Surface bundles, monodromy, & arithmetic groups

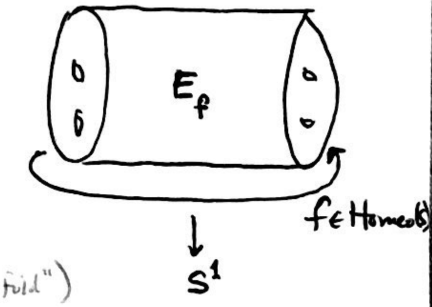
jt/w N. Salter

I. Atiyah-Kodaira bundles

Surface bundle $S \longrightarrow E \longrightarrow B$

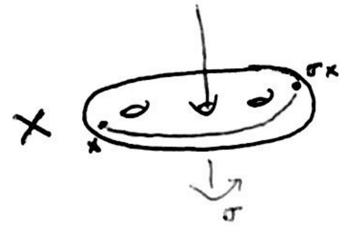
- S: closed surface genus g
- $E \longrightarrow B$: fiber bundle str. gp. Homeo(S)
- B: manifold

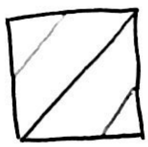
e.g.



<u>Main Ex</u> (AK) $S_g \to E \longrightarrow B^2$ (Andy: "some funny 4-manifold")

<u>idea of construction</u>





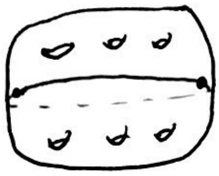
$\Gamma_{id} = \{(x,x)\}$

$\Gamma_\sigma = \{(x,\sigma x)\}$.

$X \times X$

take 2-fold cover $E \longrightarrow X \times X$ branched over $\Gamma_{id} \cup \Gamma_\sigma$.

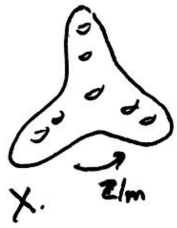
<u>output</u> $S_6 \longrightarrow E \longrightarrow S_{129}$



$\pi_1(S_{129}) = \cancel{\text{Ker}}\,[\pi_1(S_3) \longrightarrow H_1(S_3;\mathbb{Z}/2)]$

construction can be done w/ any X with free $\mathbb{Z}/m$ action.

$S_g \xrightarrow[\text{branched.}]{\mathbb{Z}/m}$



$\longrightarrow$

$\rightsquigarrow S_g \longrightarrow E_{h,m} \longrightarrow B^2$

<u>Interesting properties</u>

- holomorphic construction (use later)
- sig(E) $\neq 0$. (Atiyah) (in context of Andy $\Rightarrow k_1 \neq 0$)
- E does not admit Riem. metric w/ $K \leq 0$ (Ballmann-Gromov-Schroeder 1985, Stadler 2015)
- (•) Conjecturally not flat $\nexists$ (Bestvina-Church-Souto)

$$\begin{tikzcd} & \mathrm{Diff}(S)^{\mathbb{Z}/m} \arrow[d] \\ \pi_1(B) \arrow[r] \arrow[ur, dashed, "\nexists"] & \mathrm{Mod}(S) \end{tikzcd}$$

- multiple fiberings

$$\begin{tikzcd} S_6 \arrow[r] & E \arrow[r, equal] \arrow[d] & E \arrow[d] & S_{321} \arrow[l] \\ & S_{129} & S_3 & \end{tikzcd}$$

<u>Q</u>: other fiberings?

II. Monodromy arithmeticity problem

(monodromy approach to fibering Q
more general problem. use to answer).

(Earle-Eells) $\chi(S)<0$. For fixed B

$$\left\{ \begin{matrix} \text{bundles} \\ S \to E \to B \end{matrix} \right\} / \text{iso} \longleftrightarrow \left\{ \begin{matrix} \text{homomorphisms} \\ \pi_1(B) \to \mathrm{Mod}(S) \end{matrix} \right\} / \text{conj.}$$

main tool for studying bundles

$\mapsto$ monodromy

$\leftarrow\!\shortmid$ miracle

(Andy: Diff $\sim$ Mod)

<u>Topology-Monodromy dictionary</u>

- (Thurston) $E_f \to S^1$ admits >1 fibering $\iff H^1(S;\mathbb{Q})^f \neq 0$.
- (Salter) $E \to B^2$ has unique fibering $\Leftarrow H^1(S;\mathbb{Q})^{\pi_1(B)} = 0$.

  (Chen) to show AK $S_g \to E \to B$ fibers in 2 ways show $H^1(S_g)^{\pi_1(B)} \cong H^1(X)$

- (Deligne) $E \xrightarrow{\text{holo}} B$ quasiproj $\Rightarrow$ Zariski closure G of $\Gamma_E < Sp_{2g}(\mathbb{C})$ is semisimple & <u>Hermitian</u>

  <u>Arapura</u>

$$\Gamma_E = \mathrm{Im}\left[\pi_1(B) \to \mathrm{Mod}(S_g) \longrightarrow \mathrm{Aut}(H_1(S_g), \text{int form}) \simeq Sp_{2g}(\mathbb{Z})\right]$$

eg $G(\mathbb{R}) = Sp_{2h}(\mathbb{R})$ or $SU(p,q)$ but not solvable $\begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix}$, or even $SL_n\mathbb{R}$.

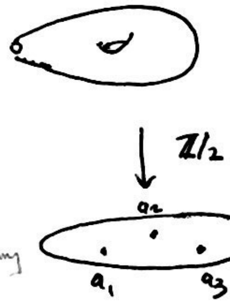
<u>Monodromy-arithmeticity question</u> (Griffiths-Schmid) Is $\Gamma_E < G(\mathbb{Z})$

<u>finite</u> index or <u>infinite</u> index? for AK manifolds?

arithmetic thin

<u>Ex</u> (braid monodromies) $k \geq 3$, $d \geq 2$

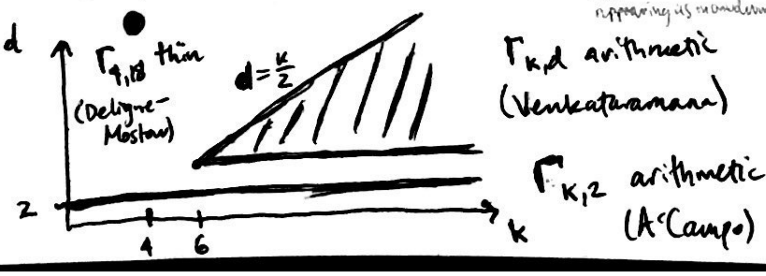
$S = \{ y^2 = \prod (x - a_i) \} \subset \mathbb{C}^2$



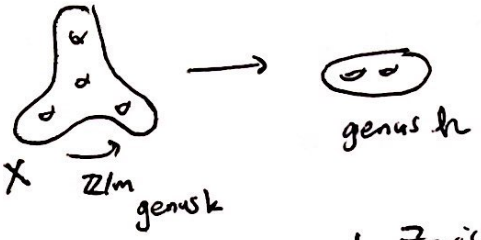
$$\begin{array}{c} S \longrightarrow E_{k,d} \\ \downarrow \\ \mathrm{Conf}_k(\mathbb{C}) \end{array}$$

$B_k \simeq \pi_1 \mathrm{Conf}_k(\mathbb{C}) \to Sp_{2g}(\mathbb{Z})$.

(Burau) same as in Dan's talk now appearing as monodromy



<u>Thm</u> (Salter–T)

$S_g \longrightarrow$  $\rightsquigarrow S_g \to E_{h,m} \to B$

If $m \geq 2$, $h \geq 5$, then Γ_E is arithmetic, and Zariski closure can be determined.

<u>Ex</u> $m=7$ Γ_E arithmetic in $SU(k,k+5) \times SU(k+1,k+4) \times SU(k+2,k+3)$

Here $\mathbb{Z}/7 \curvearrowright H^1(S_g;\mathbb{Q}) = \mathbb{Q}^{2k} \oplus \mathbb{Q}(\zeta_7)^{2k+5}$ int. form $\rightsquigarrow$ herm. form. $SU(V,\beta)$ alg gp over $\mathbb{Q}^+(\zeta_7)$

<u>Cor</u> AK bundles fiber in exactly two ways.

"<u>Pf sketch</u> of Thm"

(1) Observe the monodromy of $S_g \to E_{h,m} \to B$ factors

$$\rho: \mathrm{Mod}(S_h,*) \dashrightarrow \underset{\text{centralizer}}{\mathrm{Mod}(S_g)^{\mathbb{Z}/m}} \longrightarrow Sp_{2g}(\mathbb{Z})^{\mathbb{Z}/m}$$

$$\triangledown$$

$$\pi_1(S_h) = \ker\left[\mathrm{Mod}(S_h,*) \to \mathrm{Mod}(S_h)\right] \quad \text{point-pushing subgp.}$$

$$\Gamma_E = \mathrm{Im}\left(\rho|_{\pi_1(S_h)}\right)$$

(2) Show $\mathrm{Im}(\rho) < Sp_{2g}(\mathbb{Z})^{\mathbb{Z}/m}$ finite index (hence arithmetic)
by showing $\mathrm{Im}(\rho)$ contains "enough" unipotents.
(extend Looijenga, Grunewald–Larsen–Lubotzky–Malestein to branched covers) (most work).

(3) $\Gamma_E \triangleleft \mathrm{Im}(\rho)$ Margulis: Λ irred arithm. lattice in ss Lie rank ≥ 2.
$N \triangleleft \Lambda \Rightarrow N$ finite or finite-index.

$\square$

enough unipotents

$SL_2\mathbb{Z} = \langle \begin{pmatrix}1&1\\0&1\end{pmatrix}, \begin{pmatrix}1&0\\1&1\end{pmatrix} \rangle$

$SL_n\mathbb{Z} \supseteq \langle E_{ij}^r \rangle$ finite index for fixed $r \geq 1$, $n \geq 3$. (Tits)

~~[illegible]~~

<u>Rmk</u> This argument does <u>not</u> identify Zariski closure of Γ_E.

Need finer understanding of how pp's lift to cover.

<u>Rmk</u> $\mathrm{Im}(\rho) < Sp_{2g}(\mathbb{Z})^{\mathbb{Z}/m}$ often <u>not</u> finite index ((2) often false!) when m composite

$$\pi_1(S_g) \longrightarrow \pi_1(X) \longrightarrow \pi_1(S_h) \quad \text{not normal cover.}$$

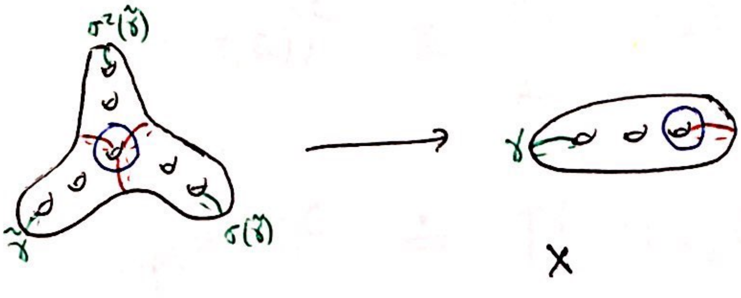
eg $m=4$ $\quad Sp_{2g}(\mathbb{R})^{\mathbb{Z}/m} = Sp_{2k}(\mathbb{R}) \times Sp_{2(k+1)}(\mathbb{R}) \times SU(k,k+2)$

$$\mathrm{Im}(\rho)(\mathbb{R}) = Sp_{k+1}(\mathbb{R}) \times Sp_{k+1}(\mathbb{R}) \times SU(k,k+2)$$

Lecture 2 Caltech 2020

$$\begin{array}{ll} Y & \text{regular cover, deck gp } \quad C \cong \underset{\langle\sigma\rangle}{\underset{\|}{\mathbb{Z}/m\mathbb{Z}}} \quad m \geq 2 \\ \downarrow & \\ X & \text{closed or. surf. genus } g \geq 3 \end{array}$$

<u>Thm</u> (Looijenga) Image of $\rho: \mathrm{Mod}(X) \dashrightarrow \mathrm{Sp}(H_1(Y))^C$ has finite index in $\mathrm{Sp}(H_1(Y))^C$



$T_\gamma \in \mathrm{Mod}(X)$ lifts to

$T_{\tilde{\gamma}} \circ T_{\sigma(\tilde{\gamma})} \circ T_{\sigma^2(\tilde{\gamma})} \in \mathrm{Mod}(Y)^C$

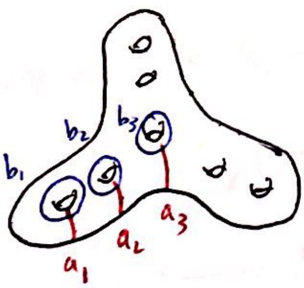
Strategy: $\mathrm{Im}(\rho)$ contains "enough" unipotents (not Looijenga's proof)

I. $\mathrm{Sp}(H_1(Y))^C$ as arithmetic group.

$d \neq 1$

$V_d = K_d\{a_1 b_1 \ldots a_{g-1} b_{g-1}\}$

• (Chevalley-Weil) $H_1(Y;\mathbb{Q}) \cong \underset{\mathbb{Q}\{a_g, b_g\}}{\underset{\|}{\mathbb{Q}^2}} \oplus \underset{\mathbb{Q}C\{a_1 b_1, \ldots, a_{g-1}, b_{g-1}\}}{\underset{\|}{(\mathbb{Q}C)^{2g-2}}} \cong \bigoplus_{d|m} V_d$



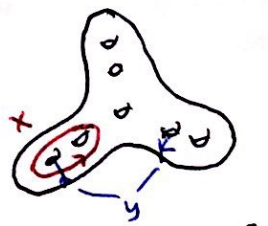
$$\mathbb{Q}C \cong \mathbb{Q}[\sigma]/(\sigma^m - 1) \cong \bigoplus_{d|m} \underbrace{\mathbb{Q}[\sigma]/\underbrace{\Phi_d(\sigma)}_{\text{cyclotomic poly}}}_{=: K_d}$$

• Reidemeister pairing $\langle \cdot, \cdot \rangle : H_1(Y) \times H_1(Y) \longrightarrow \mathbb{Z}C$

$$\langle x, y \rangle := \sum_{c \in C} (x, cy)\, c$$

$\uparrow$ $\mathbb{Z}$-valued alg intersection on $H_1(Y)$

track intersection & sheet it occurs in



$\langle x, y \rangle = 1 - \sigma^2$

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skew Hermitian wrt involution $\bar{c} = c^{-1}$ on $\mathbb{Z}C$.

$$\langle ax, y\rangle = a\langle x,y\rangle \qquad \langle y,x\rangle = -\overline{\langle x,y\rangle}$$

extends to $H_1(Y;\mathbb{Q})$, restricts to $V_d \times V_d \longrightarrow K_d$

$U(V_d)$ unitary group, alg group over $k_d = \{\bar{z} = z\} \subset K_d$.

$$\begin{array}{ccc} Sp(H_1(Y;\mathbb{Q}))^C & \cong & \prod_{d|m} U(V_d) \\ \text{VI} & & \text{VI} \\ Sp(H_1(Y))^C & \doteq & \prod U(V_d, \mathcal{O}_d) \\ & \text{commensurable} & \end{array}$$

$\mathcal{O}_d \subset K_d$ ring of integers

<u>Lemma</u> $\Gamma < \prod U(V_d, \mathcal{O}_d)$ finite index $\Leftrightarrow$ $\mathrm{proj}_d(\Gamma) < U(V_d, \mathcal{O}_d)$ f.i. $\forall d$.

<u>Pf idea of ($\Leftarrow$)</u> Key fact: $U(V_d)$ simple, rank $k \geq 2$
$\quad\quad$ $\shortparallel$ max dim isotropic subspace

Suppose $\Gamma < \Lambda_1 \times \Lambda_2$ $\quad \Lambda_i < G_i$ lattice in simple, rank ≥ 2.

$$\begin{tikzcd} N_2 \arrow[dr, hook] & & \Gamma_1 \subset \Lambda_1 \\ & \Gamma \arrow[ur, two heads] \arrow[r, hook] \arrow[dr, two heads] & \Lambda_1 \times \Lambda_2 \arrow[u] \arrow[d] \\ N_1 \arrow[ur, hook] & & \Gamma_2 \subset \Lambda_2 \end{tikzcd}$$

$N_1 \triangleleft \Gamma_2$, $N_2 \triangleleft \Gamma_1$ $\qquad (\gamma_1,\gamma_2)(e,n)(\gamma_1,\gamma_2)^{-1} = (e, \gamma_2 n \gamma_2^{-1})$

$\underset{\text{Margulis}}{\Rightarrow}$ N_i finite or finite index

if $N_1 = \{1\}$ $\quad \Gamma \cong \Gamma_1$ $\underset{\text{Margulis}}{\Rightarrow}$

$$\begin{tikzcd} \Gamma \arrow[r] \arrow[d, hook] & \Gamma_2 \arrow[d, hook] \\ G_1 \arrow[r, "\cong"] & G_2 \end{tikzcd}$$

extends

simple $\qquad$ image contains lattice

$U(V_d) \not\cong U(V_{d'})$ for $d \neq d'$. $\qquad \square$

II. Enough unipotents

Fix $d \mid m$, $d \neq 1$ $\quad V = V_d$, $K = K_d$

$\quad\quad\quad\quad\quad\quad \overset{\shortparallel}{K\{a_1, b_1, \ldots, a_{g-1}, b_{g-1}\}}$

$P < U(V)$ (parabolic) subgp preserving ~~flg~~ flag $0 \subset K\{a_1\} \subset K\{a_1\}^\perp \subset V$

$\vee$

$\mathcal{U}$ unipotent radical (trivial on successive quotients) $\quad$ wrt basis $\left(\begin{array}{c|c|c} 1 & * & z \\ \hline 0 & I & y \\ \hline 0 & 0 & 1 \end{array}\right)$ $\quad \bar{z} - z = \langle y, y \rangle$

$$0 \to k \to \mathcal{U} \to K\{a_1, b_1\}^\perp \to 0$$

nonsplit central extension.

$$0 \to \mathcal{O}_k \to \mathcal{U}(\mathcal{O}) \xrightarrow{\pi} \mathcal{O}_K^{2g-4} \to 0$$

flag $0 \subset K\{b_1\} \subset K\{b_1\}^\perp \subset V$ $\rightsquigarrow$ opposite parabolic $P' > \mathcal{U}'$

Thm ("enough unipotents") (Venkataramana) $\Gamma < U(V, \mathcal{O})$ f.i. $\Longleftrightarrow$

$\pi(\Gamma \cap \mathcal{U}(\mathcal{O})),\ \pi(\Gamma \cap \mathcal{U}'(\mathcal{O})) < \mathcal{O}_K^{2g-4}$ f.i.

III Producing unipotents in image $\mathrm{Mod}(X) \dashrightarrow \underset{U(V, \mathcal{O})}{\cancel{Sp(H_1(Y))^C}}$

Need unipotents shearing b_1 by $y \in \mathcal{O}_K\{a_1 b_1, \ldots, a_{g-1}, b_{g-1}\}$.

- lifting Dehn twists. if $\gamma \subset X$ scc lifts to $\tilde{\gamma} \subset Y$, T_γ lifts to

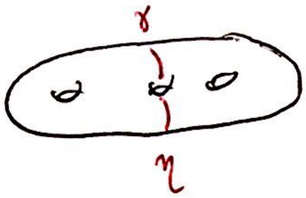
$$\tilde{T}_\gamma = \prod_{c \in C} T_{c(\tilde{\gamma})}$$

$T_{c(\tilde{\gamma})}$ acts on $H_1(Y)$ by $x \mapsto x + (x, c(\tilde{\gamma}))\, c(\tilde{\gamma})$

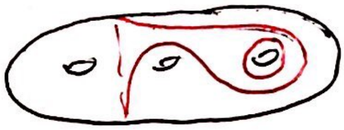
$\Rightarrow \tilde{T}_\gamma$ acts by $x \longmapsto x + \langle x, \tilde{\gamma} \rangle \tilde{\gamma}$

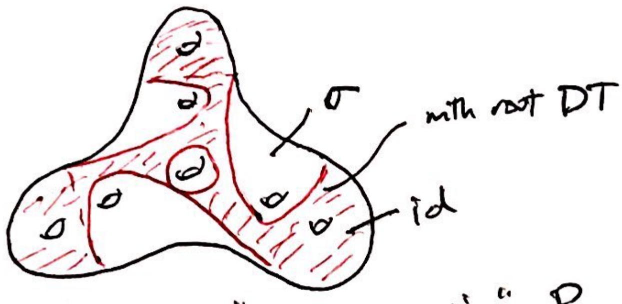
$[\gamma] = [a_1] + [a_2]$ $\quad \tilde{T}_\gamma \in P$

<u>Key computation</u> lifting bounding pair



$T_\gamma \circ T_\eta^{-1}$ bounding pair lifts to Y b/c in Torelli

special ex. 



bp lifts to "partial rotation" R

action on $H_1(Y)$

$$R(x) = \begin{cases} x & x \in \mathbb{Q}C\{a_1, b_1\} \oplus \mathbb{Q}\{a_g, b_g\}. \\ \sigma(x) & x \in \mathbb{Q}C\{a_2 b_2 - a_{g-1} b_{g-1}\} \end{cases}$$

- commutator trick: $\tilde{T}_\gamma \in \Gamma$ as before $[\tilde{\gamma}] = [a_1] + [a_2]$

  $[\tilde{T}_\gamma, R]$ unipotent and $\pi([\tilde{T}_\gamma, R]) = (\sigma - 1) a_2$

- conjugation of $[\tilde{T}_\gamma, R]$ by R $\pi(R[\tilde{T}_\gamma, R]R^{-1}) = \sigma(\sigma-1)a_2$

$\Rightarrow \pi(\Gamma \cap \mathcal{U}(\mathcal{O}))$ contains

$\{\sigma^i(\sigma - 1)v \mid v = a_2 b_2, \dots, a_{g-1} b_{g-1}\} < \mathcal{O}_K^{2g-2}$ finite index //.

Salter-T extends computation to (certain)

regular <u>branched</u> cover $Y \to X$ w/ <u>nilpotent</u> deck group

rep theory more complicated

producing unipotents more complicated