# Braid Groups & Nielsen realization

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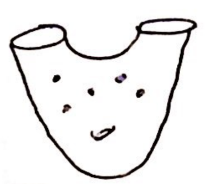
## I. The Problem

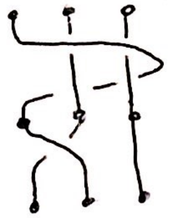


**Setup**
- $S$ cpt surface $X_n = \{x_1, \dots, x_n\} \subset S$.
- $\mathrm{Diff}(S, X_n)$ or. $C^1$ diffeos, $f|_{\partial S} = \mathrm{id}$, $f(X_n) = X_n$.
- $\mathrm{Mod}(S, X_n) := \pi_0 \mathrm{Diff}(S, X_n)$ mapping class group.
- $B_n(S)$ surface braid group
- $P : B_n(S) \longrightarrow \mathrm{Mod}(S, X_n)$ Push homomorphism

**Defn** Config. space $\mathrm{Conf}_n(S) = \{(x_1, \dots, x_n) : x_i \in \mathring{S},\ x_i \neq x_j \text{ if } i \neq j\} / S_n$.

Braid group $B_n(S) = \pi_1\left(\mathrm{Conf}_n(S)\right)$

**E.g.** • $B_n(\mathbb{D}) = B_n$. • $B_1(S) \simeq \pi_1(S)$.



**Defn** fibration $\mathrm{Diff}(S, X_n) \longrightarrow \mathrm{Diff}(S) \longrightarrow \mathrm{Conf}_n(S)$

$$f \longmapsto f(X_n)$$

induces "Birman exact seq"

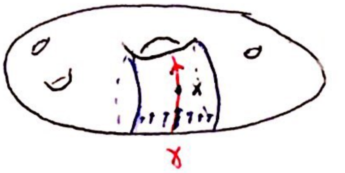
$$\pi_1 \mathrm{Diff}(S) \longrightarrow B_n(S) \xrightarrow{P} \mathrm{Mod}(S, X_n) \longrightarrow \mathrm{Mod}(S) \to 1.$$

**E.g.** • $(S = \mathbb{D})$ $\mathrm{Diff}(\mathbb{D}^2) \sim *$ Smale $\Rightarrow B_n(\mathbb{D}) \simeq \mathrm{Mod}(\mathbb{D}, X_n)$

• $(n=1)$ $P : \pi_1(S, x) \simeq B_1(S) \longrightarrow \mathrm{Mod}(S, \{x\})$

$$[\gamma] \longmapsto [f_1]$$

where $f_t$ is flow that pushes $x$ around $\gamma$.



Main Question (MO, Scott P), $S=\mathbb{D}$, by homeos

"isn't it great that this random person I never heard of is interested in similar probs to me?"

Does there exist $\tilde{P} : B_n(S) \longrightarrow \mathrm{Diff}(S, X_n)$ s.t.

$$\begin{tikzcd}
 & \mathrm{Diff}(S, X_n) \arrow[d] \\
B_n(S) \arrow[r, "P"'] \arrow[ur, dashed, "\tilde{P}"] & \mathrm{Mod}(S, X_n)
\end{tikzcd}$$

Commutes?

If $\tilde{P}$ exists, say $P$ is realized by diffeos. Kerckhoff / Morita / flat bundles.

Rmk This is example of Nielsen realization problem.

II. Tension (Is $P : B_n(S) \longrightarrow \mathrm{Mod}(S, X_n)$ realized?)

A. Evidence against realization

Thm (Bestvina-Church-Souto '09) $S = S_g$ closed, $g \geq 2$, $n \geq 1$.

$\Rightarrow$ $P$ not realized.

Q: What about $g=0,1$ or $\partial S \neq \emptyset$?

~~[illegible] $B_n(\mathbb{D}) \simeq \mathrm{Mod}(\mathbb{D}, X_n)$.~~

B. Evidence for realization ~~(focus on $S=\mathbb{D}$ $B_n(\mathbb{D}) \simeq \mathrm{Mod}(\mathbb{D}, X_n)$)~~

(i) $\mathrm{Mod}(\mathbb{T}^2, 0) \simeq SL_2\mathbb{Z} \longrightarrow \mathrm{Diff}(\mathbb{T}^2, 0)$ realization.

(ii) Cor (Thurston) $P : B_3(\mathbb{D}) \longrightarrow \mathrm{Mod}(\mathbb{D}, X_3)$ realized by homeos.

Pf sketch

- $SL_2\mathbb{Z} \curvearrowright$  $\rightsquigarrow$ $PSL_2\mathbb{Z} \curvearrowright$  (prob: wrong group & nontrivial on $\partial$)

* $PSL_2\mathbb{Z} = \mathbb{Z}/2 * \mathbb{Z}/3 = \langle a, b \mid a^2 = b^3 = 1 \rangle$.

$B_3 = \widetilde{SL_2\mathbb{Z}} = \langle x, y \mid x^2 = y^3 \rangle$. $(1 \to \mathbb{Z} \to B_3 \to PSL_2\mathbb{Z} \to 1$

- add annulus to homotop $b|_{\partial D}$ thru order 3 rots to a rot commuting w/ $a$.
- add annulus to homotop $a|_\partial, b|_\partial$ to identity preserving $a^2 = b^3$. □.

(iii) For $S = D$, if $\mathcal{P}$ realized get

$$\begin{tikzcd} & H^*(\mathrm{Diff}(D, X_n)) \arrow[dl] \\ H^*(B_n D) & H^\cdot(\mathrm{Mod}(D, X_n)) \arrow[l, "\cong"'] \arrow[u, "\pi^*"'] \end{tikzcd}$$

$\Rightarrow$ $\pi^*$ injective.

Thm (Morita) $S_g$ closed $g \gg 1$ $\Rightarrow$ $\pi^*: H^\cdot(\mathrm{Mod}(S)) \longrightarrow H^\cdot(\mathrm{Diff}(S))$ not injective

Thm (Nariman '15) $H^\cdot(\mathrm{Mod}(D, X_n)) \longrightarrow H^\cdot(\mathrm{Diff}(D \setminus X_n))$ injective (!)

(so no cohomological obstruction - maybe low genus is special and realizations exist!)

## III. Resolution

Thm (Salter–T) $S$ compact. $n \geq 6$. Then $\mathcal{P}$ : ~~$\mathrm{Mod}(S, X_n)$~~ $B_n(S) \longrightarrow \mathrm{Mod}(S, X_n)$ not realized by diffeos.

Pf outline

Step 1 (The obstruction)

Defn A group $G$ is locally indicable if every f.g. $\{1\} \neq \Gamma < G$ admits surj $\Gamma \twoheadrightarrow \mathbb{Z}$.

Thm (Thurston stability, 1974) The group

$$\mathrm{Diff}(S, T_x S) = \{ f \in \mathrm{Diff}(S) : f(x) = x, (df)_x = id \}$$

is locally indicable.

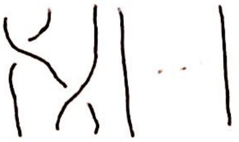
Strategy Show if $\mathcal{P}$ realized, then $\mathrm{Diff}(S, T_x S)$ would contain (image of) $\Gamma < B_n(S)$ f.g. perfect group. (ie $\Gamma = [\Gamma, \Gamma]$), impossible by Thurston ~~a contradiction~~

## Step2 (Perfect subgroup of $B_n(S)$)

Rmk $B_n$ is not perfect $\quad [B_n,B_n] \longrightarrow \overset{\langle\sigma_1,\dots,\sigma_{n-1}\rangle}{\overset{\shortparallel}{B_n}} \twoheadrightarrow \mathbb{Z}$

$\sigma_i \longmapsto 1.$

- e.g. $\sigma_1 =$  not a commutator.

- OTOH $\sigma_1\sigma_2^{-1} =$  is

$\sigma_2\sigma_1\sigma_2 = \sigma_1\sigma_2\sigma_1$

$\Rightarrow \sigma_2^{-1}\sigma_1^{-1}\sigma_2\sigma_1 = \sigma_1\sigma_2^{-1}$

$\sigma_1\sigma_2^{-1} = [\sigma_2^{-1},\sigma_1^{-1}].$

- Moreover if $n \geq 5$, have $\sigma_4 \in B_n$ and

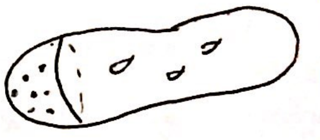
$$\sigma_1\sigma_2^{-1} = [\sigma_4\sigma_2^{-1}, \sigma_4\sigma_1^{-1}].$$

Thm (Gorin-Lin) For $n \geq 5$ $[B_n,B_n]$ f.g. perfect group.

$$[B_n,B_n] = [[B_n,B_n],[B_n,B_n]].$$

Rmk False for $n=3,4$. $(B_4 \twoheadrightarrow B_3 \twoheadrightarrow PSL_2\mathbb{Z} \twoheadrightarrow \mathbb{Z})$

Rmk ~~$B_n(S)$ contains image of~~ $B_n(D) \longrightarrow B_n(S)$.
$(D, X_n) \hookrightarrow (S, X_n)$ induces



## Step3 (Reduction to Thurston stability)

Easy Lemma Fix $n \geq 5$. Every $\alpha: B_n \longrightarrow GL_2^+\mathbb{R}$ has abelian image.

If $\tilde{P}$ exists

$$\begin{tikzcd} B_n(S) \arrow[r, "\tilde{P}"] \arrow[d, phantom, "\cup"] & \mathrm{Diff}(S, X_n) \arrow[d, phantom, "\cup"] & \\ B_{n-1}(S) \arrow[r] \arrow[rr, bend right, "D"'] & \mathrm{Diff}(S, X_n) \cap \mathrm{Diff}(S, \{x_n\}) \arrow[r] & GL(T_{x_n}S) \end{tikzcd}$$

Lemma $\Rightarrow$ ~~$\tilde{P}$~~
$[B_{n-1}, B_{n-1}] < \ker D$

$\Rightarrow \tilde{P}([B_{n-1}, B_{n-1}]) < \mathrm{Diff}(S, T_{x_n}S)$

$\underset{\text{Thurston}}{\Rightarrow} \exists \ [B_{n-1}, B_{n-1}] \longrightarrow \tilde{P}([B_{n-1}, B_{n-1}]) \twoheadrightarrow \mathbb{Z}$ impossible for $n \geq 6$. □

## IV. Application

**Cor** $S = S_g$ closed, $g \geq 2$. $\mathrm{Diff}(S) \to \mathrm{Mod}(S)$ does not split.

**Rmk** • Due to

- (Morita '87) in case of $C^2$ diffeos, $g \geq 18$
- (Franks-Handel '09) $C^1$ diffeos, $g \geq 3$
- (Markovic, Markovic-Saric 08) homeos, $g \geq 2$.

(Our proof is elementary)

**Pf Sketch** - Suppose $s : \mathrm{Mod} \to \mathrm{Diff}$ exists.

- Idea: consider action of hyperelliptic involution $\tau \in \mathrm{Mod}(S)$ and its centralizer $C(\tau)$

• hyperelliptic involution

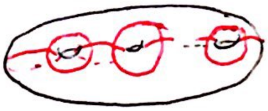
**Claim** $s(\tau)$ has $2g+2$ fixed pts (of course $\exists$ ~~diffeo~~ lift w/ $2g+2$ f.p. but $s(\tau)$ is a random lift)

Pf: Lefschetz f.p.

$$\# \mathrm{Fix}(s(\tau)) = \sum (-1)^i \, \mathrm{tr}\left(\tau \mid H_i(S)\right) = 1-(-2g)+1$$
$$= 2+2g.$$

Let $\mathrm{Fix}(s(\tau)) = \{x_1, \dots, x_{2g+2}\} =: X_{2g+2}$

• Centralizer $C(\tau)$ not locally indicable:

 defines $B_{2g+2} \to C(\tau)$.

• **Claim** The induced action $B_{2g+2} \to C(\tau) \xrightarrow{s} \mathrm{Diff}(S, X_{2g+2}) \to \overset{\mathrm{Sym}(2g+2)}{\cancel{\mathrm{Perm}(X_{2g+2})}}$ is the standard one. (Birman-Hilden: isotopic $\Rightarrow$ symmetrically isotopic)

•
$$\begin{tikzcd}
B_{2g+2} \arrow[r] & \mathrm{Diff}(S, X_{2g+2}) & \\
B_{2g+1} \arrow[u, phantom, "\cup" rotate=90] \arrow[r] & \mathrm{Diff}(S, X_{2g+2}) \cap \mathrm{Diff}(S, \{x_{2g+2}\}) \arrow[u, phantom, "\cup" rotate=90] \arrow[r] & GL_2(T_{x_{2g+2}} S).
\end{tikzcd}$$

repeat as before $\square$