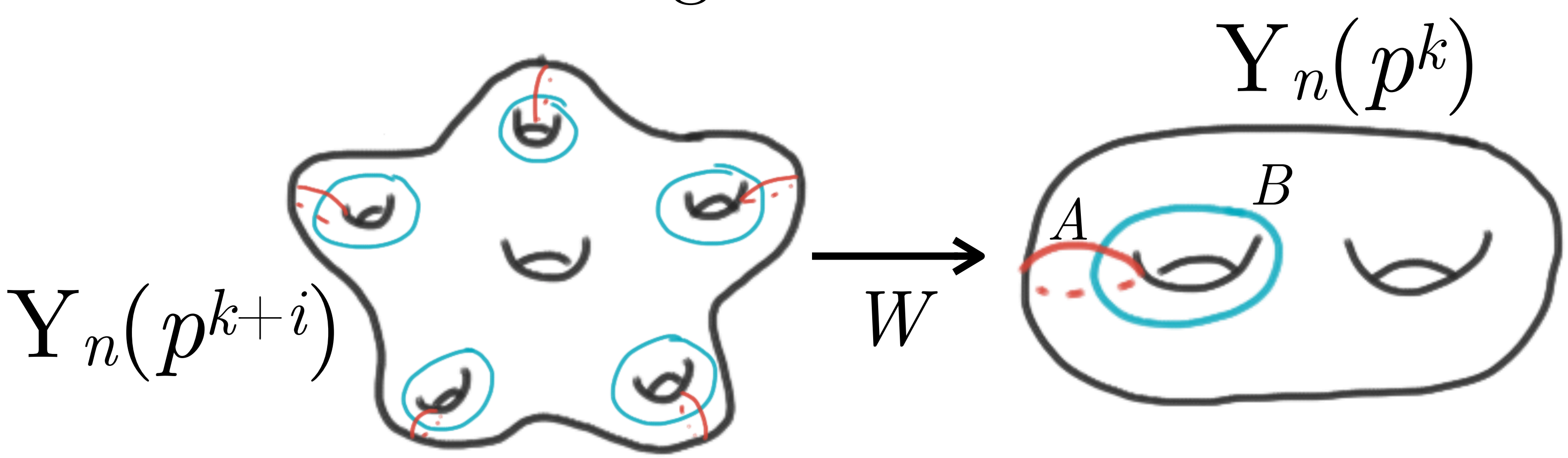
Idea: Choose regular cover



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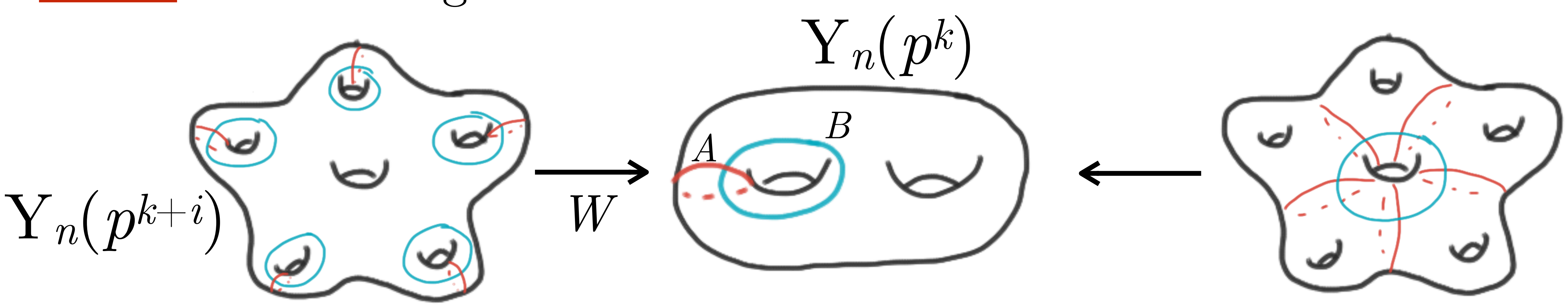

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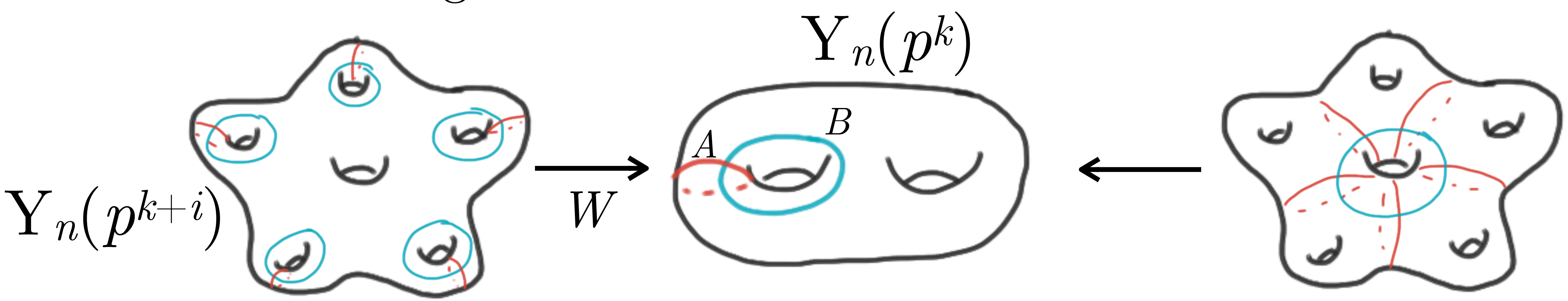

Key: compute size of image of $\pi_1(A)$, $\pi_1(B)$ in deck group W.

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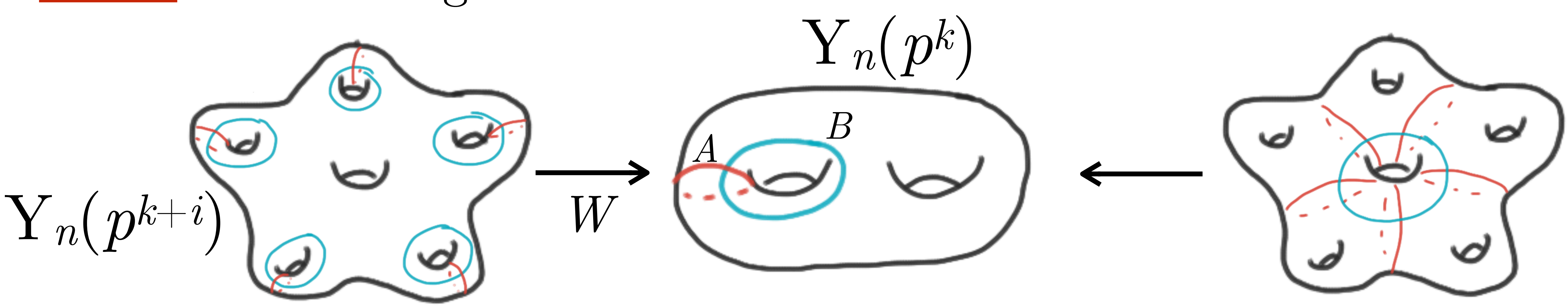

Key: compute size of image of $\overset{\mathrm{GL}_{n-1}(\mathbb{R})}{\overset{\vee}{\pi_1(A)}}$, $\overset{\mathrm{R}_{K/\mathbb{Q}}(G_m)}{\overset{\vee}{\pi_1(B)}}$ in deck group W.

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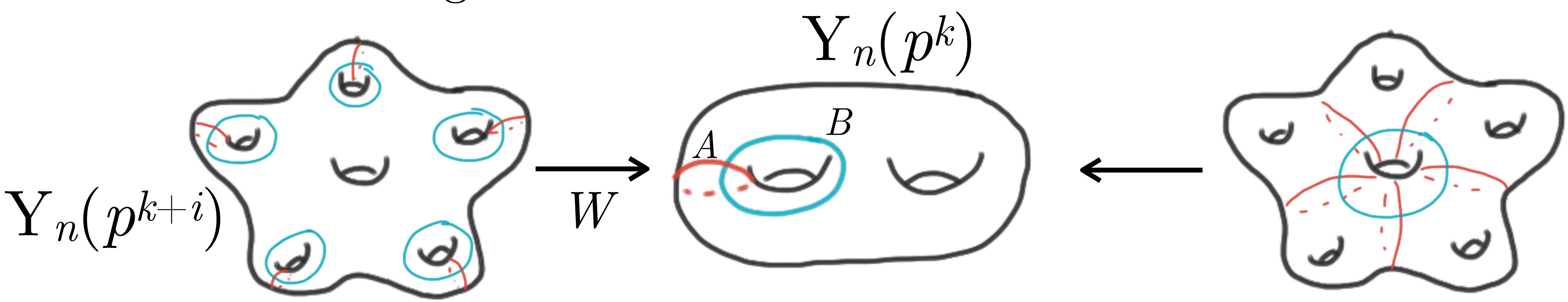

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Tools: strong approximation, . . .

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Question: What is the growth rate of H_n and the subspace of H_n spanned by flat cycles?

Thank you