Obstructions to Nielsen realization

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Joint with Nick Salter

I. Realization problem for braid groups

Braid group (3 interpretations)

$B_n \coloneqq \langle \text{braids} \rangle$

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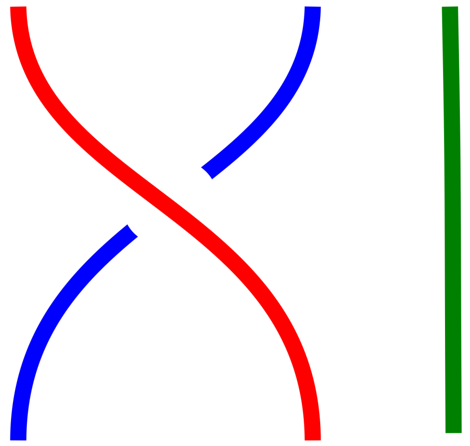

a

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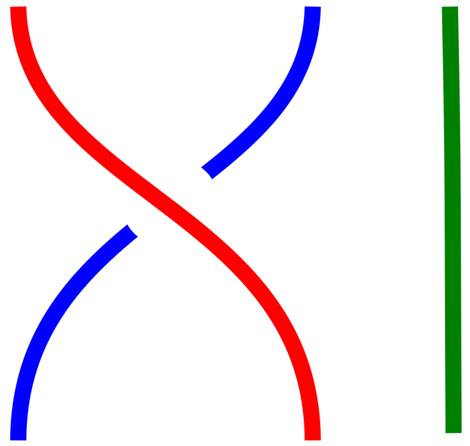
$B_n \coloneqq \langle \text{braids} \rangle \simeq \pi_1 \operatorname{Conf}_n(D^2)$



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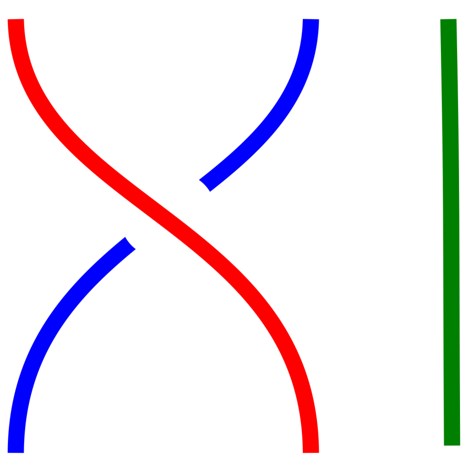
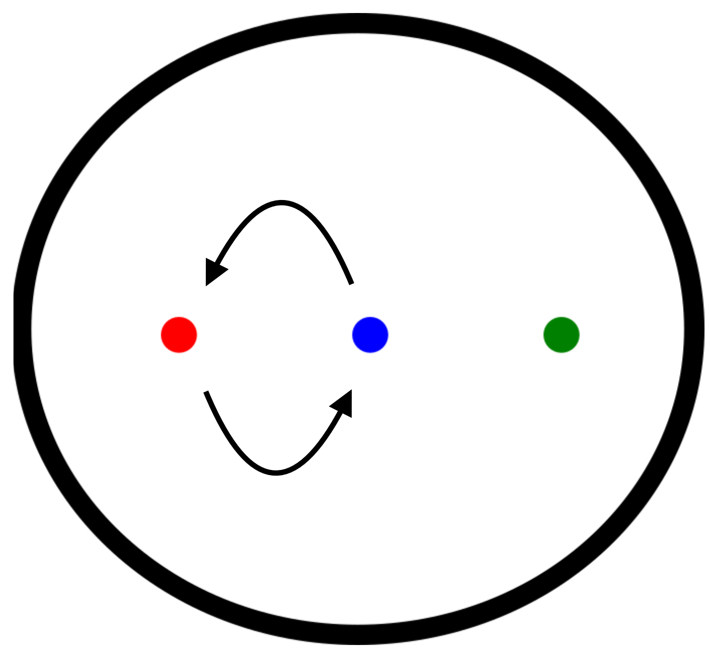


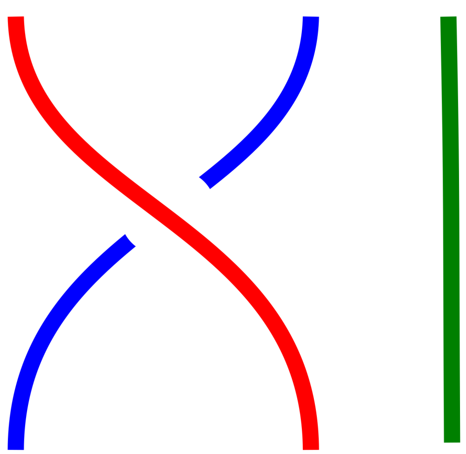
$\leftrightarrow$



a $\qquad$ $[\gamma_a]$

Braid group (3 interpretations)

$$B_n \coloneqq \langle \text{braids} \rangle \simeq \pi_1 \, \mathrm{Conf}_n(D^2) \simeq \pi_0 \, \mathrm{Diff}(D^2, \, n \text{ points})$$



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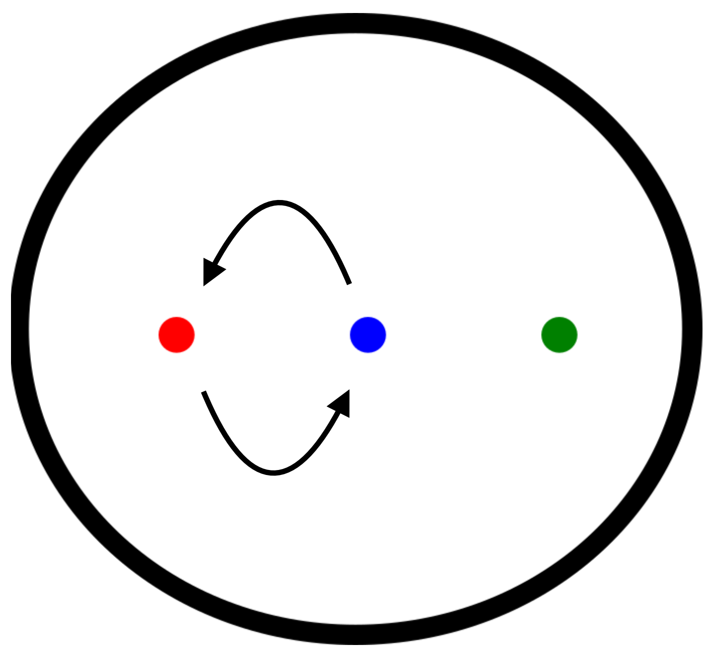

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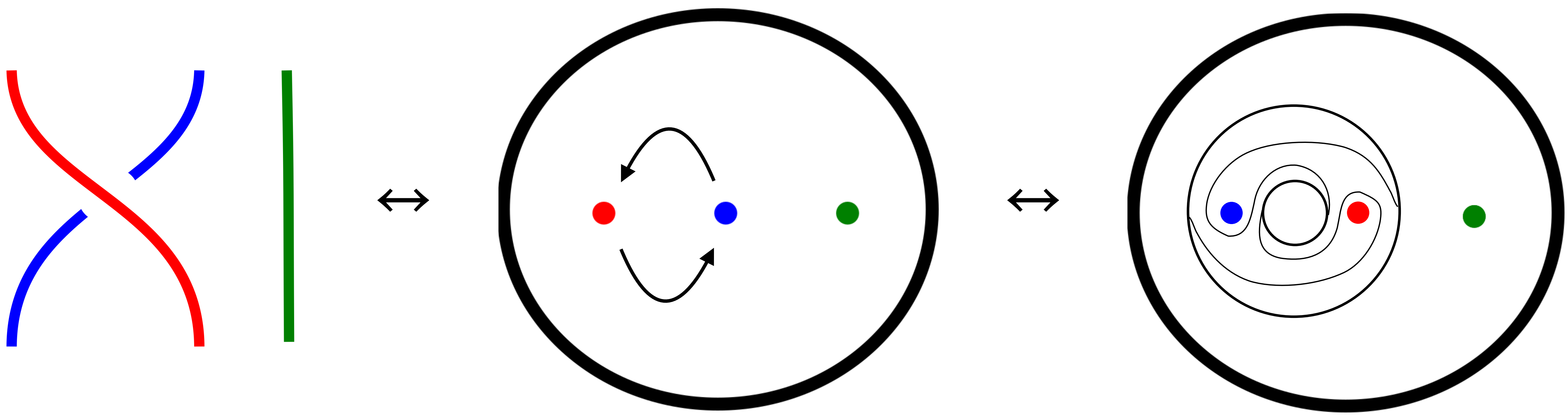

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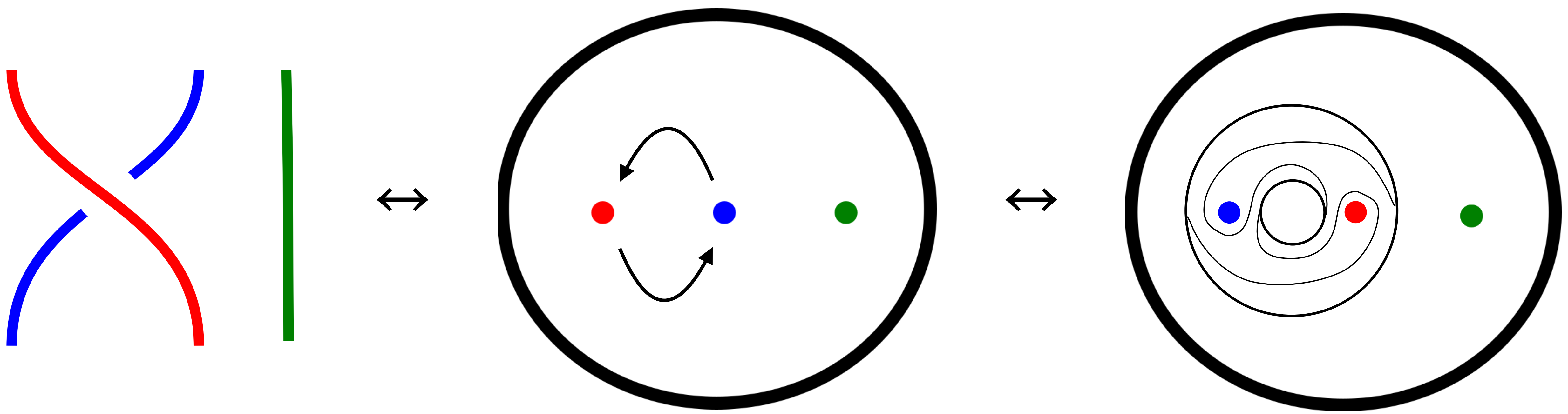

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$$B_3 = \langle \; a, b \mid aba = bab \; \rangle$$

Question

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1. $F \sim \Phi_a$, $G \sim \Phi_b$ (isotopic)

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2. $F \circ G \circ F = G \circ F \circ G$?

Question 2.0

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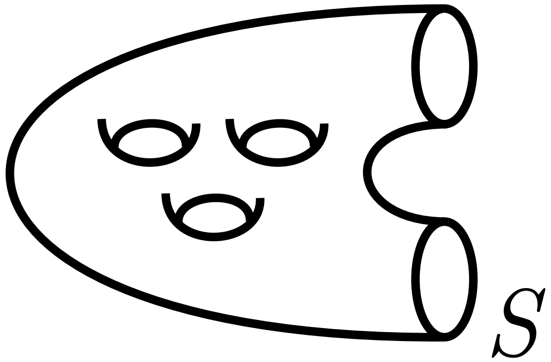

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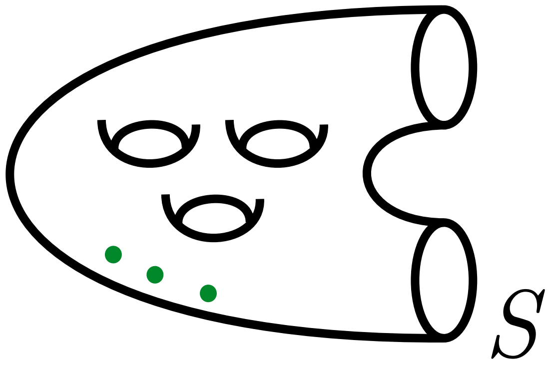

$$\mathrm{Conf}_n(S) = \{(x_1, \ldots, x_n) \mid x_i \in S,\ x_i \neq x_j \text{ for } i \neq j\ \}/S_n$$

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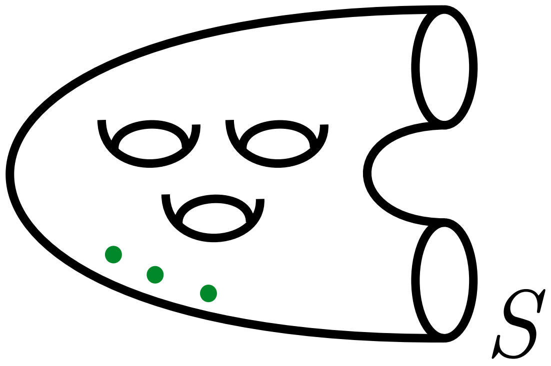

surface braid group: $B_n(S) = \pi_1(\mathrm{Conf}_n(S))$

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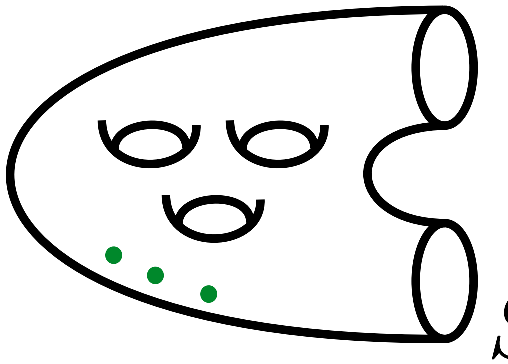

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$\mathrm{Diff}(S, X_n)$

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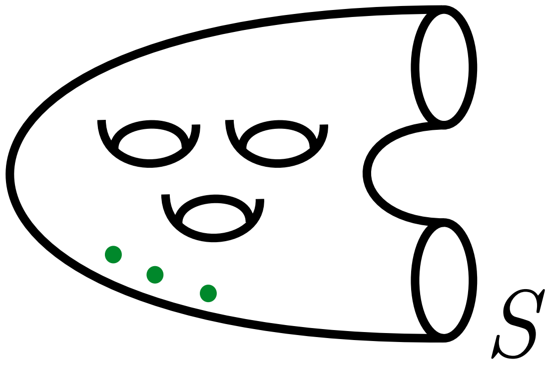

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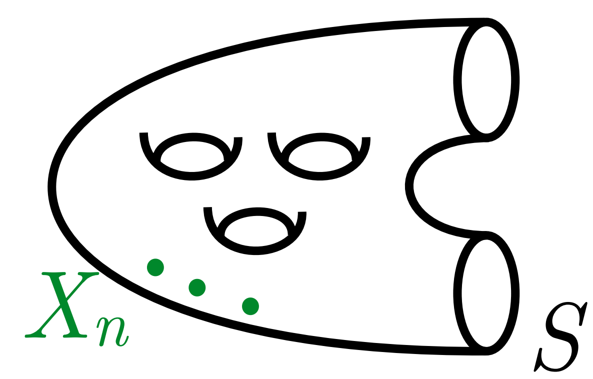

$\mathrm{Diff}(S, X_n)$ C^1 or. pres. diffeos, $g|_{\partial S} = \mathrm{Id}$ and $g(X_n) = X_n$

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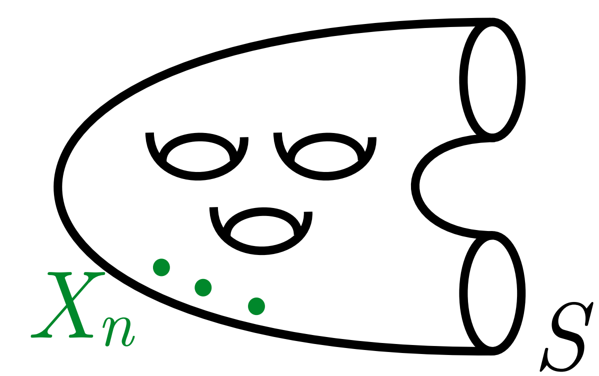

$\mathrm{Mod}(S, X_n) \coloneqq \pi_0 \, \mathrm{Diff}(S, X_n)$

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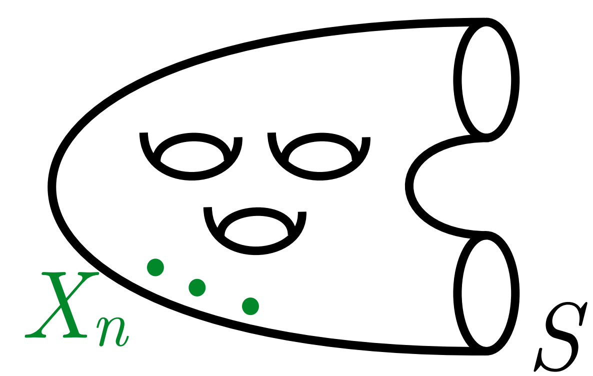

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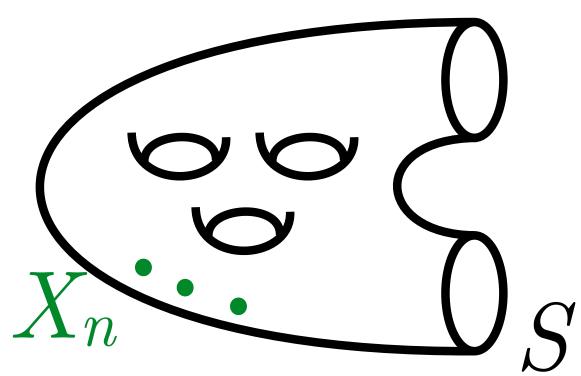

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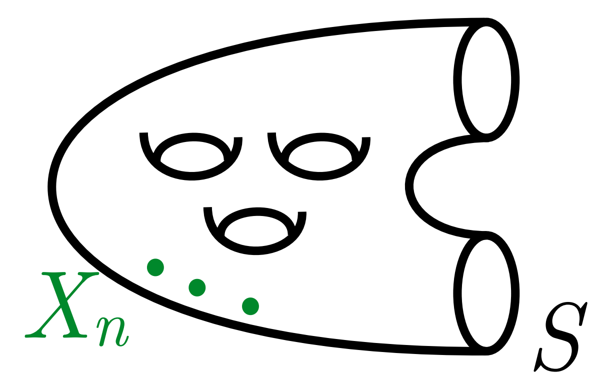
Push = point-pushing homomorphism



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If σ exists, say Push is *realized by diffeomorphisms.*

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- Theorem (Bestvina-Church-Souto, 2009). S closed, genus $\geq 2 \Rightarrow$ Push : $\pi_1(S) \to \mathrm{Mod}(S, *)$ not realized by diffeos.

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[†] G $\neq$ SO(n,1) for $n \geq 3$, but e.g. G $=$ $SL_n(\mathbb{R})$, SU(n,1), $E_{8(-24)}$

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- Theorem (Nariman). $\mathrm{Diff}(D \setminus X_n) \twoheadrightarrow \mathrm{Mod}(D,X_n) \simeq B_n$ splits *cohomologically*.

Main theorem

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Theorem (Salter-T, 2015). S compact, genus ≥ 0.

For $n \geq 6$, Push is not realized by diffeomorphisms.

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Example. $\mathrm{Diff}(D, X_n) \twoheadrightarrow B_n$ does not split for $n \geq 6$.

Corollary (of proof). S closed, genus ≥ 2.

Then $\mathrm{Diff}(S) \twoheadrightarrow \mathrm{Mod}(S)$ does not split.

Aside: Flat surface bundles

Open Question. Does there exist $S_g \to E \to S_h$ not *flat*?

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Equivalently, does there exist ρ that has no lift?

$$
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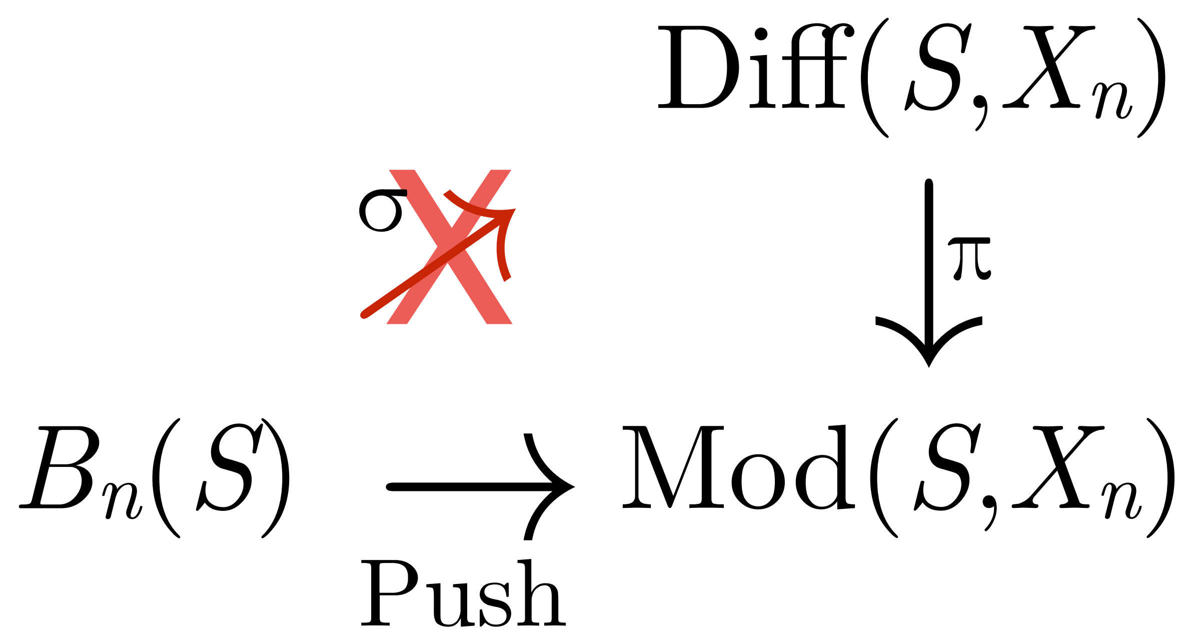
 & & \mathrm{Diff}(S_g) \\
 & \not\nearrow & \big\downarrow \pi \\
\pi_1(S_h) & \xrightarrow{\ \rho\ } & \mathrm{Mod}(S_g)
\end{array}
$$

II. Proof of Main Theorem

Main theorem



<u>Theorem</u> (Salter-T, 2015). S compact, genus ≥ 0.
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Strategy

Show: $B_n(S) \xrightarrow{\sigma} \mathrm{Diff}(S, X_n)$ $\Rightarrow$

exists

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Strategy

Show: $B_n(S) \xrightarrow{\sigma} \mathrm{Diff}(S, X_n)$ exists $\Rightarrow$ $\mathrm{Diff}(S, T_xS)$ contains f.g. perfect subgroup $\Gamma = [\Gamma,\Gamma]$.

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$$\langle \sigma_1 , \ldots , \sigma_{n\text{-}1} \rangle$$

$$||$$

$$B_n \to \mathbb{Z}$$

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$$\langle \sigma_1, \ldots, \sigma_{n-1} \rangle$$

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$$\sigma_i \mapsto 1$$

Proof idea: Perfect subgroup

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$$\begin{array}{c} \langle\, \sigma_1\,,\, \ldots\,,\, \sigma_{n\text{-}1}\,\rangle \\ \| \\ 1 \to [B_n, B_n] \to B_n \to \mathbb{Z} \to 1 \\ \sigma_i \mapsto 1 \end{array}$$

Proof idea: Perfect subgroup

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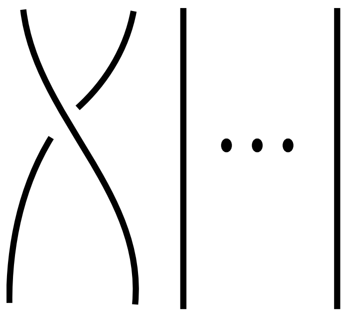
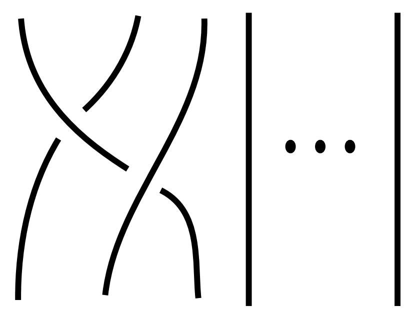
$$||$$

$$1 \to [B_n, B_n] \to B_n \to \mathbb{Z} \to 1$$

$$\sigma_i \mapsto 1$$

e.g. $\sigma_1 =$  not a commutator, but

$\sigma_1 \sigma_2{}^{-1} =$  is a commutator.

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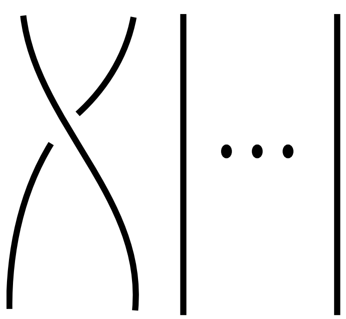
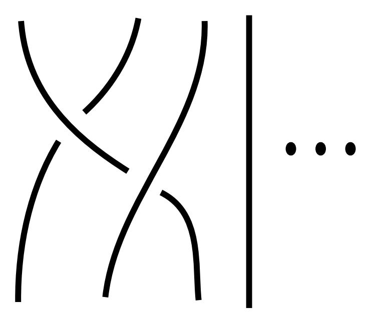
e.g. $\sigma_1 =$  not a commutator, but

$\sigma_1\, \sigma_2{}^{-1} =$  is a commutator.

$$\sigma_1\, \sigma_2{}^{-1} = [\, \sigma_2{}^{-1},\ \sigma_1{}^{-1}\,] = [\, \sigma_4\, \sigma_2{}^{-1},\ \sigma_4\, \sigma_1{}^{-1}\,]$$

Proof idea: Perfect subgroup

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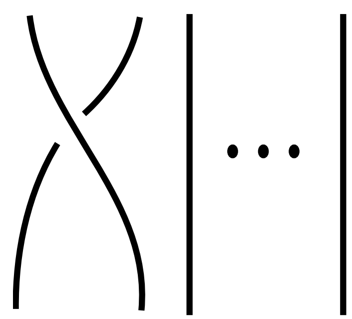
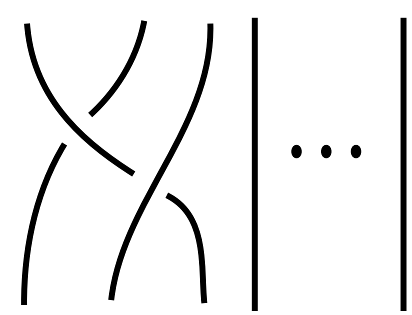
e.g. $\sigma_1 =$  not a commutator, but

$\sigma_1 \sigma_2^{\,\text{-}1} =$  is a commutator.

$$\sigma_1 \sigma_2^{\,\text{-}1} = [\, \sigma_2^{\,\text{-}1}, \sigma_1^{\,\text{-}1} \,] = [\, \sigma_4 \sigma_2^{\,\text{-}1}, \sigma_4 \sigma_1^{\,\text{-}1} \,]$$

Theorem (Gorin-Lin). For $n \geq 5$, $[B_n, B_n]$ is f.g. perfect group.

Thank you.

Corollary (Thurston). $\text{Diff}(D, X_3) \to \text{Mod}(D, X_3) \simeq B_3$ splits.

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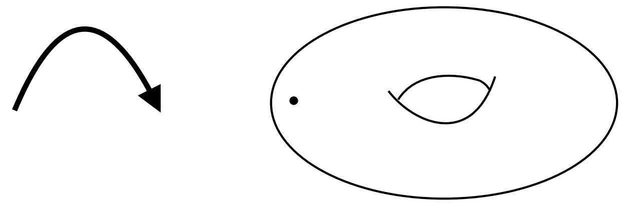
$\mathrm{SL}_2(\mathrm{Z})$



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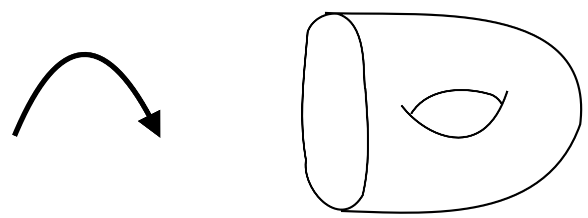
$\mathrm{SL}_2(\mathbb{Z})$ 

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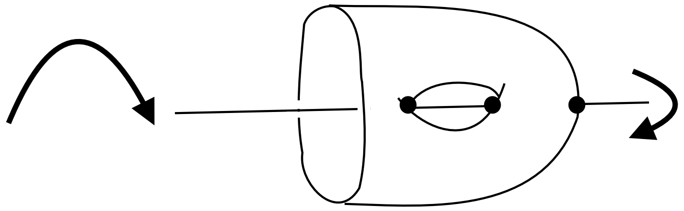
$\mathrm{SL}_2(\mathbb{Z})$ 

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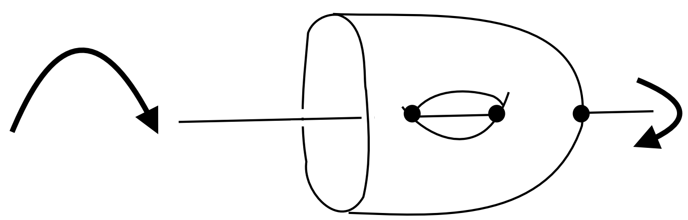
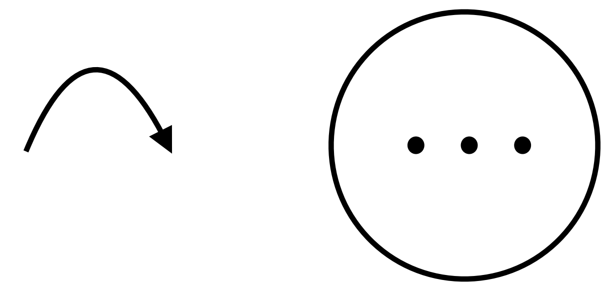


$PSL_2(\mathbb{Z})$



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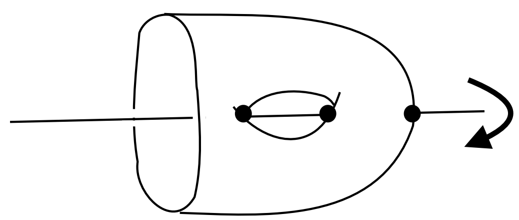
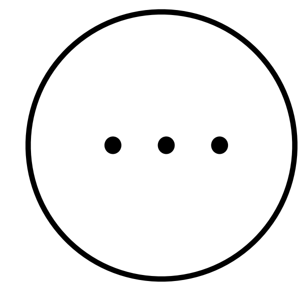
$\mathrm{SL}_2(\mathbb{Z}) \curvearrowright$ 

$\mathrm{PSL}_2(\mathbb{Z}) \curvearrowright$ 

$$1 \to \mathbb{Z} \to B_3 \to \mathrm{PSL}_2(\mathbb{Z}) \to 1$$

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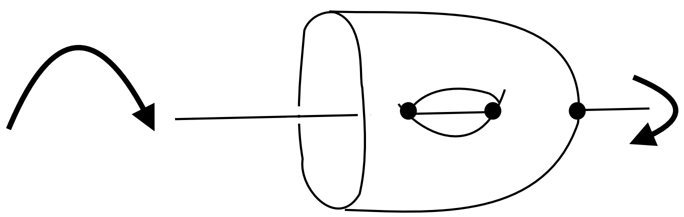
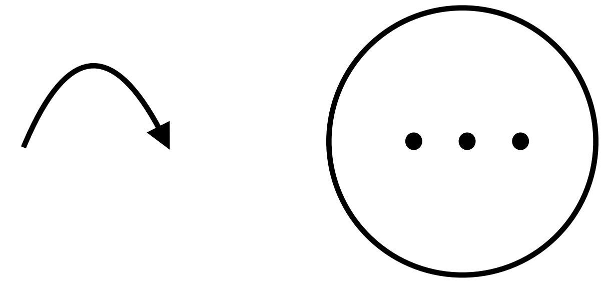
$SL_2(\mathbb{Z})$  $PSL_2(\mathbb{Z})$ 

$$1 \to \mathbb{Z} \to B_3 \to PSL_2(\mathbb{Z}) \to 1$$

$$PSL_2(\mathbb{Z}) = \langle\, x, y \mid x^2 = y^3 = 1 \,\rangle$$

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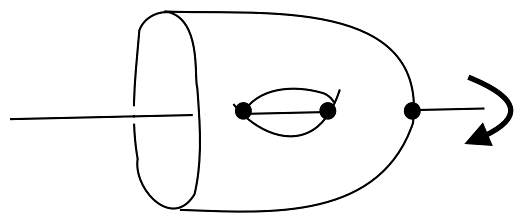
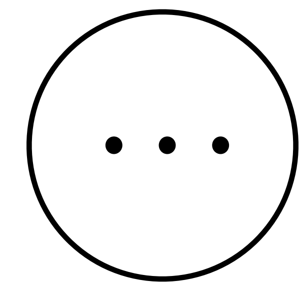
$\mathrm{SL}_2(\mathrm{Z})$ ↷  $\mathrm{PSL}_2(\mathrm{Z})$ ↷ 

$$1 \to \mathrm{Z} \to \mathrm{B}_3 \to \mathrm{PSL}_2(\mathrm{Z}) \to 1$$

$$\mathrm{B}_3 = \langle\, a, b \mid a^2 = b^3 \,\rangle \qquad \mathrm{PSL}_2(\mathrm{Z}) = \langle\, x, y \mid x^2 = y^3 = 1 \,\rangle$$

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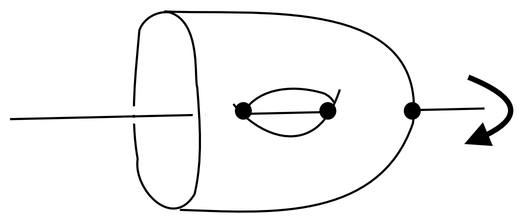
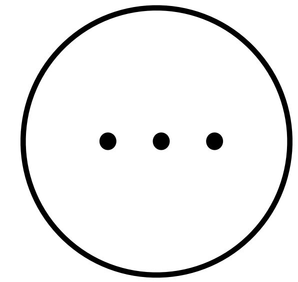
$\mathrm{SL}_2(\mathrm{Z}) \curvearrowright$  $\mathrm{PSL}_2(\mathrm{Z}) \curvearrowright$ 

$$
\begin{array}{ccccccccc}
1 \to \mathrm{Z} \to & \mathrm{B}_3 & \to & \mathrm{PSL}_2(\mathrm{Z}) & \to 1 \\
& \| & & \| & \\
& \langle\, a, b \mid a^2 = b^3 \,\rangle & & \langle\, x, y \mid x^2 = y^3 = 1 \,\rangle &
\end{array}
$$

Homotope $x|_\partial$, $y|_\partial$ to identity preserving relation $x^2 = y^3$.