Multiple fiberings of surface bundles over surfaces

Bena Tshishiku

Purdue AMS Sectional Meeting

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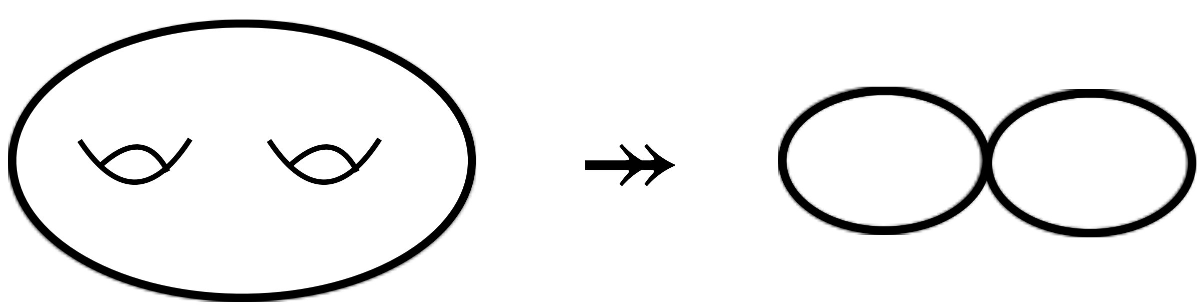

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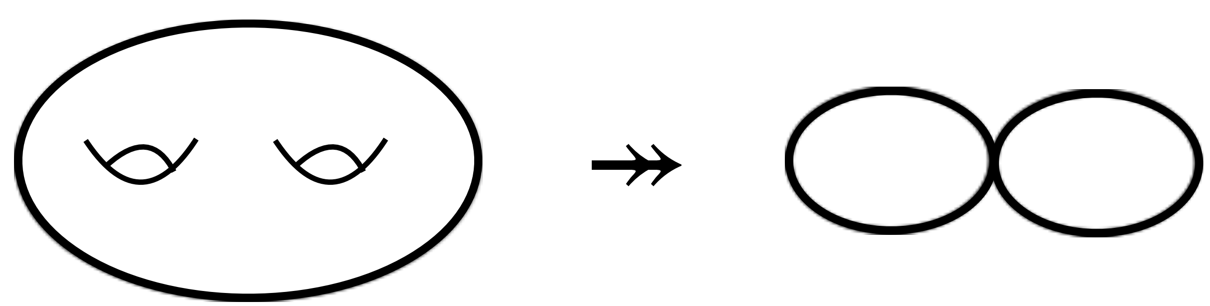

Main Question. How many ways can E fiber as a surface bundle over a surface?

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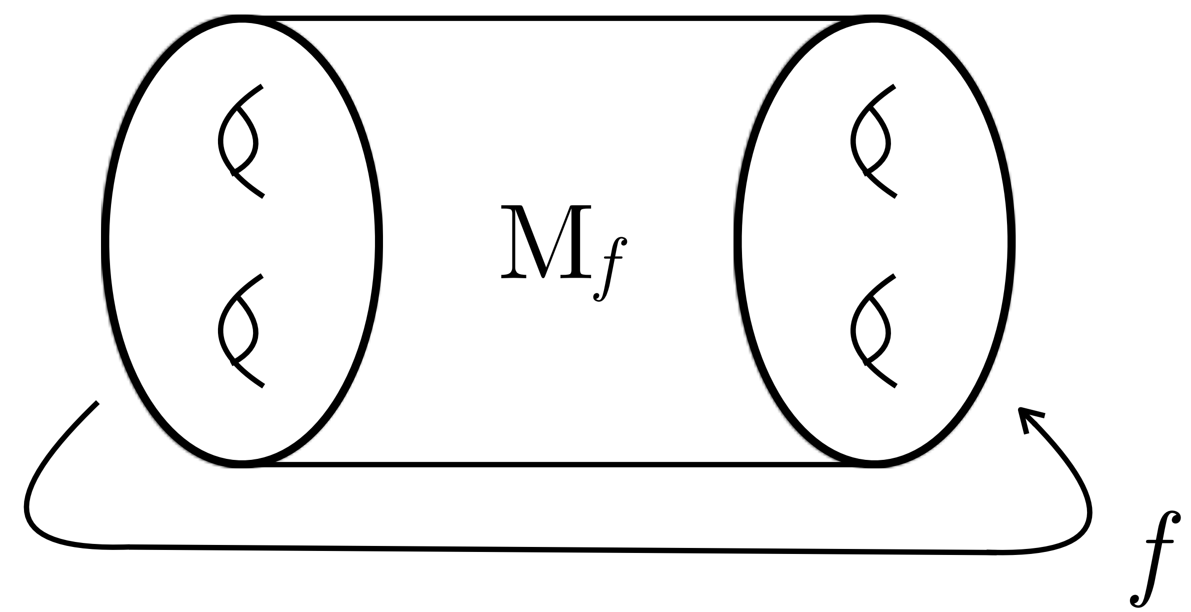

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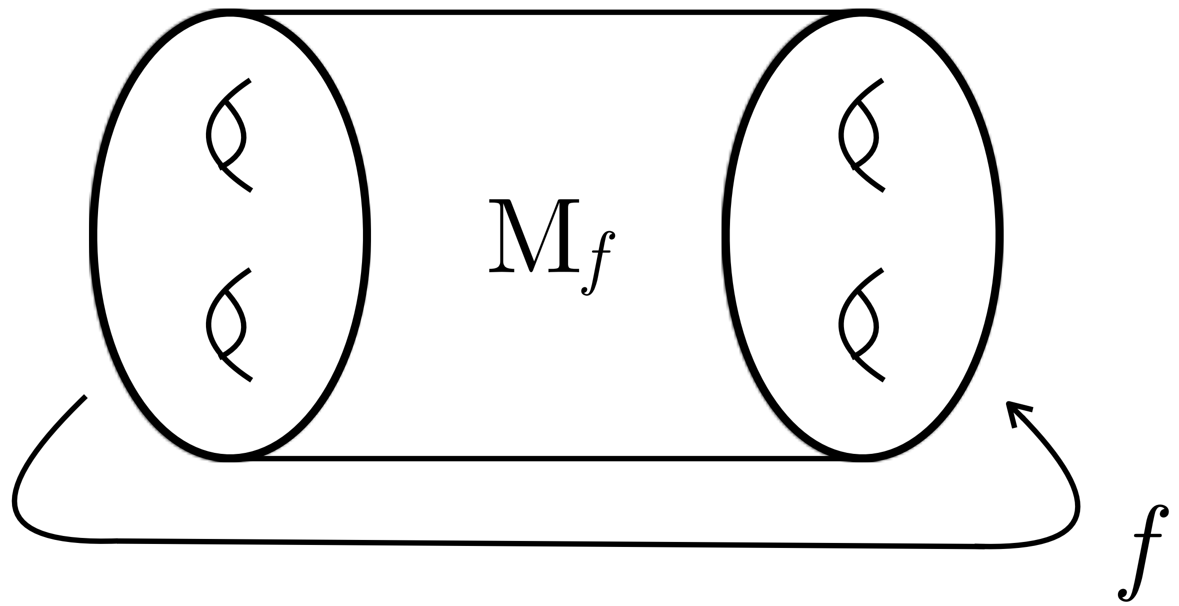

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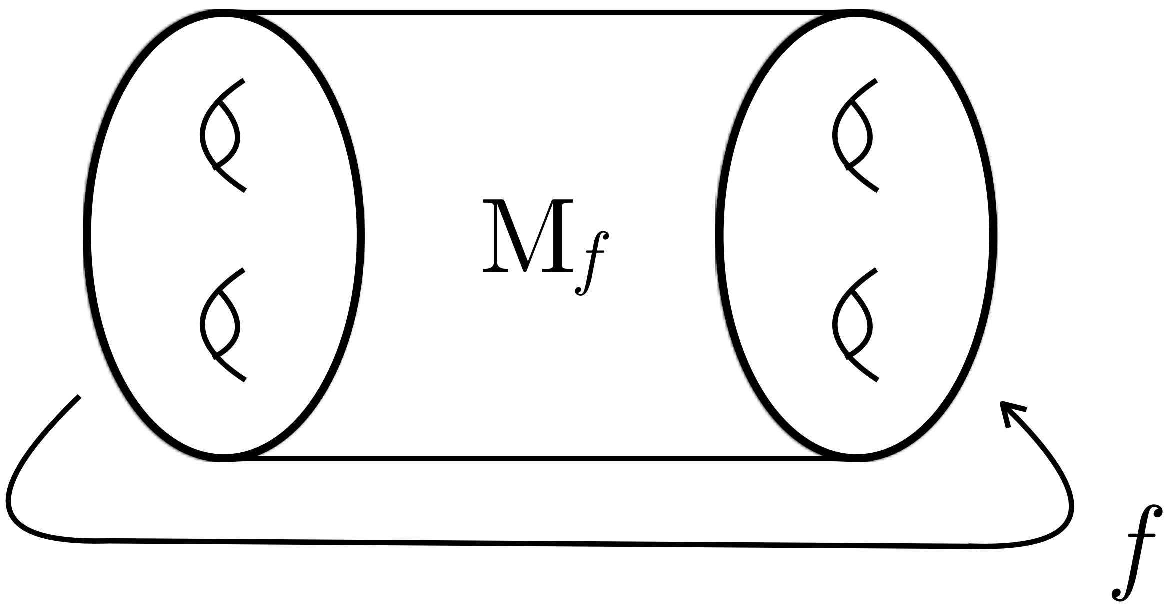

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These fiberings are organized by the Thurston norm.

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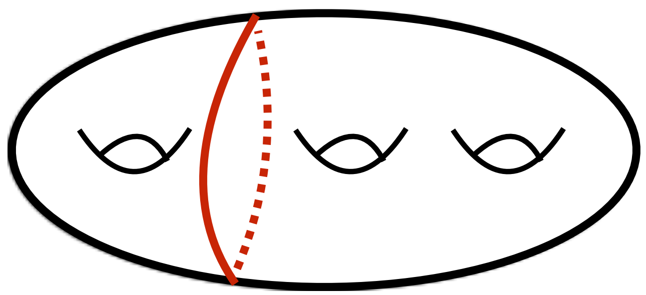

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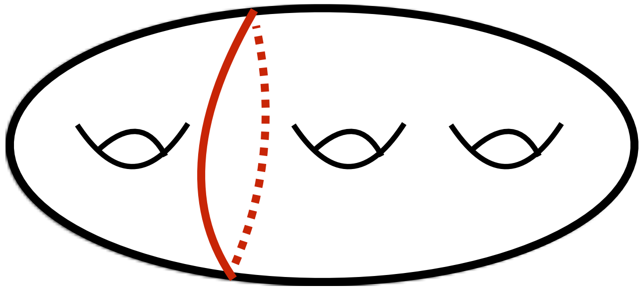

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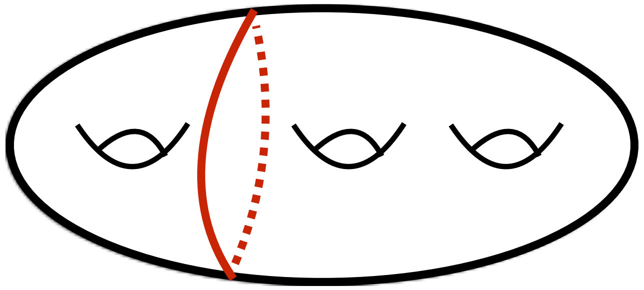

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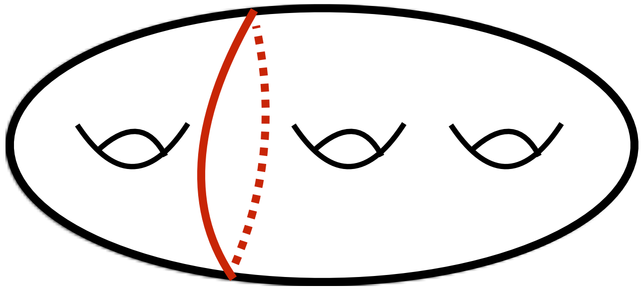

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Question. $\exists$ E^4 that fibers in exactly 3 ways?

Atiyah-Kodaira bundle

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branched covers of products of surfaces

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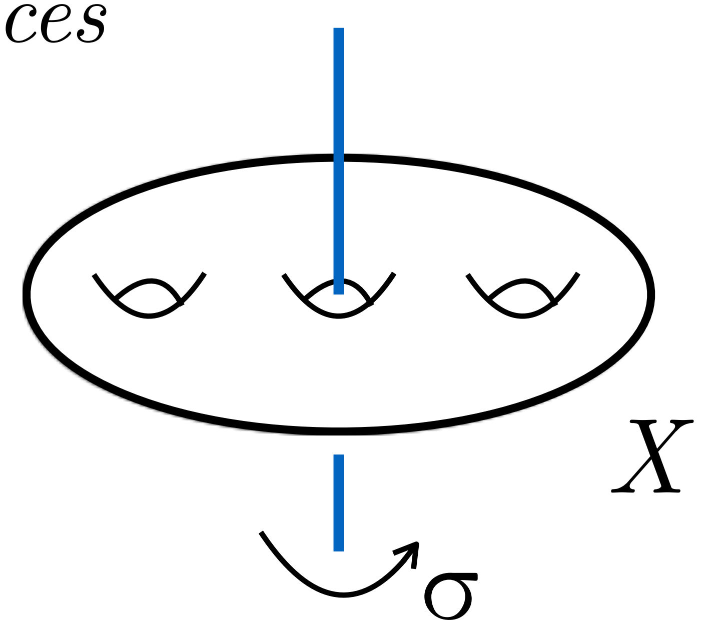

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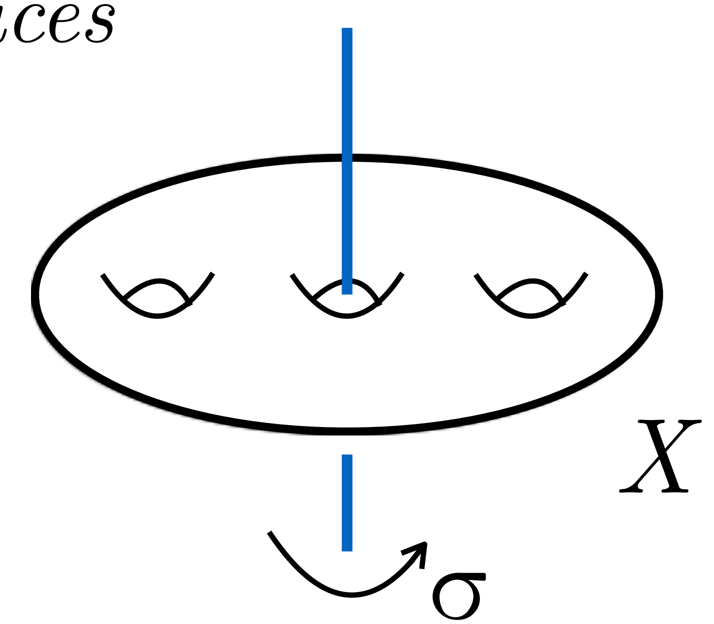

Idea. Take m-fold cover $E_{k,m} \to X \times X$

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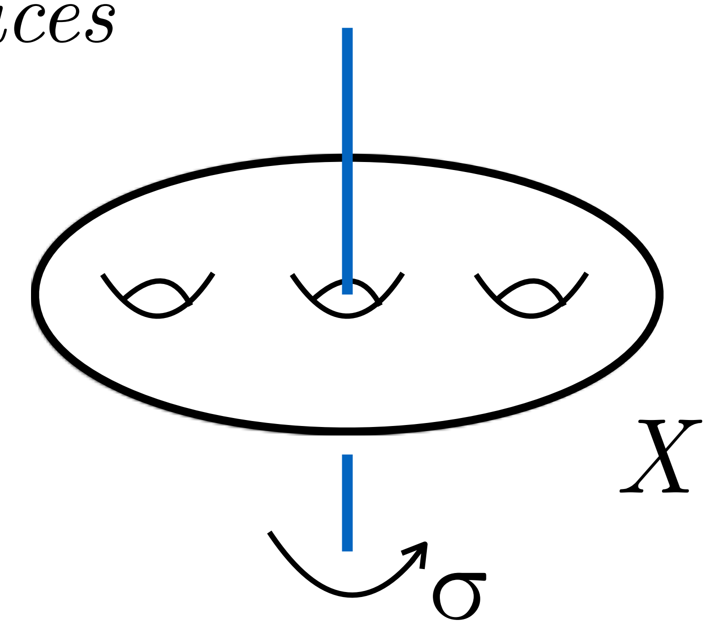

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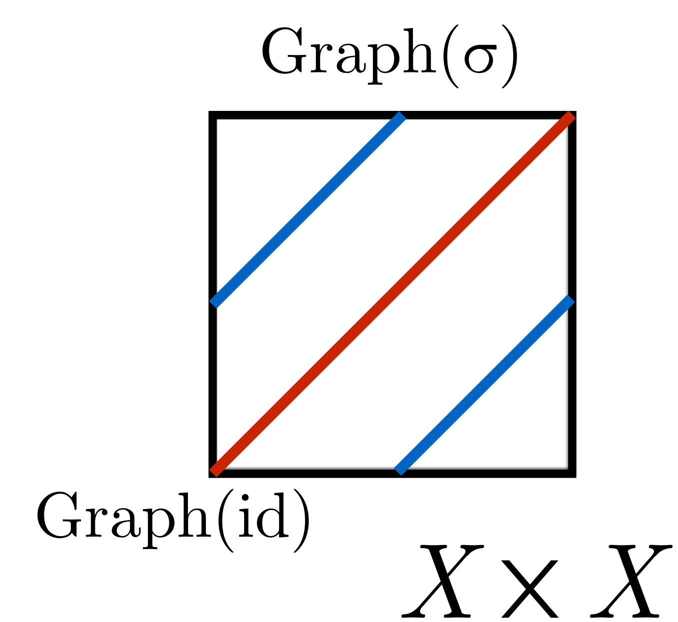

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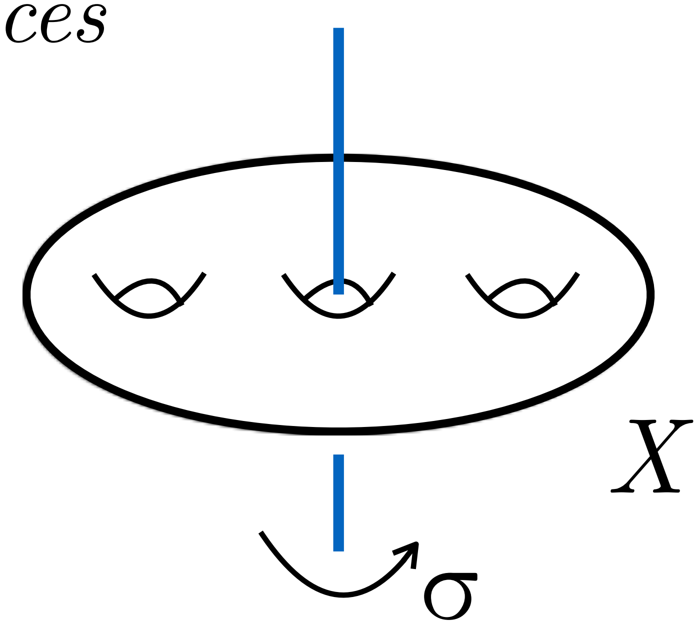

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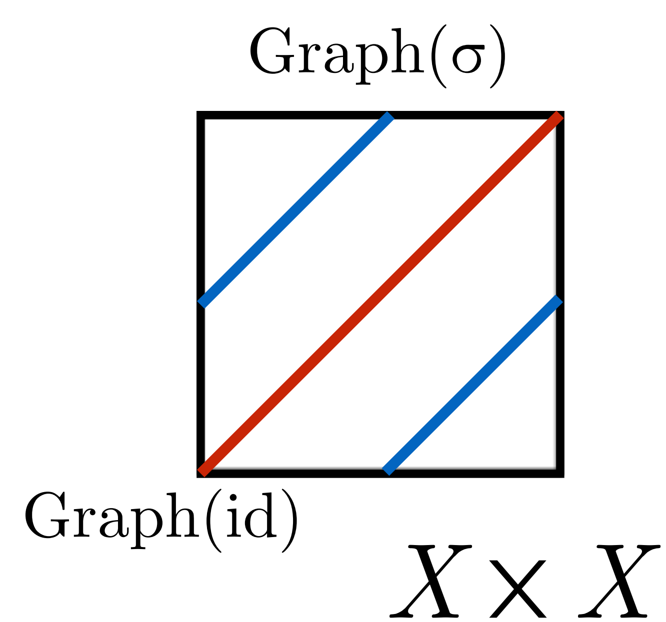

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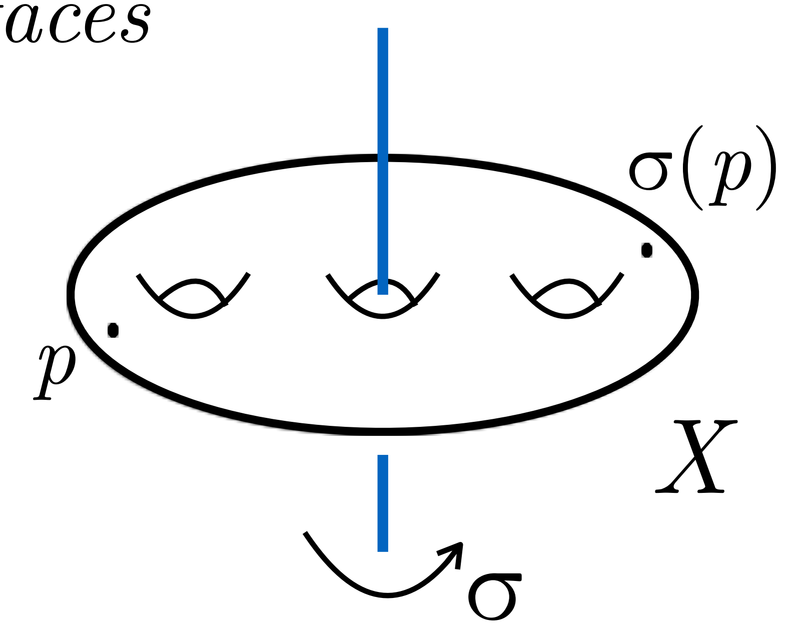

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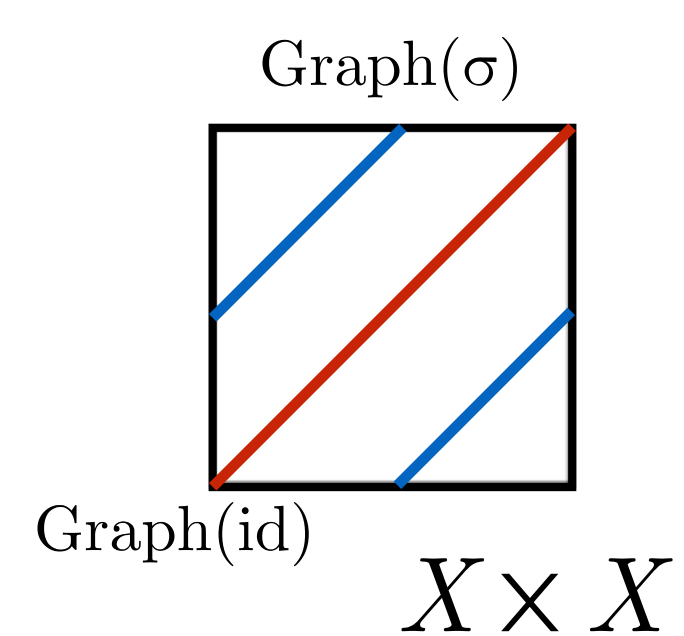

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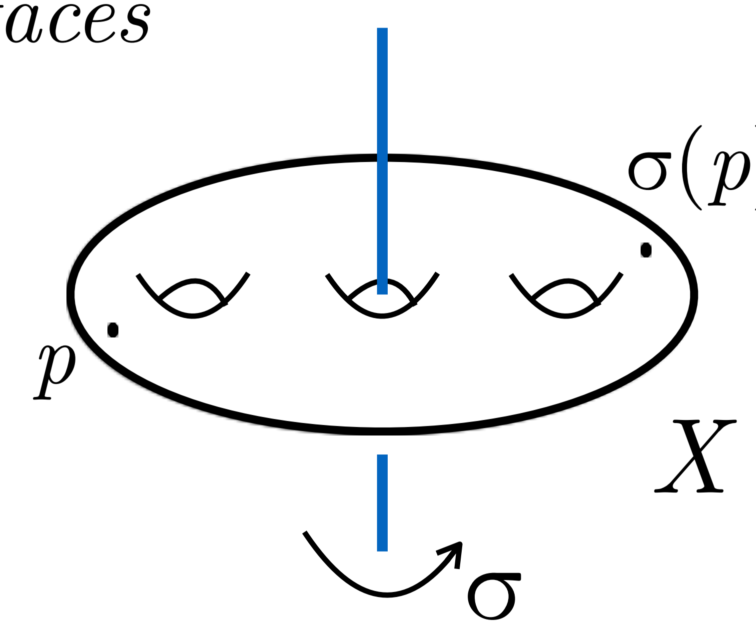

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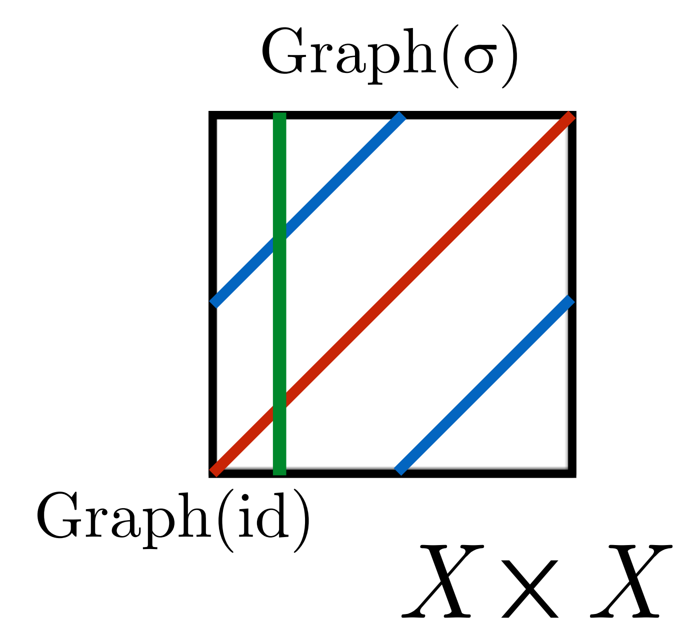

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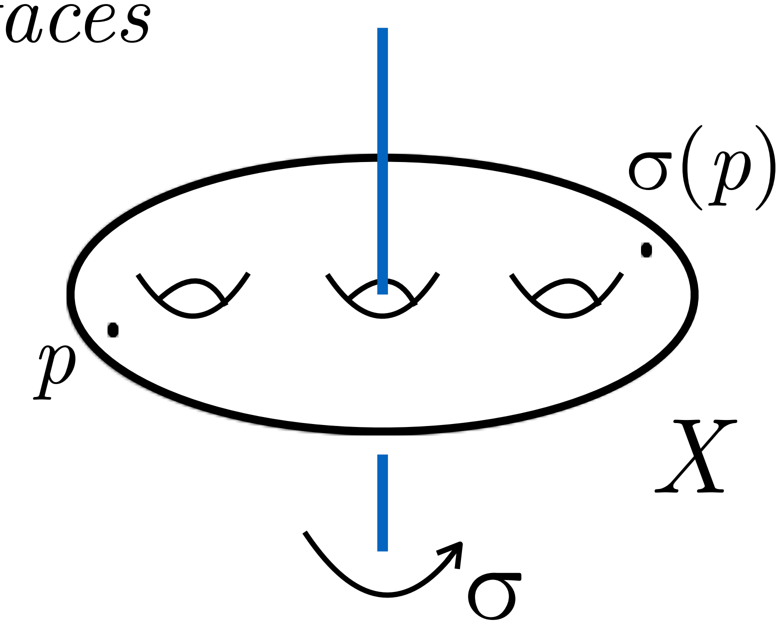

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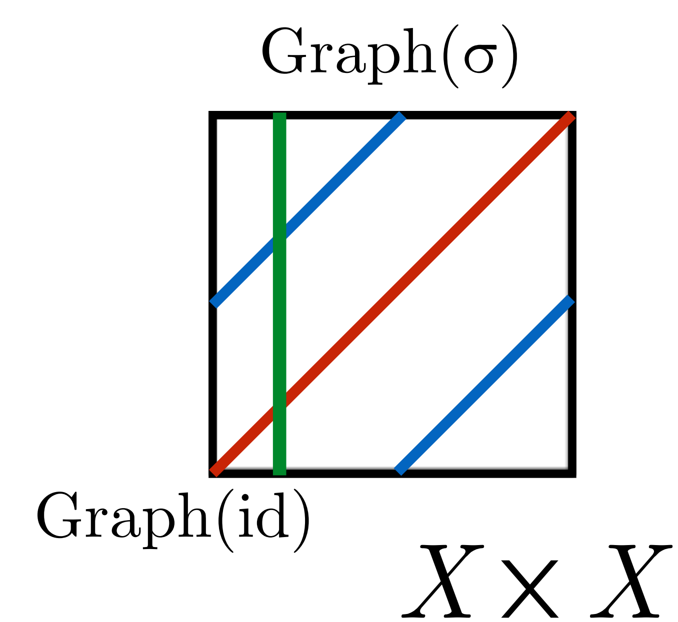

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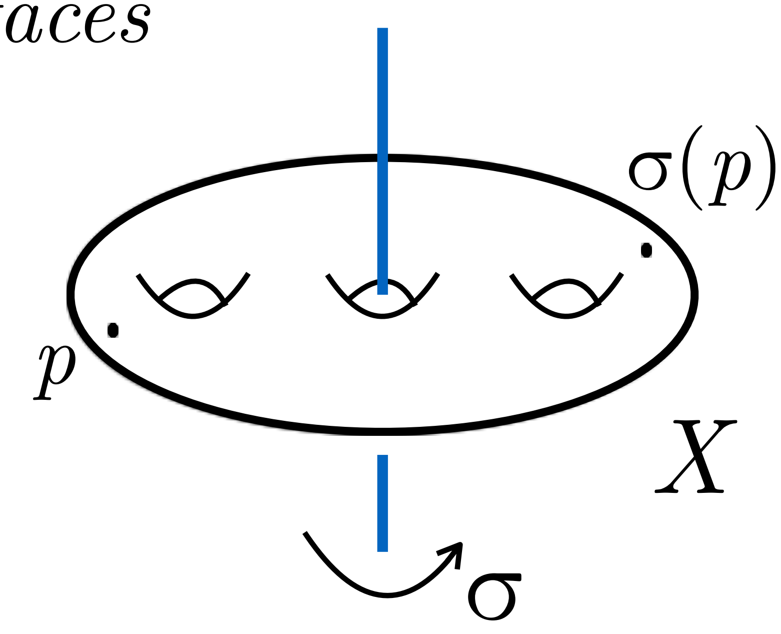

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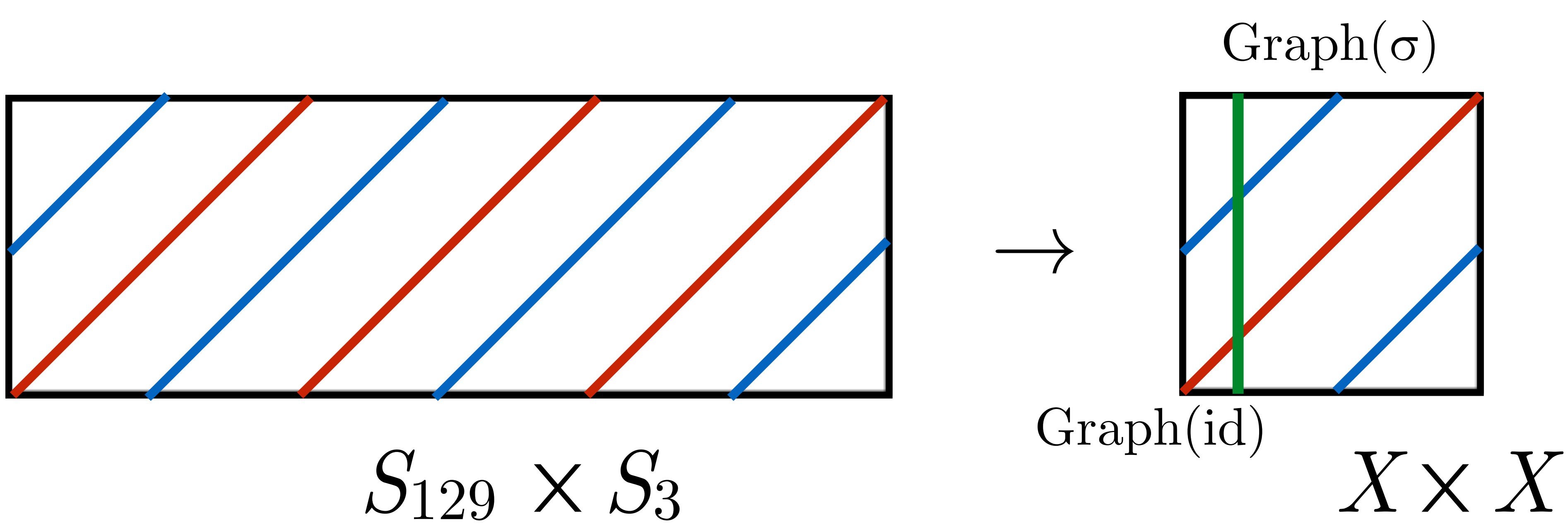

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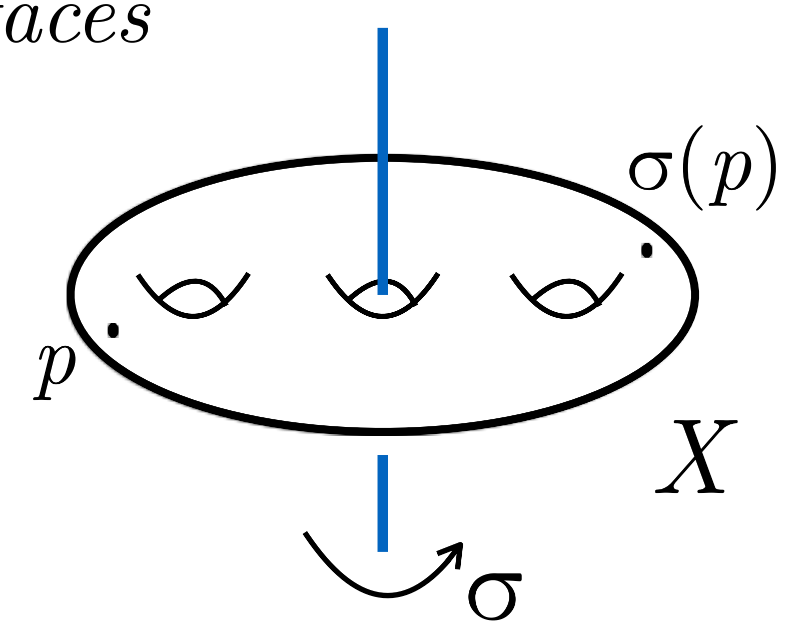

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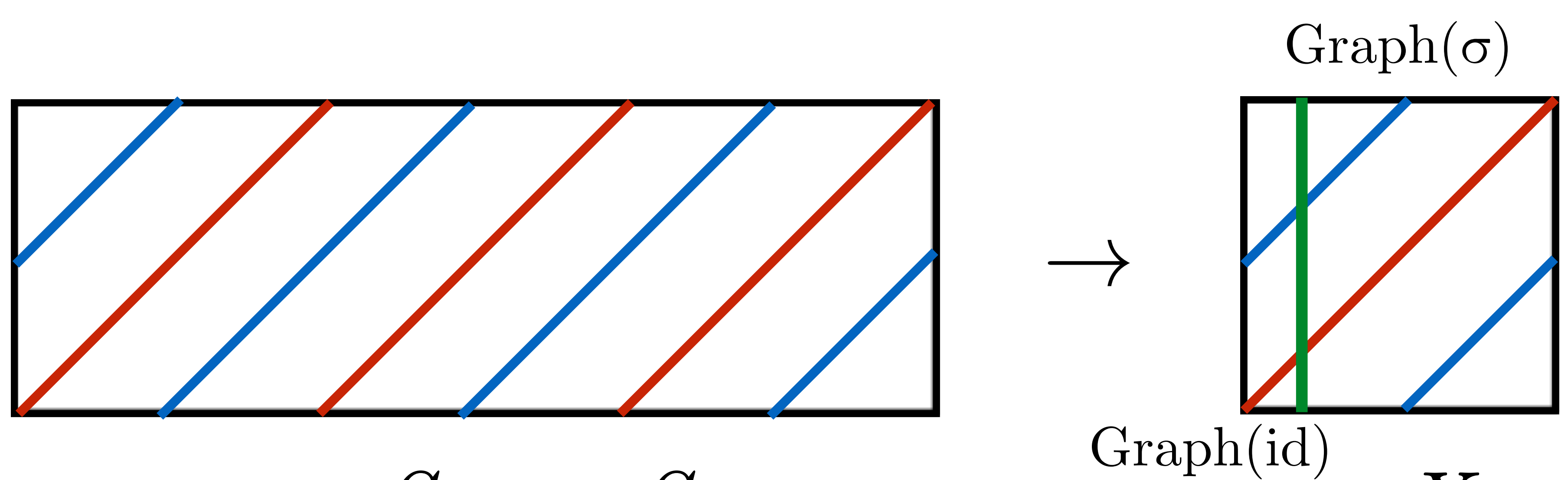

$S_{129} \times S_3$ $\qquad X \times X$

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<u>Question.</u> Does $E_{k,m}$ have > 2 fiberings?

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$\mathbb{Z}/7\mathbb{Z} \curvearrowright H^1(F;\mathbb{Q}) \cong \mathbb{Q}^{2k} \times \mathbb{Q}(\zeta_7)^{2k+5}$

Thank you