Arithmeticity of groups $\mathbb{Z}^n \rtimes \mathbb{Z}$

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<u>Arithmeticity question</u> Given $A \in GL_n(\mathbb{Z})$, is $\Gamma_A = \mathbb{Z}^n \rtimes_A \mathbb{Z}$ arithmetic?

ie $\Gamma_A \doteq G(\mathbb{Z})$ where G algebra group $/\mathbb{Q}$

<u>Assume</u> A hyperbolic (no eigenvals on S^1) & irreducible

<u>Ex</u> • $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ $\Gamma_A = \left\{ \begin{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^x & \begin{matrix} z \\ y \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 1 \end{pmatrix} : x, y, z \in \mathbb{Z} \right\} = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \right\} (\mathbb{Z})$

• $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ $\Gamma_A = \left\{ \begin{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^x & \begin{matrix} z \\ y \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 1 \end{pmatrix} \right\} \doteq \left\{ \left(\begin{array}{c|c} B & \begin{matrix} z \\ y \end{matrix} \\ \hline \begin{matrix} 0 & 0 \end{matrix} & 1 \end{array} \right) : BA = AB \right\} (\mathbb{Z})$

$n=2$ Γ_A always arithmetic.

Solvable vs semisimple

$\Gamma_A = \mathbb{Z}^n \rtimes_A \mathbb{Z}$ is lattice in solvable Lie group.

(Margulis) $\Gamma < G(\mathbb{R})$ lattice is semisimple Lie group.

- if $\mathrm{rank}_{\mathbb{R}} G \geq 2$ (eg $G = SL_n$, $n \geq 3$), then Γ arithmetic
- Γ arithmetic $\iff [\mathrm{Comm}_G(\Gamma) : \Gamma] = \infty$

$$\mathrm{Comm}_G(\Gamma) = \{g \in G \mid g\Gamma g^{-1} \cap \Gamma \text{ finite index in } \Gamma \,\&\, g\Gamma g^{-1}\}.$$

(Studenmund) $\Gamma < G(\mathbb{R})$ lattice in solvable

$\Rightarrow [\mathrm{Comm}(\Gamma) : \Gamma] = \infty$.

Semisimple example

$A \in SL_2(\mathbb{Z}) = \mathrm{Out}(F_2)$ hyperbolic, $F_2 \rtimes_A \mathbb{Z} \hookrightarrow PSL_2(\mathbb{C})$

Thm (Bowditch-Maclachlan-Reid)

$F_2 \rtimes_A \mathbb{Z}$ arithmetic $\iff A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}$

up to powers, conj.

Basic Q: How rare is arithmeticity of $\mathbb{Z}^n \rtimes_A \mathbb{Z}$?

$\{n : \exists\, A \in GL_n\mathbb{Z} \text{ st. } \Gamma_A \text{ arith}\} \subset \{n \geq 2\}$

finite or infinite?

Complement: finite or infinite?

Number theory examples

$K/\mathbb{Q}$, $[K:\mathbb{Q}]=n$, $\mathcal{O} \subset K$ ring of integers

$\lambda \in \mathcal{O}^\times$, $A_\lambda =$ matrix of $\lambda \curvearrowright \mathcal{O} \cong \mathbb{Z}^n$

Ex $K = \mathbb{Q}[t]/(t^3+t^2-2t-1)$

$\mathcal{O} = \mathbb{Z}\{1, t, t^2\}$ $\quad \mathcal{O}^\times \cong \mathbb{Z}/2\mathbb{Z} \times \underbrace{\mathbb{Z}^2}_{\langle \varepsilon_1, \varepsilon_2 \rangle}$

$\varepsilon_1 = t^2+t-1$ acts on $\mathcal{O}$ by $\begin{pmatrix} -1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$

$\varepsilon_2 = -t^2+2$ ——— " ——— $\begin{pmatrix} 2 & -1 & 1 \\ 0 & 0 & 1 \\ -1 & 1 & -1 \end{pmatrix}$

For $\lambda \in \langle \varepsilon_1, \varepsilon_2 \rangle$ is Γ_{A_λ} arithmetic? Never arithmetic.

Number theory examples

$K/\mathbb{Q}$, $[K:\mathbb{Q}]=n$, $\mathcal{O}\subset K$ ring of integers

$\lambda\in\mathcal{O}^\times$, $A_\lambda =$ matrix of $\lambda \curvearrowright \mathcal{O}\cong\mathbb{Z}^n$

<u>Ex</u> $K=\mathbb{Q}[t]/(t^3-t^2+1)$

$\langle\varepsilon\rangle$

$\mathcal{O}=\mathbb{Z}\{1,t,t^2\}$ $\qquad$ $\mathcal{O}^\times\cong\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}$

$\varepsilon=-t^2+t$ acts on $\mathcal{O}$ by $\begin{pmatrix}0&1&0\\1&0&1\\-1&0&0\end{pmatrix}$

Γ_ε arithmetic.

Ex. $K = \mathbb{Q}(\sqrt{2}, \sqrt{3})$ $\quad \mathcal{O} = \mathbb{Z}\left\{1, \sqrt{2}, \sqrt{3}, \frac{\sqrt{2}+\sqrt{6}}{2}\right\}$

$$\mathcal{O}^\times \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}^3$$

$\varepsilon_1 = 1+\sqrt{2}$ $\quad \varepsilon_2 = \frac{\sqrt{2}+\sqrt{6}}{2}$ $\quad \varepsilon_3 = \sqrt{2}+\sqrt{3}$

act by
$$\begin{pmatrix} 1 & 2 & 0 & 1 \\ 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 2 & 3 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 2 & 2 & 1 \end{pmatrix}$$

$\lambda \in \langle \varepsilon_1, \varepsilon_2, \varepsilon_3 \rangle$

$\Gamma_\lambda = \mathcal{O} \rtimes \langle \lambda \rangle$ arithmetic $\Longleftrightarrow$ $\lambda \in \langle \varepsilon_i \rangle$ $\quad i = 1, 2, 3$

Rmk. $A \in GL_n(\mathbb{Z})$ hyp. irred, λ eigenval, $K = \mathbb{Q}(\lambda)$

$$\Gamma_A = \mathbb{Z}^n \rtimes_A \mathbb{Z} \iff \Gamma_\lambda = \mathcal{O} \rtimes \langle \lambda \rangle = \mathbb{Z}^n \rtimes_{A_\lambda} \mathbb{Z}$$

arith   arith.

Thm (T)

- $\forall N \; \exists n \geq N$ and $A \in GL_n(\mathbb{Z})$ hyperbolic, (fully) irreducible st Γ_A arithmetic
- if $n \geq 5$ prime, $\forall A \in GL_n(\mathbb{Z})$ hyp., irred Γ_A non-arithmetic.

$\Rightarrow \{n \mid \exists \mathbb{Z}^n \rtimes \mathbb{Z} \text{ arith}\}$ ∞ w/ ∞ complement.

Arithmeticity criterion $K/\mathbb{Q}$, $\lambda \in \mathcal{O}^\times$

$$\Gamma_\lambda = \mathcal{O} \rtimes \langle \lambda \rangle \subset \mathcal{O} \rtimes \mathcal{O}^\times = G(\mathbb{Z})$$

where $G = R_{K/\mathbb{Q}}(\mathbb{G}_a) \rtimes R_{K/\mathbb{Q}}(\mathbb{G}_m)$

$\mathbb{G}_a = \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\}$, $\mathbb{G}_m = GL_1$, $R_{K/\mathbb{Q}}(-)$ restriction of scalars

$S :=$ Zariski closure of $\langle \lambda \rangle \subset R_{K/\mathbb{Q}}(\mathbb{G}_m)$

Thm (Grunewald – Platonov)

Γ_λ arithmetic $\iff \langle \lambda \rangle \doteq S(\mathbb{Z}) \iff \operatorname{rank} S(\mathbb{Z}) = 1.$

Algebraic tori : T linear alg gp $/\mathbb{Q}$, diagonalizable over $\mathbb{C}$

Splitting field of T : smallest $P \subset \mathbb{C}$

s.t. T diagonalizable $/P$. $P/\mathbb{Q}$ Galois

Character group : $X(T) = \mathrm{Hom}(T, \mathbb{G}_m) \cong \mathbb{Z}^d \circlearrowleft \mathrm{Gal}(P/\mathbb{Q})$

Fact $\exists$ equivalence of categories

$$\left\{ \begin{array}{c} P\text{-split alg.} \\ \text{tori } / \mathbb{Q} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{f.g free abelian} \\ \mathbb{Z}[\mathrm{Gal}(P/\mathbb{Q})]\text{-mod} \end{array} \right\}$$

$$T \longmapsto X(T).$$

__Example__. $T = R_{K/\mathbb{Q}}(\mathbb{G}_m)$

e.g.

$K = \mathbb{Q}(\sqrt{2}) \curvearrowright \mathbb{Q}(\sqrt{2}) = \mathbb{Q}\{1, \sqrt{2}\} \cong \mathbb{Q}^2$

$K \longrightarrow M_2(\mathbb{Q}) \qquad a+b\sqrt{2} \longmapsto \begin{pmatrix} a & 2b \\ b & a \end{pmatrix}$

$\underline{R^1_{K/\mathbb{Q}}(\mathbb{G}_m)} = \ker\left[R_{K/\mathbb{Q}}(\mathbb{G}_m) = \left\{ \begin{pmatrix} a & 2b \\ b & a \end{pmatrix} \right\} \xrightarrow{\text{norm}} \mathbb{G}_m \right]$.

splitting field = Galois closure P of $K/\mathbb{Q}$

character group $X(T) \cong \mathbb{Z}[G/H]$

$\circlearrowleft G = \mathrm{Gal}(P/\mathbb{Q}) \quad H = \mathrm{Gal}(P/K)$.

Computing rank $T(\mathbb{Z})$:

<u>Thm</u> T any torus$/\mathbb{Q}$, splitting field P

- $\operatorname{rank} T(\mathbb{Z}) = \operatorname{rank}_{\mathbb{R}}(T) - \operatorname{rank}_{\mathbb{Q}}(T)$.

- $\operatorname{rank}_{\mathbb{Q}}(T) = \operatorname{rank} X(T)^{\mathrm{Gal}(P/\mathbb{Q})}$

  $\operatorname{rank}_{\mathbb{R}}(T) = \operatorname{rank} X(T)^{\tau}$ $\quad \tau \in \mathrm{Gal}(P/\mathbb{Q})$ complex conj.

<u>Ex.</u> $T = R_{K/\mathbb{Q}}(\mathbb{G}_m)$ $\quad \operatorname{rank}_{\mathbb{Q}} = 1$, $\operatorname{rank}_{\mathbb{R}} = r+s$

$\operatorname{rank}\left(T(\mathbb{Z}) = \mathcal{O}_K^{\times}\right) = r+s-1$ (Dirichlet)

Thm $n \geq 5$ prime $\implies \Gamma_A$ non arith $\forall A \in GL_n(\mathbb{Z})$ hyp. irred.

Proof sketch

- Reduction $K/\mathbb{Q}$, $[K:\mathbb{Q}] = n$, $T = R_{K/\mathbb{Q}}(\mathbb{G}_m)$

Suffices to show $\nexists$ $S < T$ w/ $\operatorname{rank} S(\mathbb{Z}) = 1$.

- P Galois closure, $G = \mathrm{Gal}(P/\mathbb{Q})$, $H = \mathrm{Gal}(P/K)$

Case P tot. imaginary

show $X(T) \otimes \mathbb{Q} = \mathbb{Q}[G/H] \cong \mathbb{Q} \oplus V$ w/ V irred

$\implies T \sim \mathbb{G}_m \times R'_{K/\mathbb{Q}}(\mathbb{G}_m) \leftarrow S$

$\operatorname{rank} S(\mathbb{Z}) = \operatorname{rank} T(\mathbb{Z}) = r + s - 1 \geq 2$ $\quad (n = r + 2s \geq 5)$

• Reduction $K/\mathbb{Q}$, $[K:\mathbb{Q}]=n$, $T = R_{K/\mathbb{Q}}(\mathbb{G}_m)$

Suffices to show $\nexists$ $S \subset T$ w/ rank $S(\mathbb{Z}) = 1$.

• P Galois closure, $G = \mathrm{Gal}(P/\mathbb{Q})$, $H = \mathrm{Gal}(P/K)$

<u>Case</u> P tot. imaginary

show $X(T) \otimes \mathbb{Q} = \mathbb{Q}[G/H] \cong \mathbb{Q} \oplus V$ w/ V irred

$G \subset S_n$ transitive (n prime)

• (Burnside) G solvable $\Rightarrow$ $G \curvearrowright G/H$ 2-transitive

• (Galois) G not solvable $\Rightarrow$ $G \subset \mathbb{Z}/n\mathbb{Z} \rtimes (\mathbb{Z}/n\mathbb{Z})^\times$

<u>Ex</u> $K_1 = \mathbb{Q}[t]/(t^3+t^2-2t-1)$

$K_2 = \mathbb{Q}[t]/(t^3-t^2+1)$ $\qquad T_i = R_{K_i/\mathbb{Q}}(\mathbb{G}_m)$

$K_1/\mathbb{Q}$ Galois $\quad X(T_1)\otimes\mathbb{Q} \cong \mathbb{Q}[\mathbb{Z}/3\mathbb{Z}] \cong \mathbb{Q}\oplus V_1$

Galois closure P of $K_2/\mathbb{Q}$ $\quad$ deg 6, $\quad \mathrm{Gal}(P/\mathbb{Q}) = S_3$

$$X(T_2)\otimes\mathbb{Q} \cong \mathbb{Q}[S_3/\mathbb{Z}/2\mathbb{Z}] \cong \mathbb{Q}\oplus V_2$$

rank $T_1(\mathbb{Z}) = 3+0-1 = 2$ $\quad$ (K totally real)

rank $T_2(\mathbb{Z}) = 1+1-1 = 1$