PONTRYAGIN CLASSES OF LOCALLY SYMMETRIC MANIFOLDS

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ABSTRACT. Pontryagin classes $p_i(M)$ are basic invariants of a smooth manifold M, and many topological problems can be reduced to computing these classes. For a locally symmetric manifold, Borel-Hirzebruch [3] gave an algorithm to determine if $p_i(M)$ is nonzero. In addition they implemented their algorithm for a few well-known M and for $i = 1, 2$. Nevertheless, there remained several M for which their algorithm was not implemented. In this note we compute low degree Pontryagin classes for every closed locally symmetric manifold of noncompact type. As a result of this computation, we answer the question: Which closed locally symmetric M have at least one nonzero Pontryagin class?

CONTENTS

1. INTRODUCTION

For a manifold M and $i > 0$, the Pontryagin class $p_i(M) \in H^{4i}(M; \mathbb{Q})$ is a diffeomorphism invariant. When these classes are nonzero, they can serve as obstructions to certain geometric problems (see for example [18]).

In this paper, we are interested in closed locally symmetric manifolds M of noncompact type. Let G be a semisimple Lie group without compact factors; let $K \subset G$ be a maximal compact subgroup; and let $\Gamma \subset G$ be a cocompact, torsion-free lattice. The manifold G/K has a G-invariant Riemannian metric and is a symmetric space of noncompact type. Γ acts freely and properly on G/K, and the closed manifold $M = \Gamma\backslash G/K$ is a locally symmetric manifold of noncompact type.

Question 1.1. For which $\Gamma\backslash G/K$ is $p_i(\Gamma\backslash G/K) \neq 0$ for some $i > 0$?

Throughout this paper all cohomology groups $H^*(\cdot)$ will be with $\mathbb{Q}$ coefficients unless otherwise specified.

Date: March 23, 2015.
Key words and phrases. Algebraic topology, differential geometry, characteristic classes.

The classical approach to determine if $p_i(\Gamma\backslash G/K) \neq 0$ is roughly as follows. Let $U \subset G_{\mathbb{C}}$ be the maximal compact subgroup of the complexification of G. By the Proportionality Principle (see [10], Section 4.14), $p_i(\Gamma\backslash G/K) \neq 0$ if and only if $p_i(U/K) \neq 0$. In [3], Borel-Hirzebruch relate $p_i(U/K)$ to the weights of the action of K on the Lie algebra $\mathrm{Lie}(U)$. From this, showing $p_i(U/K) \neq 0$ reduces to showing that a polynomial is nonzero modulo an ideal (see [3] for details). The paper [3] also contains explicit computations, including computations that $p_1(F_4/\mathrm{Spin}_9) \neq 0$, $p_1(G_2/\mathrm{SO}_4) \neq 0$, and that the first Chern class $c_1(U/K) \neq 0$, where U/K is a compact Hermitian symmetric manifold and U is *not* of exceptional type.

The objective of this paper is three-fold:

(1) Give a stream-lined way to determine if $p_i(\Gamma\backslash G/K) \neq 0$. One feature of our approach is that it does not use the Proportionality Principle mentioned above to reduce to computations for the compact dual U/K. Our main idea is to use the action of Γ on the visual boundary $\partial(G/K)$ to study Pontryagin classes. This differs from the approach of Borel-Hirzebruch but ultimately reduces to the same problem: determining if a polynomial is nonzero modulo an ideal.

(2) Compute low dimensional Pontryagin classes $p_i(\Gamma\backslash G/K)$ for every locally symmetric manifold of noncompact type, including every exceptional example. As mentioned above, for a handful of G these computations follow from computations done by Borel-Hirzebruch in [3]. Computations for $G = E_{6(6)}$ and $G = E_{6(-26)}$ were done by Takeuchi [17]; however, the author was unable to find computations for every semisimple Lie group. The purpose of the present paper is to have a single source for computations for every $\Gamma\backslash G/K$.

(3) Answer Question 1.1. A priori the answer depends on the choice of both Γ and G, but it follows from the Proportionality Principle that the answer is independent of Γ (here it is important that Γ is cocompact). The author's interest in Question 1.1 arose from the work [18], in which Theorem 1.2 below is used to address a Nielsen realization problem: if $p_i(\Gamma\backslash G/K) \neq 0$ for some $i > 0$, then the natural homomorphism $\pi_1(M) \to \pi_0\mathrm{Diff}(M, *)$ does not lift to $\mathrm{Diff}(M, *)$.

Main results. A complete list of the simple real Lie groups of noncompact type are contained in Tables 1 and 2 below. We use subscripts instead of parentheses wherever possible; for example, we use O_n instead of $O(n)$ to denote the orthogonal group. The examples in Table 1 are complex Lie groups, viewed here as real Lie groups. For any locally symmetric space M of noncompact type, the universal cover $\widetilde{M}$ is a symmetric space. Up to isogeny the isometry group of $\widetilde{M}$ has identity component equal to a product of groups in Tables 1 and 2. See [8] for further information.

For G in Table 1, we can conclude $p_i(\Gamma\backslash G/K) = 0$ for all $i > 0$ without any computation (see Section 5). This is why we have separated the simple, real Lie groups into two separate tables. The main work involved in the present paper is to determine those G in Table 2 for which $p_i(\Gamma\backslash G/K) = 0$ for all $i > 0$. Here is a summary of our results.

G	K
$\mathrm{SL}_n\mathbb{C}$	SU_n
$\mathrm{SO}_n\mathbb{C}$	SO_n
$\mathrm{Sp}_{2n}\mathbb{C}$	Sp_n
$E_6^{\mathbb{C}}$	E_6
$E_7^{\mathbb{C}}$	E_7
$E_8^{\mathbb{C}}$	E_8
$F_4^{\mathbb{C}}$	F_4
$G_2^{\mathbb{C}}$	G_2

TABLE 1. Complex noncompact simple Lie groups

	G	K
AI	$\mathrm{SL}_n\mathbb{R}$	SO_n
AII	SU^*_{2n}	Sp_n
AIII	$\mathrm{SU}_{p,q}$	$S(U_p \times U_q)$
BDI	$\mathrm{SO}_{p,q}$	$\mathrm{SO}_p \times \mathrm{SO}_q$
DIII	SO^*_{2n}	U_n
CI	$\mathrm{Sp}_{2n}\mathbb{R}$	U_n
CII	$\mathrm{Sp}_{p,q}$	$\mathrm{Sp}_p \times \mathrm{Sp}_q$
E_6I	$E_{6(6)}$	$\mathfrak{sp}_4$
E_6II	$E_{6(2)}$	$\mathfrak{su}_6 \times \mathfrak{su}_2$
E_6III	$E_{6(-14)}$	$\mathfrak{so}_{10} \times \mathfrak{so}_2$
E_6IV	$E_{6(-26)}$	$\mathfrak{f}_4$
E_7V	$E_{7(7)}$	$\mathfrak{su}_8$
E_7VI	$E_{7(-5)}$	$\mathfrak{so}_{12} \times \mathfrak{su}_2$
E_7VII	$E_{7(-25)}$	$\mathfrak{e}_6 \times \mathfrak{so}_2$
E_8VIII	$E_{8(8)}$	$\mathfrak{so}_{16}$
E_8IX	$E_{8(-24)}$	$\mathfrak{e}_7 \times \mathfrak{su}_2$
F_4I	$F_{4(4)}$	$\mathfrak{sp}_3 \times \mathfrak{su}_2$
F_4II	$F_{4(-20)}$	$\mathfrak{so}_9$
G_2	$G_{2(2)}$	$\mathfrak{su}_2 \times \mathfrak{su}_2$

TABLE 2. Real noncompact simple Lie groups

Theorem 1.2. *Let G be any real, simple, noncompact Lie group and let $\Gamma \subset G$ be a cocompact lattice. Then $p_i(\Gamma\backslash G/K) = 0$ for all $i > 0$ if and only if G is*

(i) *one of the Lie groups in Table* 1*, or*
(ii) *one of* $\mathrm{SL}_n(\mathbb{R})$*,* SU^*_{2n}*,* $\mathrm{SO}_{p,1}$*, or* $E_{6(-26)}$*.*

Theorem 1.2 and its proof (in Section 6) show that the answer to Question 1.1 is somewhat subtle. For example, let $G = \mathrm{SO}_{p,q}$ and assume $p \geq q$. If $p, q \geq 2$, then $p_1(\Gamma\backslash G/K) \neq 0$ if and only if $p \neq q$. If $p = q$, then $p_2(\Gamma\backslash G/K) \neq 0$ as long as $p \geq 4$. If either $p = q = 2$ or $p = q = 3$ or $p > q = 1$, then $p_i(\Gamma\backslash G/K) = 0$ for all $i > 0$.

It turns out that if G is not one of the groups from (i) or (ii) in Theorem 1.2, and $\Gamma \subset G$ is cocompact, then either $p_1(\Gamma\backslash G/K) \neq 0$ or $p_2(\Gamma\backslash G/K) \neq 0$. Thus to answer Question 1.1, we need only consider low dimensional Pontryagin classes. To determine if $p_i(\Gamma\backslash G/K) \neq 0$ for $i \geq 3$ using the methods of this paper would be feasible, but more computationally intensive.

The proof of Theorem 1.2 immediately extends to the case when G is semisimple to give a complete classification of which $\Gamma\backslash G/K$ have $p_i(\Gamma\backslash G/K) = 0$ for all $i > 0$.

Corollary 1.3. *Let $G = \prod G_i$ be a semisimple Lie group with simple, noncompact factors G_i, and let $\Gamma \subset G$ be a cocompact lattice. Then $p_i(\Gamma\backslash G/K) = 0$ for all $i > 0$ if and only if each G_i is one of the groups from (i) or (ii) in Theorem 1.2.*

Instead of asking for $\Gamma\backslash G/K$ with a nonzero Pontryagin *class*, one could ask for $\Gamma\backslash G/K$ with a nonzero Pontryagin *number*. Specifically, let M be a manifold with $\dim M = 4k$, and let $i_1, \dots, i_m \in \mathbb{N}$ such that $i_1 + \cdots + i_m = k$. The cup product $p_{i_1}(M) \cup \cdots \cup p_{i_m}(M) \in H^{4k}(M)$ can be evaluated on the fundamental class $[M] \in H_{4k}(M)$, and the resulting integer

$$\big\langle p_{i_1}(M) \cup \cdots \cup p_{i_m}(M), [M] \big\rangle \in \mathbb{Z}$$

is called a *Pontryagin number*. These integers are *topological* invariants of M by Novikov's theorem (see [12], for example). We remark that a manifold can have zero Pontryagin numbers but have some nonzero Pontryagin classes. For example, take $\Gamma \subset G = \mathrm{SO}_{p,q}$ with p, q both odd. As explained in [12] Theorem B, the Pontryagin numbers of $\Gamma\backslash G/K$ are all zero. On the other hand, we show that if $p > q > 2$, then $p_1(\Gamma\backslash G/K) \neq 0$ (see Section 6). For more information about Pontryagin numbers of locally symmetric spaces, see [12]. For recent results about the Euler characteristic of homogeneous spaces, see Mostow [16].

Method of proof. We compute $p_i(\Gamma\backslash G/K)$ by the following procedure. Let $n = \dim M$. The unit tangent bundle $T^1M \to M$ has a flat $\mathrm{Homeo}(S^{n-1})$ structure with monodromy $\mu : \Gamma \to \mathrm{Homeo}(S^{n-1})$ given by the action of Γ on the visual boundary $\partial(G/K) \simeq S^{n-1}$. The homomorphism μ induces a map of classifying spaces $M \approx B\Gamma \to B\mathrm{Homeo}(S^{n-1})$ and hence a map

$$\mu^* : H^*\big(B\mathrm{Homeo}(S^{n-1})\big) \to H^*(M).$$

There are classes $q_i \in H^{4i}\big(B\mathrm{Homeo}(S^{n-1})\big)$ for which $\mu^*(q_i) = p_i(M)$ (see Section 2). To determine if $\mu^*(q_i) = 0$, note that μ factors $\mu = \alpha_1 \circ \alpha_2 \circ \alpha_3$, where

$$\Gamma \xrightarrow{\alpha_3} G^\delta \xrightarrow{\alpha_2} G \xrightarrow{\alpha_1} \mathrm{Homeo}(S^{n-1}). \tag{1.1}$$

Here G^δ is the group G viewed as a Lie group with the discrete topology. The map α_3 is the inclusion, α_2 is the identity (which is continuous), and α_1 is the action of G on its visual boundary.

To understand $\mu^* = (\alpha_1 \circ \alpha_2 \circ \alpha_3)^*$, we study the individual maps

$$\boxed{H^*\big(B\mathrm{Homeo}(S^{n-1})\big) \xrightarrow{\alpha_1^*} H^*(BG) \xrightarrow{\alpha_2^*} H^*(BG^\delta) \xrightarrow{\alpha_3^*} H^*(M).} \tag{1.2}$$

Step 1. Computing α_1^* reduces to computing weights of the adjoint action of K on $\mathfrak{g} = \mathrm{Lie}(G)$ (see Section 4).

Step 2. Computing α_2^* reduces to computing the map $H^*(BG_{\mathbb{C}}) \to H^*(BG)$ induced by complexification $G \to G_{\mathbb{C}}$. This follows from Chern-Weil theory (see Section 5).

Step 3. α_3^* is injective on the image of $H^*(BG) \to H^*(BG^\delta)$ by a transfer argument (see Section 6).

Structure of the paper. In Section 2 we define the classes q_i mentioned above and show that $\mu^*(q_i) = p_i(M)$. In Section 3 we recall Borel's computation of $H^*(BK)$ for K a compact Lie group, and we recall how characteristic classes of

a representation can be computed in terms of the weights of that representation. In Sections 4 and 5 we complete Steps 1 and 2, respectively. In Section 6 we explain Step 3 and combine Steps 1, 2, and 3 to conclude which $\Gamma\backslash G/K$ have $p_i(\Gamma\backslash G/K) \neq 0$ for some $i > 0$ and thus prove Theorem 1.2.

1.1. **Acknowledgements.** The author would like to thank his advisor B. Farb for his gracious and ceaseless support, for his encouragement to complete this project, and for extensive comments that significantly improved a draft of this paper. The author also thanks the referee for carefully reading this paper and pointing out several typographical errors.

2. An algorithm for computing Pontryagin classes

This section has two goals. First we recall the definition of the Pontryagin classes of topological sphere bundles $q_i \in H^{4i}\big(B\mathrm{Homeo}(S^{n-1})\big)$. Then we explain why $\mu^*(q_i) = p_i(M)$. This will follow from the construction of a flat $\mathrm{Homeo}(S^{n-1})$ structure on the unit tangent bundle $T^1M \to M$.

2.1. **Pontryagin classes of sphere bundles.** The Pontryagin classes $p_i \in H^*(BO_n)$ are classically defined as invariants of real vector bundles (see [15]). The following proposition shows that these invariants can also be defined for topological $\mathbb{R}^n$-bundles.

Proposition 2.1. *The inclusion $g : O_n \hookrightarrow \mathrm{Homeo}(\mathbb{R}^n)$ induces a surjection*

$$g^* : H^*\big(B\mathrm{Homeo}(\mathbb{R}^n);\mathbb{Q}\big) \to H^*\big(BO_n;\mathbb{Q}\big).$$

Proposition 2.1 can be proved using results from Kirby-Siebenmann [11]. The argument (which the author learned from A. Hatcher) is given in [18].

From Proposition 2.1, Pontryagin classes of sphere bundles can be defined as follows. Define a homomorphism $\delta : \mathrm{Homeo}(S^{n-1}) \to \mathrm{Homeo}(\mathbb{R}^n)$ using the Alexander trick: $\delta(f)$ performs the homeomorphism f on the sphere of radius r for every $r > 0$, and $\delta(f)$ fixes the origin. This induces maps between classifying spaces and hence a map

$$\delta^* : H^*\big(B\mathrm{Homeo}(\mathbb{R}^n)\big) \to H^*\big(B\mathrm{Homeo}(S^{n-1})\big)$$

The restriction of δ to the subgroup $O_n \subset \mathrm{Homeo}(S^{n-1})$ is the standard action $O_n \to \mathrm{Homeo}(\mathbb{R}^n)$, so there is a commutative diagram

$$\begin{array}{ccc} H^*\big(B\mathrm{Homeo}(\mathbb{R}^n)\big) & \xrightarrow{\delta^*} & H^*\big(B\mathrm{Homeo}(S^{n-1})\big) \\ & {\scriptstyle g^*}\searrow \quad \swarrow{\scriptstyle r^*} & \\ & H^*(BO_n) & \end{array} \tag{2.1}$$

By Proposition 2.1, there is a class $\widetilde{p}_i \in H^{4i}\big(B\mathrm{Homeo}(\mathbb{R}^n)\big)$ with $g^*(\widetilde{p}_i) = p_i$. Since Diagram 2.1 commutes, $\delta^*(\widetilde{p}_i) \in H^{4i}\big(B\mathrm{Homeo}(S^{n-1})\big)$ is nontrivial. We refer to the classes $q_i = \delta^*(\widetilde{p}_i)$ as the *Pontryagin classes of topological sphere bundles.*

2.2. **Flat structure on the unit tangent bundle.** We continue to assume that G is a semisimple Lie group without compact factors and that $K \subset G$ is a maximal compact subgroup. With these assumptions G/K is contractible and has a metric of nonpositive curvature such that G acts by isometries on G/K. In addition, G acts on the visual boundary $\partial(G/K) \simeq S^{n-1}$ (see e.g. [2]). If G has rank 1, then the action of G on $\partial(G/K)$ is smooth, but this is not known in general. Thus,

even though G/K is an algebraic example of a contractible, nonpositively curved manifold, the action on $\partial(G/K)$ is a priori only an action by homeomorphisms. By restriction to $\Gamma \subset G$, we obtain an action $\Gamma \to \mathrm{Homeo}(S^{n-1})$, and from this action we can build an S^{n-1}-bundle $E \to M$, where E is the quotient of $(G/K) \times S^{n-1}$ by the diagonal action of Γ.

Lemma 2.2. *Let M be a complete Riemannian manifold of nonpositive curvature with universal cover $\widetilde{M}$. The sphere bundle whose monodromy is the action of the deck group $\pi_1(M)$ on the visual boundary $\partial\widetilde{M} \simeq S^{n-1}$ is isomorphic to the unit tangent bundle $T^1M \to M$.*

Lemma 2.2 shows that $T^1M \to M$ is flat. It also explains why the monodromy $\mu : \Gamma \to \mathrm{Homeo}(S^{n-1})$ factors as claimed in (1.1): μ is the Γ-action on $\partial(G/K)$, which is the restriction of the G-action on $\partial(G/K)$.

Proof of Lemma 2.2. Since M has nonpositive curvature, $\widetilde{M}$ is contractible, and the unit tangent bundle $\pi : T^1\widetilde{M} \to \widetilde{M}$ is trivial. An explicit trivialization can be defined as follows. Define a map

$$\phi : T^1\widetilde{M} \to \partial\widetilde{M}$$

by associating a geodesic ray to a vector via the exponential map. Now define

$$\begin{array}{rccc} \tau : & T^1\widetilde{M} & \to & \widetilde{M} \times \partial\widetilde{M} \\ & z & \mapsto & (\pi(z), \phi(z)). \end{array}$$

Since $\partial\widetilde{M}$ is homeomorphic to S^{n-1}, this gives the desired trivialization.

It is not hard to see that τ is equivariant with respect to the actions of the deck group $\pi_1(M)$ on $T^1\widetilde{M}$ and $\widetilde{M} \times \partial\widetilde{M}$. Note that $T^1\widetilde{M}/\pi_1(M) \simeq T^1M$. Then the quotient by the $\pi_1(M)$-action induces a bundle isomorphism

$$\begin{array}{ccc} T^1M & \xrightarrow{\tau'} & \widetilde{M} \times_{\pi_1(M)} \partial\widetilde{M} \\ & \searrow \quad \swarrow & \\ & M & \end{array}$$

□

3. Compact Lie groups, characteristic classes, and representations

Let K be a compact Lie group and let BK be its classifying space. In this section we recall Borel's computation of the cohomology $H^*(BK)$ (see Theorem 3.1). For a representation $\rho : K \to \mathrm{GL}_m(\mathbb{C})$, we recall how to use Borel's computation to determine the image of $\rho^* : H^*(B\mathrm{GL}_m(\mathbb{C})) \to H^*(BK)$ (specifically the image of the Chern classes) in terms of the weights of ρ. For more details, see [3].

3.1. **The cohomology of BK.** Let K be a compact Lie group. Let $S \subset K$ be a maximal abelian subgroup. S is homeomorphic to an r-torus $(S^1)^r$ for some integer r, which is called the *rank* of K. Let $N_K(S)$ denote the normalizer of S in K. The *Weyl group* is defined as $W = N_K(S)/S$.

Theorem 3.1 (Borel [4])**.** *Let K be a compact Lie group with maximal torus S and Weyl group W. The inclusion $S \hookrightarrow K$ induces an isomorphism*

$$H^*(BK; \mathbb{Q}) \simeq H^*(BS; \mathbb{Q})^W.$$

3.2. **Weights, transgression, and characteristic classes.** Let K be a compact Lie group with maximal torus S, and let $\rho : K \to \mathrm{GL}_n(\mathbb{C})$ be a representation. The restriction $\rho\big|_S$ is a sum of 1-dimensional representations $\lambda_i : S \to \mathbb{C}^\times$ called the *weights* of ρ. By the identification $\mathbb{C}^\times = K(\mathbb{Z}, 1)$ and the fact that cohomology is a represented functor, the weights can be viewed as elements of $H^1(S;\mathbb{Z})$. Since we are interested in cohomology with $\mathbb{Q}$-coefficients, we view the weights as elements of $H^1(S) = H^1(S;\mathbb{Q})$.

For the representation ρ, we are interested in computing the induced map

$$\rho^* : H^*\big(B\mathrm{GL}_n(\mathbb{C});\mathbb{Q}\big) \to H^*(BK;\mathbb{Q})$$

in terms of the weights $\lambda_1, \ldots, \lambda_n \in H^1(S;\mathbb{Q})$. Let $S \to ES \to BS$ be the universal principal S-bundle. The edge map

$$\tau : H^1(S;\mathbb{Q}) \to H^2(BS;\mathbb{Q})$$

on the E_2-page of the Serre spectral sequence of this fibration is an isomorphism, called the *transgression*. Let $w_i = -\tau(\lambda_i)$. The *total Chern class* $c(\rho) \in H^*(BS;\mathbb{Q})$ is defined by

$$c(\rho) = 1 + c_1(\rho) + \cdots + c_n(\rho) = \prod_{i=1}^{n}(1 + w_i). \tag{3.1}$$

The Weyl group W permutes the weights of ρ, so $c(\rho) \in H^*(BS)^W \simeq H^*(BK)$.

If $\rho : K \to \mathrm{GL}_n(\mathbb{R})$ is a real representation and $\rho_{\mathbb{C}} : K \to \mathrm{GL}_n(\mathbb{C})$ is the complexification, one defines the i-th *Pontryagin class* $p_i(\rho)$ of ρ by the formula

$$p_i(\rho) = (-1)^i c_{2i}(\rho_{\mathbb{C}}).$$

Remark 3.2. The transgression τ can be given concretely as follows. Identify $H^1(H;\mathbb{Z}) \simeq \mathrm{Hom}(H, \mathbb{C}^\times)$, and define $\mathrm{Hom}(S, \mathbb{C}^\times) \to H^2(BS)$: Given $\varphi : S \to \mathbb{C}^\times$, form the space

$$E_\varphi = \frac{EH \times \mathbb{C}^\times}{H}$$

which is the quotient of $EH \times \mathbb{C}^\times$ by the diagonal action of H, where H acts on $\mathbb{C}^\times$ by φ. Now E_φ has a natural projection to $EH/H = BH$, and this makes E_φ a $\mathbb{C}^\times$ bundle over BH. The first Chern class $c_1(E_\varphi)$ lives in $H^2(BS;\mathbb{Z})$, and the transgression is given by $\tau(\varphi) = c_1(E_\varphi)$.

3.3. **The invariant polynomials $H^*(BS)^W$.** Let K be a compact Lie group with maximal torus S. The ring of invariant polynomials $H^*(BS)^W$ is well-known (see [9], Ch. 3). For the exceptional groups, explicit polynomial generators for $H^*(BS)^W$ can be computed as follows. Let K be one of exceptional compact Lie groups: E_8, E_7, E_6, F_4, G_2. Let V be a fundamental representation of K of minimal dimension. Denote $\lambda_1, \ldots, \lambda_d \in H^1(S)$ the weights of this representation, so that $\tau(\lambda_i) \in H^2(BS)$. In [13], Mehta shows that power sums

$$I_k = \sum_{i=1}^{d} \tau(\lambda_i)^k$$

generate the invariant polynomials $H^*(BS)^W$. In the remainder of this section we recall the descriptions of $H^*(BS)^W$ for the different compact Lie groups, and we record explicit generators of $H^*(BS)^W$ that will be used in Sections 5 and 6.

To express the weights $\lambda_1, \dots, \lambda_d$ for the exceptional K, we use the descriptions from Adams [1]. We remark that our expressions for I_k for E_8, E_7, E_6, and F_4 agree with [13] up to a change of basis. Let $\mathrm{Sym}(x_1, \dots, x_n) \subset \mathbb{Q}[x_1, \dots, x_n]$ denote the subring of symmetric polynomials.

$\underline{B\mathrm{SO}_n}$: Let $k = \lfloor n/2 \rfloor$. As a subring of $\mathbb{Q}[y_1, \dots, y_k]$,

$$H^*(B\mathrm{SO}_n) \simeq H^*(BS)^W = \begin{cases} \mathrm{Sym}(y_1^2, \dots, y_k^2) & n = 2k+1 \\ \langle \mathrm{Sym}(y_1^2, \dots, y_k^2), y_1 \cdots y_k \rangle & n = 2k \end{cases}$$

$\underline{BU_n}$:

$$H^*(BU_n) \simeq H^*(BS)^W = \mathrm{Sym}(y_1, \dots, y_n).$$

Note that $H^*(B\mathrm{SU}_n)$ is the quotient of $\mathrm{Sym}(y_1, \dots, y_n)$ by the ideal generated by $y_1 + \cdots + y_n$. Similarly, $H^*\big(BS(U_p \times U_q)\big)$ is the quotient of

$$\mathrm{Sym}(y_1, \dots, y_p) \otimes \mathrm{Sym}(z_1, \dots, z_q)$$

by the ideal generated by $y_1 + \cdots + y_p + z_1 + \cdots + z_q$.

$\underline{B\mathrm{Sp}_n}$: $H^*\big(B\mathrm{Sp}(n)\big) \simeq H^*(BS)^W = \mathrm{Sym}(y_1^2, \dots, y_n^2)$.

$\underline{BE_8}$: Let $S \subset E_8$ be a maximal torus with Weyl group W. Up to conjugation, we can assume $S \subset \mathrm{Spin}_{16} \subset E_8$ and that S is a maximal torus of Spin_{16}. This allows us to identify $H^1(S) \simeq \mathbb{Q}\{J_1, \dots, J_8\}$ and $H^2(BS) \simeq \mathbb{Q}[z_1, \dots, z_8]$ in such a way that the roots of E_8 are

$$\begin{cases} \pm J_i \pm J_j & 1 \le i < j \le 8, \\ \frac{1}{2}(\pm J_1 \pm \cdots \pm J_8) & \text{even number of } -\text{'s} \end{cases} \tag{3.2}$$

See [1] (pg. 56). W preserves the roots, and $H^*(BS)^W \simeq H^*(BE_8)$ is generated by

$$I_{2k} = \sum_{1 \le i < j \le 8} (z_i + z_j)^{2k} + (z_i - z_j)^{2k} + \sum \frac{1}{2^{2k}} (z_1 \pm \cdots \pm z_8)^{2k},$$

where the second sum is over all terms with an even number of $-$ signs. According to [13] (pg. 1088), $H^*(BS)^W \simeq \mathbb{Q}[I_2, I_8, I_{12}, I_{14}, I_{18}, I_{20}, I_{24}, I_{30}]$. We record here that $I_2 = 30(z_1^2 + \cdots + z_8^2)$.

$\underline{BE_7}$: For any embedding $\mathrm{SU}_2 \hookrightarrow E_8$, the identity component of the centralizer of SU_2 is isomorphic to E_7 (see [1], pg. 49). Choose SU_2 so that its roots are $\pm(J_7 - J_8)$ in E_8 (cf. 3.2). Let $S \subset E_7$ be a maximal torus with Weyl group W. Since the roots of E_7 are orthogonal to the roots of SU_2, we can identify $H^1(S) \simeq \mathbb{Q}\{J_1, \dots, J_6, J_7 + J_8\}$. Let $z_i = \tau(J_i)$ in $H^2(BS)$ for $1 \le i \le 6$ and let $z_7 = \tau(J_7 + J_8)$.

Following [1] (pg. 52), the fundamental representation of E_7 is 56-dimensional, and by restricting this representation to $\mathfrak{so}_{12} \times \mathfrak{su}_2 \subset \mathfrak{e}_7$, one can compute the weights:

$$\begin{cases} \pm J_i \pm (\frac{J_7 + J_8}{2}) & 1 \le i \le 6 \\ \frac{1}{2}(\pm J_1 \pm \cdots \pm J_6) & \text{odd number of } J_i \text{ have } - \text{ sign} \end{cases}$$

W preserves these weights, and $H^*(BS)^W \simeq H^*(BE_7)$ is generated by

$$I_{2k} = \sum_{i=1}^{6} (\frac{z_7}{2} + z_i)^{2k} + (\frac{z_7}{2} - z_i)^{2k} + \frac{1}{2^{2k}} \sum (z_1 \pm \cdots \pm z_6)^{2k},$$

where the second sum is over all terms with an odd number of $-$ signs. According to [13] (pg. 1086), $H^*(BS)^W \simeq \mathbb{Q}[I_2, I_6, I_8, I_{10}, I_{12}, I_{14}, I_{18}]$. Note that for this description of E_7,

$$I_2 = 6(z_1^2 + \cdots + z_6^2) + 3z_7^2.$$

A second computation for BE_7: We will also need the following description of $H^*(BE_7)$. This time choose $\mathrm{SU}_2 \hookrightarrow E_8$ so that its roots are $\pm(J_1 + \cdots + J_8)$. Let $S \subset E_7$ be a maximal torus with Weyl group W. $H^1(S)$ is the orthogonal complement of $J_1 + \cdots + J_8$ in $\mathbb{Q}\{J_1, \ldots, J_8\}$. One can compute the weights (with respect to this description of $H^1(S)$) of the fundamental representation of E_7 by restricting to $\mathrm{SU}_8 \subset E_7$ (see [1], pg. 69). The weights are

$$\pm\left(J_i + J_j - \frac{1}{4}(J_1 + \cdots + J_8)\right), \quad 1 \leq i < j \leq 8.$$

Let $z_i = \tau(J_i) \in H^2(BS)$ for $1 \leq i \leq 8$. (Technically J_i does not live in $H^1(S)$, so we really mean the restriction of the weight J_i for E_8 to a weight for E_7.) $H^*(BS)^W \simeq H^*(BE_7)$ is generated by

$$I_{2k} = \sum_{1 \leq i < j \leq 8} \left((z_i + z_j) - \frac{1}{4}(z_1 + \cdots + z_8)\right)^{2k}.$$

As above, $H^*(BS)^W = \mathbb{Q}[I_2, I_6, I_8, I_{10}, I_{12}, I_{14}, I_{18}]$, and one computes

$$I_2 = \frac{3}{4}\left[7\left(\sum_{1 \leq i \leq 8} z_i^2\right) + 2\left(\sum_{1 \leq i < j \leq 8} z_i z_j\right)\right]$$

BE_6: For any embedding $\mathrm{SU}_3 \hookrightarrow E_8$, the identity component of the centralizer of SU_3 is isomorphic to E_6 (see [1], pg. 49). Choose SU_3 so that its roots are $\pm(J_6 - J_7)$, $\pm(J_6 - J_8)$, and $\pm(J_7 - J_8)$ (cf. 3.2). Let $S \subset E_6$ be a maximal torus with Weyl group W. Since the roots of E_6 are orthogonal to the roots of SU_3, we can identify

$$H^1(S) \simeq \mathbb{Q}\{J_1, \ldots, J_5, J_6 + J_7 + J_8\}.$$

Let $z_i = \tau(J_i) \in H^2(BS)$ for $1 \leq i \leq 5$, and let $z_6 = \tau(J_6 + J_7 + J_8)$.

$\mathrm{Lie}(E_6) = \mathfrak{e}_6$ has two 27-dimensional fundamental representations U_1, U_2. One can compute the weights of U_i by restricting to $\mathfrak{so}_{10} \times \mathfrak{so}_2 \subset \mathfrak{e}_6$. Following [1] (pg. 53), $U_1 = \xi^{-4} + \lambda_{10}^1 \otimes \xi^2 + \Delta^+ \otimes \xi^{-1}$ and its weights are

$$\begin{cases} \pm J_i + \frac{1}{3}(J_6 + J_7 + J_8) & 1 \leq i \leq 5 \\ \frac{1}{2}\left(\pm J_1 \pm \cdots \pm J_5 - \frac{1}{3}(J_6 + J_7 + J_8)\right) & \text{even number of } J_i \text{ have } - \text{ sign} \\ -\frac{2}{3}(J_6 + J_7 + J_8) & \end{cases}$$

W preserves these weights, and $H^*(BS)^W \simeq H^*(BE_6)$ is generated by

(3.3)

$$I_k = \left(-\frac{2}{3}z_6\right)^k + \sum_{i=1}^{5}\left(\frac{1}{3}z_6 + z_i\right)^k + \left(\frac{1}{3}z_6 - z_i\right)^k + \frac{1}{2^k}\sum\left(\pm z_1 \pm \cdots \pm z_5 - \frac{1}{3}z_6\right)^k.$$

In particular, $I_1 = I_3 = 0$ and $I_2 = 6(z_1^2 + z_2^2 + z_3^2 + z_4^2 + z_5^2) + 2z_6^2$ and $I_4 = \frac{1}{12}(I_2)^2$. According to [13] (pg. 1086), $H^*(BS)^W \simeq \mathbb{Q}[I_2, I_5, I_6, I_8, I_9, I_{12}]$.

A second computation for BE_6: We will also need the following description of $H^*(BE_6)$. This time choose $\mathrm{SU}_3 \hookrightarrow E_8$ so that the roots are

$$\left\{ \pm \frac{J_1 + \cdots + J_6}{2} \pm \frac{J_7 + J_8}{2}, \quad \pm(J_7 + J_8) \right\}. \tag{3.4}$$

Let $S \subset E_6$ be a maximal torus with Weyl group W. $H^1(S)$ is the subspace of $\mathbb{Q}\{J_1, \ldots, J_8\}$ that is orthogonal to the roots in (3.4).

For this description of $\mathfrak{e}_6$, we compute the weights of the fundamental representations U_1, U_2 by restricting to $\mathfrak{su}_6 \times \mathfrak{su}_2 \subset \mathfrak{e}_6$. As a $(\mathfrak{su}_6 \times \mathfrak{su}_2)$-representation,

$$U_1 = \lambda_6^2 + \lambda_6^5 \otimes \lambda_2^1 \quad \text{and} \quad U_2 = \lambda_6^4 + \lambda_6^1 \otimes \lambda_2^1,$$

where λ_m^i denotes the i-exterior power of the standard representation of $\mathfrak{su}_m$. Since a Cartan subalgebra for $\mathfrak{su}_6 \times \mathfrak{su}_2$ is also a Cartan subalgebra for $\mathfrak{e}_6$, we can identify the weights of U_i as a $\mathfrak{su}_6 \times \mathfrak{su}_2$-representation with the weights of U_i as an $\mathfrak{e}_6$-representation.

Let $u = \frac{J_1+\cdots+J_6}{2} + \frac{J_7+J_8}{2}$ and $v = \frac{J_1+\cdots+J_6}{2} - \frac{J_7+J_8}{2}$. One computes that the weights of U_1 are

$$\begin{aligned} J_i + J_j - \tfrac{1}{3}(u+v) \qquad & 1 \le i < j \le 6 \qquad \text{(corresponding to } \lambda_6^2\text{)} \\ J_i \pm \tfrac{J_7 - J_8}{2} - \tfrac{5}{6}(u+v) \qquad & 1 \le i \le 6 \qquad \text{(corresponding to } \lambda_6^5 \otimes \lambda_2^1\text{)} \end{aligned}$$

W preserves these weights. Let $z_i = \tau(J_i) \in H^2(BS)$ for $1 \le i \le 6$ and let $z_7 = \tau(J_7 - J_8)$. (As above, by J_i we really mean the restriction of the weight for E_8 to a weight for E_7.) With respect to this basis $H^*(BS)^W \simeq H^*(BE_6)$ is generated by

$$I_k = \sum_{1 \le i < j \le 6} \left(z_i + z_j - \frac{1}{3}(z_1 + \cdots + z_6) \right)^k + \sum_{\substack{1 \le i \le 6 \\ \epsilon \in \{\pm 1\}}} \left(z_i - \frac{5}{6}(z_1 + \cdots + z_6) + \frac{\epsilon}{2} z_7 \right)^k$$

As above, $H^*(BS)^W \simeq \mathbb{Q}[I_2, I_5, I_6, I_8, I_9, I_{12}]$, and one computes

$$I_2 = 5(z_1^2 + \cdots + z_6^2) + 3z_7^2 - 2 \left(\sum_{1 \le i < j \le 6} z_i z_j \right)$$

BF_4: Let $S \subset F_4$ be a maximal torus with Weyl group W. Up to conjugation we can assume $S \subset \mathrm{Spin}_9 \subset F_4$ and that S is a maximal torus of Spin_9 (see [1], pg. 53). This allows us to identify $H^1(S) \simeq \mathbb{Q}\{L_1, \ldots, L_4\}$ and $H^2(BS) \simeq \mathbb{Q}[z_1, \ldots, z_4]$. The 26-dimensional fundamental representation of F_4 has the following nonzero weights (these are the short roots of F_4, see [1] pg. 55).

$$\begin{cases} \pm L_i & 1 \le i \le 4 \\ \frac{1}{2}(\pm L_1 \pm L_2 \pm L_3 \pm L_4) \end{cases}$$

W preserves these weights, and $H^*(BS)^W \simeq H^*(BF_4)$ is generated by

$$I_{2k} = \sum_{i=1}^{4} z_i^{2k} + \frac{1}{2^{2k}} \sum (\epsilon_1 z_1 + \epsilon_2 z_2 + \epsilon_3 z_3 + z_4)^{2k}. \tag{3.5}$$

The second sum is over all tuples $(\epsilon_1, \epsilon_2, \epsilon_3) \in \{\pm 1\}^3$. According to [13] (pg. 1091), $H^*(BS)^W = \mathbb{Q}[I_2, I_6, I_8, I_{12}]$. Note that $I_2 = 3(z_1^2 + z_2^2 + z_3^2 + z_4^2)$.

$\underline{BG_2}$: Let $S \subset G_2$ be a maximal torus (its dimension is 2). Let $J_1, J_2 \in H^1(S)$ be simple roots for G_2. Denote $z_i = \tau(J_i) \in H^2(BS)$. The Weyl group W is the dihedral group of order 12 and it permutes the nonzero weights of the 7-dimensional fundamental representation. These weights are

$$\big\{ \pm J_1,\ \pm(J_1 + J_2),\ \pm(2J_1 + J_2) \big\}.$$

See [6] (Lecture 22), for example. $H^*(BS)^W$ is generated the polynomials

$$I_{2k} = z_1^{2k} + (z_1 + z_2)^{2k} + (2z_1 + z_2)^{2k}.$$

According to [13] (pg. 1094) $H^*(BS)^W \simeq \mathbb{Q}[I_2, I_6]$. Note that

$$I_2 = 2(3z_1^2 + 3z_1z_2 + z_2^2).$$

4. Computing $\alpha_1^* : H^*\big(B\text{Homeo}(S^{n-1})\big) \to H^*(BG)$

Since the inclusion $K \hookrightarrow G$ is a homotopy equivalence, it induces an isomorphism $H^*(BG) \xrightarrow{\sim} H^*(BK)$. To understand α_1^*, we study $\alpha_1\big|_K : K \to \text{Homeo}(S^{n-1})$.

Let $T^1_{eK}(G/K)$ be the space of rays through the origin in $T_{eK}(G/K)$. The action of K on G/K induces an action on $T^1_{eK}(G/K)$. The exponential map defines a K-equivariant homeomorphism

$$s : T^1_{eK}(G/K) \to \partial(G/K).$$

The K-action on $T_{eK}(G/K)$ can be described as follows. The adjoint action of K on $\mathfrak{g} = \text{Lie}(G)$ decomposes into invariant subspaces $\mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{k} = \text{Lie}(K)$ and $\mathfrak{p} \simeq T_{eK}(G/K)$. This implies the following lemma. For more details, see [18].

Lemma 4.1. *The action of $K \subset G$ on $\partial(G/K)$ is induced by a linear representation $\iota : K \to \text{Aut}(\mathfrak{p})$.*

We refer to the representation $\iota : K \to \text{Aut}(\mathfrak{p}) \simeq \text{GL}_n(\mathbb{R})$ as the *isotropy representation*. Let $r : \text{GL}_n(\mathbb{R}) \to \text{Homeo}(S^{n-1})$ be the $\text{GL}_n(\mathbb{R})$-action on the space of rays through the origin in $\mathbb{R}^n$. By Lemma 4.1, $\alpha_1\big|_K = r \circ \iota$. Since the map

$$r^* : H^*\big(B\text{Homeo}(S^{n-1})\big) \to H^*(B\text{GL}_n(\mathbb{R}))$$

is understood via Diagram (2.1), it remains only to understand

$$\iota^* : H^*\big(B\text{GL}_n(\mathbb{R})\big) \to H^*(BK).$$

As described in Section 3.2, ι^* can be computed using the weights of ι. This will be carried out in Sections 4.1 – 4.12 as follows. Let $S \subset K$ be a maximal torus. The isotropy representation $\iota : K \to \text{Aut}(\mathfrak{p})$ is a real representation, so we complexify to get a representation $\iota_{\mathbb{C}} : K_{\mathbb{C}} \to \text{Aut}(\mathfrak{p}_{\mathbb{C}})$. On the maximal abelian subgroup $H \subset K_{\mathbb{C}}$, we obtain weights $\lambda_i : H \to \mathbb{C}^\times$. To compute the weights explicitly, we pass to the Lie algebra $\mathfrak{h}$ of H and view the weights as elements of $\mathfrak{h}^* = \text{Hom}(\mathfrak{h}, \mathbb{C})$. After computing the weights of $\iota_{\mathbb{C}}$, we use Equation (3.1) to express the total Chern class $c(\iota_{\mathbb{C}})$ as a polynomial in $H^*(BH)$. Finally, since $S \hookrightarrow H$ induces an isomorphism $H^*(BH) \to H^*(BS)$, we obtain the Pontryagin classes $p_i(\iota) = (-1)^i c_{2i}(\iota_{\mathbb{C}})$ as polynomials in $H^*(BS)^W \simeq H^*(BK) \simeq H^*(BG)$. This computes α_1^* since $\alpha_1^*(q_i) = p_i(\iota)$.

Below, V_n will denote the standard representation of $\mathfrak{gl}_n(\mathbb{C})$, $\mathfrak{sl}_n(\mathbb{C})$, $\mathfrak{so}_n(\mathbb{C})$, or $\mathfrak{sp}_{2k}(\mathbb{C})$ (for $n = 2k$).

4.1. **Isotropy representation for $\mathbf{SL}_n(\mathbb{R})$.** Upon complexification, we need to study the isotropy representation of $\mathrm{SO}_n(\mathbb{C}) \subset \mathrm{SL}_n(\mathbb{C})$. The adjoint representation of $\mathfrak{sl}_n(\mathbb{C})$ is isomorphic to the kernel of the contraction $V_n \otimes V_n^* \to \mathbb{C}$. As a $\mathfrak{so}_n(\mathbb{C})$-representation,

$$V_n \otimes V_n^* \simeq \Lambda^2(V_n) + \mathrm{Sym}^2(V_n).$$

$\Lambda^2(V_n)$ is the adjoint representation of $\mathfrak{so}_n(\mathbb{C})$. The representation $\mathrm{Sym}^2(V_n)$ is not irreducible because $\mathfrak{so}_n(\mathbb{C})$ preserves a symmetric bilinear form $B : \mathrm{Sym}^2(V_n) \to \mathbb{C}$. The kernel of B is the isotropy representation $\mathfrak{p}$.

There is a standard form B for which the diagonal subgroup $H \subset \mathrm{SO}(B) \simeq \mathrm{SO}_n(\mathbb{C})$ is a Cartan subgroup (see [6], pg. 268). For this choice, we have a standard basis $\mathfrak{h}^* = \langle L_1, \dots, L_k \rangle$, where $k = [n/2]$. If n is even, the weights of $\mathfrak{p}$ are $\pm L_i \pm L_j$ for $1 \leq i, j \leq k$. If n is odd, we have the additional weights $\pm L_i$ for $1 \leq i \leq k$.

As elements of $H^2(BH) \simeq \mathbb{Q}[y_1, \dots, y_k]$, we have the following the total Chern classes for the isotropy representation. If n is even, then

$$c(\iota_{\mathbb{C}}) = \prod_{i,j} (1 + y_i + y_j)(1 + y_i - y_j)(1 - y_i + y_j)(1 - y_i - y_j) \tag{4.1}$$

and if n is odd, then

$$c(\iota_{\mathbb{C}}) = \prod_{i} (1 - y_i)(1 + y_i) \prod_{i,j} (1 + y_i + y_j)(1 + y_i - y_j)(1 - y_i + y_j)(1 - y_i - y_j). \tag{4.2}$$

4.2. **Isotropy representation for $\mathbf{SU}_{p,q}$.** Let $n = p + q$. Upon complexification we need to study the isotropy representation of $K_{\mathbb{C}} \subset \mathrm{SL}_n(\mathbb{C})$, where $K_{\mathbb{C}}$ is the block diagonal subgroup

$$K_{\mathbb{C}} = (\mathrm{GL}_p(\mathbb{C}) \times \mathrm{GL}_q(\mathbb{C})) \cap \mathrm{SL}_n(\mathbb{C}).$$

As described in Section 4.1, the adjoint representation of $\mathfrak{sl}_n(\mathbb{C})$ is a subspace of $V_n \otimes V_n^*$. As a $\mathfrak{k}_{\mathbb{C}}$-representation

$$V_n \otimes V_n^* = \big(V_p \otimes V_p^* + V_q \otimes V_q^*\big) + \big(V_p \otimes V_q^* + V_p^* \otimes V_q\big).$$

The adjoint representation of $\mathfrak{k}_{\mathbb{C}}$ is a codimension-1 subspace of the first summand. The second summand is the isotropy representation $\mathfrak{p}$.

The diagonal subgroup $H \subset K_{\mathbb{C}}$ coincides with the diagonal subgroup of $\mathrm{SL}_n(\mathbb{C})$, so we identify $\mathfrak{h}^*$ as the quotient of $\langle L_1, \dots, L_n \rangle$ by the subspace generated by $L_1 + \cdots + L_n$. The weights of the isotropy representation are $\pm(L_i - L_j)$ for $1 \leq i \leq p$ and $p + 1 \leq j \leq p + q$.

As an element of $H^2(BH) \subset \mathbb{Q}[y_1 \dots, y_p, z_1, \dots, z_q]$, the total Chern class for the complexified isotropy representation is

$$c(\iota_{\mathbb{C}}) = \prod_{\substack{1 \leq i \leq p \\ 1 \leq j \leq q}} \big(1 + (y_i - z_j)\big)\big(1 - (y_i - z_j)\big) = \prod_{\substack{1 \leq i \leq p \\ 1 \leq j \leq q}} \big(1 - (y_i - z_j)^2\big). \tag{4.3}$$

4.3. **Isotropy representation for $\mathbf{Sp}_{2n}(\mathbb{R})$.** Upon complexification we need to study the isotropy representation of $\mathrm{GL}_n(\mathbb{C}) \subset \mathrm{Sp}_{2n}(\mathbb{C})$, where $\mathrm{GL}_n(\mathbb{C})$ is the subgroup of matrices of the form $\begin{pmatrix} A & \\ & (A^t)^{-1} \end{pmatrix}$ for $A \in \mathrm{GL}_n(\mathbb{C})$.

As a $\mathfrak{gl}_n(\mathbb{C})$-representation, the standard representation of $\mathfrak{sp}_{2n}(\mathbb{C})$ decomposes $V_{2n} = V_n + V_n^*$. The adjoint representation of $\mathfrak{sp}_{2n}(\mathbb{C})$ is isomorphic to $\mathrm{Sym}^2(V)$,

and as a $\mathfrak{gl}_n(\mathbb{C})$-representation

$$\mathrm{Sym}^2(V) \simeq \big(V_n \otimes V_n^*\big) + \big(\mathrm{Sym}^2(V_n) + \mathrm{Sym}^2(V_n^*)\big).$$

$V_n \otimes V_n^*$ is the adjoint representation of $\mathfrak{gl}_n(\mathbb{C})$ and $\mathrm{Sym}^2(V_n) + \mathrm{Sym}^2(V_n^*)$ is the isotropy representation.

Let $H \subset \mathrm{GL}_n(\mathbb{C})$ be the diagonal subgroup and identify $\mathfrak{h}^* = \langle L_1, \dots, L_n\rangle$ using the standard basis. The weights of the isotropy representation are $\pm(L_i + L_j)$ for $1 \le i, j \le n$. As an element of $H^2(BH) \simeq \mathbb{Q}[y_1, \dots, y_n]$, the total Chern class of the complexified isotropy representation is

$$\begin{aligned} c(\iota_{\mathbb{C}}) &= \textstyle\prod_{i<j}(1 + y_i + y_j)(1 - (y_i + y_j)) \prod_i (1 + 2y_i)(1 - 2y_i) \\ &= \textstyle\prod_{i<j}\big(1 - (y_i + y_j)^2\big) \prod_i \big(1 - 4y_i^2\big). \end{aligned} \tag{4.4}$$

4.4. **Isotropy representation for $\mathbf{SO}_{p,q}$.** Let $n = p + q$ and $a = [p/2]$ and $b = [q/2]$. Upon complexification, we need to study the isotropy representation of $\mathrm{SO}_p(\mathbb{C}) \times \mathrm{SO}_q(\mathbb{C}) \subset \mathrm{SO}_n(\mathbb{C})$.

The adjoint representation of $\mathfrak{so}_n(\mathbb{C})$ is $\Lambda^2(V_n)$, and as a $\mathfrak{so}_p(\mathbb{C}) \times \mathfrak{so}_q(\mathbb{C})$-representation,

$$\Lambda^2(V_n) \simeq \big(\Lambda^2(V_p) + \Lambda^2(V_q)\big) + V_p \otimes V_q.$$

$\Lambda^2(V_p) + \Lambda^2(V_q)$ is the adjoint representation of $\mathfrak{so}_p(\mathbb{C}) \times \mathfrak{so}_q(\mathbb{C})$, and $V_p \otimes V_q$ is the isotropy representation.

Let $H \subset \mathrm{SO}_p(\mathbb{C}) \times \mathrm{SO}_q(\mathbb{C})$ be the standard Cartan subgroup, and identify $\mathfrak{h}^* = \langle L_1, \dots, L_a, L_{a+1}, \dots, L_{a+b}\rangle$. In this basis, the weights of the isotropy representation are

$$\begin{cases} p, q \text{ even} & \pm L_i \pm L_j \\ p \text{ even}, q \text{ odd} & \pm L_i \pm L_j, \quad \pm L_i \\ p \text{ odd}, q \text{ even} & \pm L_i \pm L_j, \quad \pm L_j \\ p \text{ odd}, q \text{ odd} & \pm L_i \pm L_j, \quad \pm L_i, \quad \pm L_j \end{cases}$$

where $1 \le i \le a$ and $a + 1 \le j \le a + b$.

As an element of $H^*(BH) \simeq \mathbb{Q}[y_1, \dots, y_a, z_1, \dots, z_b]$, we have the following total Chern classes. If p and q are both even, then

$$\begin{aligned} c(\iota_{\mathbb{C}}) &= \textstyle\prod (1 + y_i + z_j)(1 + y_i - z_j)(1 - y_i + z_j)(1 - y_i - z_j) \\ &= \textstyle\prod_{\substack{1 \le i \le a \\ 1 \le j \le b}} \big(1 - (y_i + z_j)^2\big)\big(1 - (y_i - z_j)^2\big). \end{aligned} \tag{4.5}$$

If p is even and q is odd, then

$$c(\iota_{\mathbb{C}}) = \prod_{i=1}^{a}(1 - y_i^2) \prod_{\substack{1 \le i \le a \\ 1 \le j \le b}} \big(1 - (y_i + z_j)^2\big)\big(1 - (y_i - z_j)^2\big). \tag{4.6}$$

If p is odd and q is even, then

$$c(\iota_{\mathbb{C}}) = \prod_{i=1}^{b}(1 - z_j^2) \prod_{\substack{1 \le i \le a \\ 1 \le j \le b}} \big(1 - (y_i + z_j)^2\big)\big(1 - (y_i - z_j)^2\big). \tag{4.7}$$

If p and q are both odd, then

$$c(\iota_{\mathbb{C}}) = \prod_{i=1}^{a}(1-y_i^2)\prod_{j=1}^{b}(1-z_j^2)\prod_{\substack{1\le i\le a\\ 1\le j\le b}}\big(1-(y_i+z_j)^2\big)\big(1-(y_i-z_j)^2\big). \tag{4.8}$$

4.5. **Isotropy representation for $\mathbf{Sp}_{p,q}$.** Let $n=p+q$. After complexifying, we need to study the isotropy representation of $\mathrm{Sp}_{2p}(\mathbb{C})\times\mathrm{Sp}_{2q}(\mathbb{C})\subset\mathrm{Sp}_{2n}(\mathbb{C})$.

The adjoint representation of $\mathfrak{sp}_{2n}(\mathbb{C})$ is $\mathrm{Sym}^2(V_{2n})$, and as a $\mathfrak{sp}_{2p}(\mathbb{C})\times\mathfrak{sp}_{2q}(\mathbb{C})$-representation,

$$\mathrm{Sym}^2(V_{2n}) \simeq \big(\mathrm{Sym}^2(V_{2p})+\mathrm{Sym}^2(V_{2q})\big) + V_{2p}\otimes V_{2q}.$$

$\mathrm{Sym}^2(V_{2p})+\mathrm{Sym}^2(V_{2q})$ is the adjoint representation of $\mathfrak{sp}_{2p}(\mathbb{C})\times\mathfrak{sp}_{2q}(\mathbb{C})$, and $V_{2p}\otimes V_{2q}$ is the isotropy representation.

Let $H\subset\mathrm{Sp}_{2p}(\mathbb{C})\times\mathrm{Sp}_{2q}(\mathbb{C})$ be the standard Cartan subgroup, and identify $\mathfrak{h}^*=\langle L_1,\dots,L_p,L_{p+1},\dots,L_{p+q}\rangle$. The weights of the isotropy representation are $\pm L_i\pm L_j$, where $1\le i\le p$ and $2p+1\le j\le 2p+q$. As an element of $H^2(BH)\simeq\mathbb{Q}[y_1,\dots,y_p,z_1,\dots,z_q]$, the total Chern class of the complexified isotropy representation is

$$c(\iota_{\mathbb{C}}) = \prod_{\substack{1\le i\le p\\ 1\le j\le q}}(1+y_i+z_j)(1+y_i-z_j)(1-y_i+z_j)(1-y_i-z_j). \tag{4.9}$$

4.6. **Isotropy representation for $\mathbf{SO}^*_{2n}$.** Define SO^*_{2n} as the subgroup of $\mathrm{GL}_{2n}(\mathbb{C})$ that preserves the Hermitian form and bilinear form defined by

$$I_{n,n}=\begin{pmatrix} I_n & \\ & -I_n\end{pmatrix} \quad\text{and}\quad B_n=\begin{pmatrix} 0 & I_n\\ I_n & 0\end{pmatrix},$$

respectively. We complexify and study the isotropy representation of $\mathrm{GL}_n(\mathbb{C})\subset\mathrm{SO}_{2n}(\mathbb{C})$, where $\mathrm{GL}_n(\mathbb{C})$ is the subgroup of matrices of the form $\begin{pmatrix} A & \\ & (A^t)^{-1}\end{pmatrix}$ for $A\in\mathrm{GL}_n(\mathbb{C})$.

As a $\mathfrak{gl}_n(\mathbb{C})$-representation, the standard representation of $\mathfrak{so}_{2n}(\mathbb{C})$ decomposes $V_{2n}=V_n+V_n^*$. The adjoint representation of $\mathfrak{so}_{2n}(\mathbb{C})$ is $\Lambda^2(V_{2n})$, and as a $\mathfrak{gl}_n(\mathbb{C})$-representation,

$$\Lambda^2(V_{2n}) \simeq V_n\otimes V_n^* + \big(\Lambda^2(V_n)+\Lambda^2(V_n^*)\big).$$

$V_n\otimes V_n^*$ is the adjoint representation of $\mathfrak{gl}_n(\mathbb{C})$, and $\Lambda^2(V_n)+\Lambda^2(V_n^*)$ is the isotropy representation.

Let $H\subset\mathrm{GL}_n(\mathbb{C})$ be the diagonal subgroup and identify $\mathfrak{h}^*=\langle L_1,\dots,L_n\rangle$. The weights of the isotropy representation are $\pm(L_i+L_j)$ for $1\le i<j\le n$. As an element of $H^2(BH)\simeq\mathbb{Q}[y_1,\dots,y_n]$, the total Chern class is

$$c(\iota_{\mathbb{C}}) = \prod_{i<j}\big(1+(y_i+y_j)\big)\big(1-(y_i+y_j)\big) = \prod_{i<j}\big(1-(y_i+y_j)^2\big). \tag{4.10}$$

4.7. **Isotropy representation for $\mathbf{SU}^*_{2n}$.** After complexifying, we need to study the isotropy representation of $\mathrm{Sp}_{2n}(\mathbb{C})\subset\mathrm{SL}_{2n}(\mathbb{C})$. As described in Section 4.1, the adjoint representation of $\mathfrak{sl}_{2n}(\mathbb{C})$ is contained in $V_{2n}\otimes V_{2n}^*$. As a $\mathfrak{sp}_{2n}(\mathbb{C})$-representation

$$V_{2n}\otimes V_{2n}^* \simeq \mathrm{Sym}^2(V_{2n})+\Lambda^2(V_{2n}).$$

$\mathrm{Sym}^2(V_{2n})$ is the adjoint representation of $\mathfrak{sp}_{2n}(\mathbb{C})$. The representation $\Lambda^2(V_{2n})$ is not irreducible because by definition $\mathfrak{sp}_{2n}(\mathbb{C})$ preserves an antisymmetric bilinear form $J : \Lambda^2(V_{2n}) \to \mathbb{C}$. The kernel of J is the isotropy representation $\mathfrak{p}$.

Let $H \subset \mathrm{Sp}_{2n}(\mathbb{C})$ be the standard Cartan subgroup, and identify $\mathfrak{h}^* = \langle L_1, \dots, L_n \rangle$. The nonzero weights of the isotropy representation are $\pm L_i \pm L_j$ for $1 \le i < j \le n$. As an element of $H^2(BH)$, the total Chern class of the isotropy representation is

$$c(\iota_{\mathbb{C}}) = \prod_{1 \le i < j \le n} (1 + y_i + y_j)(1 + y_i - y_j)(1 - y_i + y_j)(1 - y_i - y_j). \tag{4.11}$$

4.8. **Isotropy representation for real forms of E_8.** Let $\mathfrak{h}^*(\mathfrak{e}_8)$ be the dual to the Cartan subalgebra of $\mathfrak{e}_8$. As in [1] (pg. 56), we identify $\mathfrak{h}_8^* = \langle J_1, \dots, J_8 \rangle$.

In the remainder of this section, we use λ_n^i to denote the i-th exterior power of the standard representations of the real Lie algebras $\mathfrak{su}_n$, $\mathfrak{so}_n$, and $\mathfrak{sp}_n$.

$\underline{E_{8(8)}}$. The maximal compact subgroup $K \subset E_{8(8)}$ has Lie algebra $\mathfrak{so}_{16}$. As an K-representation, $\mathfrak{e}_8$ decomposes

$$\mathfrak{e}_8 = \mathfrak{so}_{16} \oplus \Delta^+$$

where Δ^+ is the positive spin representation of $\mathfrak{so}_{16}$ (see [1], chapter 6). Then the isotropy representation of $K \subset E_8$ is the spin representation. Let $\mathfrak{h}^*(\mathfrak{so}_{16})$ denote the dual to the Cartan subalgebra of $\mathfrak{so}_{16}$. We identify $\mathfrak{h}^*(\mathfrak{so}_{16}) = \langle L_1, \dots, L_8 \rangle$. The weights of the isotropy representation are

$$\frac{1}{2}(\pm L_1 \pm L_2 \pm \cdots \pm L_8)$$

where the number of $+$ signs is even.

Let $S \subset K$ be a maximal torus. As an element of $H^*(BS) \simeq \mathbb{Q}[y_1, \dots, y_8]$, the total Chern class of the complexified isotropy representation is

$$c(\iota_{\mathbb{C}}) = \prod \left(1 - \frac{1}{2}(\epsilon_1 y_1 + \cdots + \epsilon_8 y_8)\right), \tag{4.12}$$

where the product is over all tuples $(\epsilon_1, \dots, \epsilon_8) \in \{\pm 1\}^8$ that have an even number of $-$'s.

$\underline{E_{8(-24)}}$. The maximal compact subgroup $K \subset E_{8(-24)}$ has Lie algebra $\mathfrak{e}_7 \times \mathfrak{su}_2$. As a K-representation, $\mathfrak{e}_8$ decomposes

$$\mathfrak{e}_8 = (\mathfrak{e}_7 \oplus \mathfrak{su}_2) \oplus (V \otimes \lambda_2^1),$$

where V is the 56-dimensional representation of E_7. See [1], pg. 54.

Since $K \subset E_8$ have the same rank, we can identify the (dual) Cartan subalgebras $\mathfrak{h}^*(\mathfrak{e}_8) = \langle J_1, \dots, J_8 \rangle$ and $\mathfrak{h}^*(K) = \langle J_1, \dots, J_6, \frac{J_7+J_8}{2} \rangle \oplus \langle \frac{J_7-J_8}{2} \rangle$.

The roots for $\mathfrak{e}_7 \times \mathfrak{su}_2$ are $\pm(J_7 - J_8)$ together with all the roots of $\mathfrak{e}_8$ that are orthogonal to $J_7 - J_8$. The roots of $\mathfrak{e}_8$ that are not roots of $\mathfrak{e}_7 \times \mathfrak{su}_2$ correspond to the weights of the isotropy representation $\mathfrak{p} = V \otimes \lambda_2^1$. Then the weights of $\mathfrak{p}$ are

$$\begin{cases} \pm J_i \pm (\frac{J_7+J_8}{2}) \pm (\frac{J_7-J_8}{2}) & 1 \le i \le 6 \\ \frac{\pm J_1 \pm \cdots \pm J_6}{2} \pm \frac{J_7-J_8}{2} & \text{even number of } J_i \text{ with } - \text{ sign} \end{cases}$$

Let $S \subset K$ be a maximal torus. Define $y_i = \tau(J_i) \in H^2(BS)$ for $1 \le i \le 6$ and $y_7 = \tau(J_7 + J_8)$ and $y_8 = \tau(J_7 - J_8)$. The total Chern class for the complexified

isotropy representation $c(\iota_{\mathbb{C}}) \in H^*(BS)^W$ is

(4.13)
$$c(\iota_{\mathbb{C}}) = \prod \Big(1 - \big(\epsilon y_i + \epsilon_7 \cdot \frac{y_7}{2} + \epsilon_8 \cdot \frac{y_8}{2}\big)\Big) \prod \left(1 - \frac{1}{2}\big(\epsilon_1 y_1 + \cdots + \epsilon_6 y_6 + \epsilon_8 y_8\big)\right)$$

The first product is over $i = 1, \dots, 6$ and all tuples $(\epsilon, \epsilon_7, \epsilon_8) \in \{\pm 1\}^3$. The second product is over the tuples $(\epsilon_1, \dots, \epsilon_6, \epsilon_8) \in \{\pm 1\}^7$ with an even number of -1's.

4.9. **Isotropy representation for real forms of E_7.**

$\underline{E_{7(7)}}$. The maximal compact subgroup $K \subset E_{7(7)}$ has Lie algebra $\mathfrak{su}_8$. Following [1] (pg. 69), as a K-representation, $\mathfrak{e}_7$ decomposes

$$\mathfrak{e}_7 = \mathfrak{su}_8 \oplus \lambda_8^4.$$

Here $\mathfrak{p} = \lambda_8^4$ is the isotropy representation, and the weights are $L_{i_1} + L_{i_2} + L_{i_3} + L_{i_4}$ for $1 \leq i_1 < \cdots < i_4 \leq 8$.

Let $S \subset K$ be a maximal torus. We identify $H^*(BS)$ with the quotient of $\mathbb{Q}[y_1, \dots, y_8]$ by the ideal generated by $y_1 + \cdots + y_8$. The total Chern class of the complexified isotropy representation is

$$c(\iota_{\mathbb{C}}) = \prod_{1 \leq i_1 < i_2 < i_3 < i_4 \leq 8} \big(1 - (y_{i_1} + y_{i_2} + y_{i_3} + y_{i_4})\big). \tag{4.14}$$

$\underline{E_{7(-5)}}$. The maximal compact subgroup $K \subset E_{7(-5)}$ has Lie algebra $\mathfrak{so}_{12} \times \mathfrak{su}_2$. Following [1] (pg. 52), $\mathfrak{e}_7$ decomposes as a K-representation as

$$\mathfrak{e}_7 = (\mathfrak{so}_{12} \oplus \mathfrak{su}_2) \ \oplus \ \Delta^+ \otimes \lambda_2^1$$

where Δ^+ is the positive spin representation of K. Here $\mathfrak{p} = \Delta^+ \otimes \lambda^1$ is the isotropy representation. Identify $\mathfrak{h}^*(K) \simeq \langle L_1, \dots, L_6 \rangle \oplus \langle L_7 \rangle$ in the standard way. The weights of the isotropy representation are $\frac{1}{2}\big(\epsilon_1 L_1 + \cdots + \epsilon_6 L_6\big) \pm L_7$, where $(\epsilon_1, \dots, \epsilon_6) \in \{\pm 1\}^6$ has an even number of -1's.

Let $S \subset K$ be a maximal torus. As an element of $H^*(BS) \simeq \mathbb{Q}[y_1, \dots, y_6] \otimes \mathbb{Q}[y_7]$, the total Chern class for the complexified isotropy representation is

$$c(\iota_{\mathbb{C}}) = \prod \left(1 - \frac{1}{4}(\epsilon_1 y_1 + \cdots + \epsilon_6 y_6 + y_7)^2\right), \tag{4.15}$$

where $(\epsilon_1, \dots, \epsilon_6) \in \{\pm 1\}^6$ has an even number of -1's.

$\underline{E_{7(-25)}}$. The maximal compact subgroup $K \subset E_{7(-25)}$ has Lie algebra $\mathfrak{e}_6 \times \mathfrak{so}_2$. In a similar way to [1] (pg. 52), one shows that $\mathfrak{e}_7$ decomposes as a K-representation:

$$\mathfrak{e}_7 = (\mathfrak{e}_6 \oplus \mathfrak{so}_2) \ \oplus \ U_1 \otimes \xi^2 \oplus U_2 \otimes \xi^{-2}$$

where U_1 and U_2 are the 27-dimensional representations of E_6. Identify

$$\mathfrak{h}^*(\mathfrak{e}_6) = \langle J_1, \dots, J_5, J_6 + J_7 + J_8 \rangle \quad \text{and} \quad \mathfrak{h}^*(\mathfrak{e}_7) = \langle J_1, \dots, J_6, J_7 + J_8 \rangle$$

as subspaces of $h^*(\mathfrak{e}_8)$ (see Section 3). The orthogonal complement of $\mathfrak{h}^*(\mathfrak{e}_6)$ in $\mathfrak{h}^*(\mathfrak{e}_7)$ is 1-dimensional and generated by $-2J_6 + J_7 + J_8$. We have

$$\mathfrak{h}^*(\mathfrak{e}_6 \times \mathfrak{so}_2) = \langle J_1, \dots, J_5, J_6 + J_7 + J_8 \rangle \oplus \langle -2J_6 + J_7 + J_8 \rangle.$$

The weights of the isotropy representation are the roots of $\mathfrak{e}_7$ that are not roots of $\mathfrak{e}_6 \times \mathfrak{so}_2$. These are $\pm(J_7 + J_8)$, $\pm J_i \pm J_6$ for $1 \le i \le 5$, and

$$\frac{1}{2}\big(\epsilon_1 J_1 + \cdots + \epsilon_5 J_5 + \epsilon_6(J_6 - J_7 - J_8)\big)$$

where $(\epsilon_1, \ldots, \epsilon_6) \in \{\pm 1\}^6$ has even number of -1's. To write these roots as weights of $\mathfrak{e}_6 \times \mathfrak{so}_2$, note that

$$J_7 + J_8 = \frac{1}{3}\big[2(J_6 + J_7 + J_8) + (-2J_6 + J_7 + J_8)\big]$$

and

$$J_6 = \frac{1}{3}\big[(J_6 + J_7 + J_8) - (-2J_6 + J_7 + J_8)\big].$$

The weights of the isotropy representation are

- $$\pm\frac{1}{3}\big[2(J_6 + J_7 + J_8) + (-2J_6 + J_7 + J_8)\big],$$
- for $1 \le i \le 5$,
$$\pm J_i \pm \frac{1}{3}\big[(J_6 + J_7 + J_8) - (-2J_6 + J_7 + J_8)\big]$$
- for $(\epsilon_1, \ldots, \epsilon_6) \in \{\pm 1\}^6$ with an even number of -1's,
$$\frac{1}{2}\left(\epsilon_1 J_1 + \cdots + \epsilon_5 J_5 + \epsilon_6 \cdot \frac{1}{3}\big[-(J_6 + J_7 + J_8) - 2(-2J_6 + J_7 + J_8)\big]\right)$$

Let $S \subset E_6 \times \mathrm{SO}_2$ be a maximal torus. Let $z_i = \tau(J_i) \in H^2(BS)$ for $1 \le i \le 5$, and let $z_6 = \tau(J_6 + J_7 + J_8)$ and let $z_7 = \tau(-2J_6 + J_7 + J_8)$. As elements of $H^2(BS)$, the weights of the isotropy representation are

- $\pm(2z_6 + z_7)$,
- $\pm z_i \pm \frac{1}{3}(z_6 - z_7)$ for $1 \le i \le 5$, and
- $\frac{1}{2}\big(\epsilon_1 z_1 + \cdots + \epsilon_5 z_5 + \epsilon_6 \cdot \frac{1}{3}(-z_6 - 2z_7)\big)$, where $(\epsilon_1, \ldots, \epsilon_6) \in \{\pm 1\}^6$ has even number of -1's.

The total Chern class is

$$\begin{aligned} c(\iota_{\mathbb{C}}) \quad = \quad & \big(1 - \tfrac{1}{9}(2z_6 + z_7)^2\big) \prod \big(1 - (z_i + \tfrac{z_6 - z_7}{3})^2\big)\big(1 - (z_i - \tfrac{z_6 - z_7}{3})^2\big) \\ & \prod \Big(1 - \tfrac{1}{4}\big(\epsilon_1 z_1 + \cdots + \epsilon_5 z_5 + \epsilon_6 \tfrac{-z_6 - 2z_7}{3}\big)^2\Big). \end{aligned} \tag{4.16}$$

The first product is over $1 \le i \le 5$. The second product is over $(\epsilon_1, \ldots, \epsilon_6) \in \{\pm 1\}^6$ with an even number of -1's.

4.10. **Isotropy representation for real forms of E_6.**

$\underline{E_{6(6)}}$ The maximal compact subgroup $K \subset E_{6(6)}$ has Lie algebra $\mathfrak{sp}_4$. As a K-representation, $\mathfrak{e}_6$ decomposes as

$$\mathfrak{e}_6 = \mathfrak{sp}_4 \oplus W$$

where $W \subset \lambda_8^4$ is the kernel of the contraction map $\lambda_8^4 \to \lambda_8^2$. It is the irreducible representation of K with highest weight $L_1 + L_2 + L_3 + L_4$. So $\mathfrak{p} = W$ and the nonzero weights of the isotropy representation are

$$\begin{aligned} \pm L_i \pm L_j \qquad & 1 \le i < j \le 4 \\ \epsilon_1 L_1 + \cdots + \epsilon_4 L_4 \quad & (\epsilon_1, \ldots, \epsilon_4) \in \{\pm 1\}^4. \end{aligned}$$

Let $S \subset \mathrm{Sp}_4$ be a maximal torus. As an element of $H^*(BS) \simeq \mathbb{Q}[y_1, \dots, y_4]$, the total Chern class is

(4.17)

$$c(\iota_{\mathbb{C}}) = \prod_{1 \leq i < j \leq 4} \big(1 - (y_i + y_j)^2\big)\big(1 - (y_i - y_j)^2\big) \prod \big(1 - (\epsilon_1 y_1 + \epsilon_2 y_2 + \epsilon_3 y_3 + y_4)^2\big).$$

The second product is over all tuples $(\epsilon_1, \epsilon_2, \epsilon_3) \in \{\pm 1\}^3$.

$\underline{E_{6(2)}}$ The maximal compact subgroup $K \subset E_{6(2)}$ has Lie algebra $\mathfrak{su}_6 \times \mathfrak{su}_2$. As a K-representation, $\mathfrak{e}_6$ decomposes as

$$\mathfrak{e}_6 = (\mathfrak{su}_6 \oplus \mathfrak{su}_2) \ \oplus \ (\lambda_6^3 \otimes \lambda_2^1).$$

With respect to the standard basis for $\mathfrak{h}^*(\mathfrak{su}_6 \times \mathfrak{su}_2)$, the isotropy representation $\mathfrak{p} = \lambda_6^3 \otimes \lambda_2^1$ has weights $L_{i_1} + L_{i_2} + L_{i_3} \pm L_7$, where $1 \leq i_1 < i_2 < i_3 \leq 6$.

Let $S \subset K$ be a maximal torus. As an element of $\mathbb{Q}[y_1, \dots, y_7] \twoheadrightarrow H^*(BS)$ the total Chern class is

$$c(\iota_{\mathbb{C}}) = \prod_{1 \leq i_1 < i_2 < i_3 \leq 6} \big(1 - (y_{i_1} + y_{i_2} + y_{i_3} + y_7)^2\big). \tag{4.18}$$

$\underline{E_{6(-14)}}$ The maximal compact subgroup $K \subset E_{6(-14)}$ has Lie algebra $\mathfrak{so}_{10} \times \mathfrak{so}_2$. According to [1] (pg. 53), as a K-representation $\mathfrak{e}_6$ decomposes as

$$\mathfrak{e}_6 = \big(\mathfrak{so}_{10} + \mathfrak{so}_2\big) \oplus \big(\Delta^+ \otimes \xi^3 + \Delta^- \otimes \xi^{-3}\big)$$

where $\xi : \mathrm{SO}_2 \to \mathbb{C}^\times$ denotes the identity representation of SO_2 (or rather the induced Lie algebra representation), and $\xi^k : \mathrm{SO}_2 \to \mathbb{C}^\times$ denotes the k-th power of ξ.

With respect to the standard basis $\mathfrak{h}^*(\mathfrak{so}_{10} \times \mathfrak{so}_2) = \langle L_1, \dots, L_5 \rangle \oplus \langle L_6 \rangle$, the isotropy representation $\mathfrak{p} = \Delta^+ \otimes \xi^3 + \Delta^- \otimes \xi^{-3}$ has weights

$$\frac{1}{2}\big(\epsilon_1 L_1 + \cdots + \epsilon_5 L_5\big) + 3\epsilon_6 L_6,$$

where $(\epsilon_1, \cdots, \epsilon_6) \in \{\pm 1\}^6$ has an even number of -1's.

Let $S \subset K$ be a maximal torus. As an element of $H^*(BS) \simeq \mathbb{Q}[y_1, \dots, y_5] \otimes \mathbb{Q}[y_6]$, the total Chern class of the isotropy representation is

$$c(\iota_{\mathbb{C}}) = \prod \left(1 - \frac{1}{4}(\epsilon_1 y_1 + \cdots + \epsilon_5 y_5 + 6y_6)^2\right), \tag{4.19}$$

with the product over the tuples $(\epsilon_1, \dots, \epsilon_5) \in \{\pm 1\}^5$ with an even number of -1's.

$\underline{E_{6(-26)}}$ The maximal compact subgroup $K \subset E_{6(-26)}$ has Lie algebra $\mathfrak{f}_4$. Following [1] (pg. 95), as a K-representation, $\mathfrak{e}_6$ decomposes

$$\mathfrak{e}_6 = \mathfrak{f}_4 \ \oplus \ U$$

where U is the 26-dimensional fundamental representation of $\mathfrak{f}_4$. With respect to the standard basis $\mathfrak{h}^*(\mathfrak{f}_4) = \langle L_1, \dots, L_4 \rangle$, the isotropy representation $\mathfrak{p} \simeq U$ has weights $\pm L_i$ for $1 \leq i \leq 4$ and $\frac{1}{2}(\pm L_1 \pm L_2 \pm L_3 \pm L_4)$.

Let $S \subset F_4$ be a maximal torus. As an element of $H^*(BS) \simeq \mathbb{Q}[y_1, y_2, y_3, y_4]$, the total Chern class of the isotropy representation is

$$c(\iota_{\mathbb{C}}) = \prod_{i=1}^{4}(1 - y_i^2) \prod_{(\epsilon_1, \epsilon_2, \epsilon_3) \in \{\pm 1\}^3} \left(1 - \frac{1}{4}(\epsilon_1 y_1 + \epsilon_2 y_2 + \epsilon_3 y_3 + y_4)^2\right). \tag{4.20}$$

4.11. **Isotropy representation for real forms of F_4.**

$\underline{F_{4(4)}}$ The maximal compact subgroup $K \subset F_{4(4)}$ has Lie algebra $\mathfrak{su}_2 \times \mathfrak{sp}_3$. As a K-representation, $\mathfrak{f}_4$ decomposes as

$$\mathfrak{f}_4 = (\mathfrak{su}_2 \oplus \mathfrak{sp}_3) \ \oplus \ \lambda_2^1 \otimes W$$

where $W \subset \lambda_6^3$ is the kernel of the contraction map $\lambda_6^3 \to \lambda_6^1$ (it is an irreducible representation of Sp_3 of dimension 14).

Since $K \subset F_{4(4)}$ have the same rank, we can identify $\mathfrak{h}^*(\mathfrak{f}_4) = \langle J_1, J_2, J_3, J_4\rangle$ and $\mathfrak{h}^*(\mathfrak{su}_2 \times \mathfrak{sp}_3) = \langle L_1\rangle \oplus \langle L_2, L_3, L_4\rangle$.

The roots of F_4 are

$$\begin{cases} \pm J_i & 1 \le i \le 4 \\ \pm J_i \pm J_j & 1 \le i < j \le 4 \\ \frac{1}{2}(\pm J_1 \pm J_2 \pm J_3 \pm J_4) & \end{cases}$$

According to [7] (pg. 390), the roots of $\mathfrak{su}_2 \times \mathfrak{sp}_3 \subset \mathfrak{f}_4$ are

$$\pm(J_1 - J_2),\ \pm(J_1 + J_2),\ \pm J_3,\ \pm J_4,\ \pm J_3 \pm J_4,$$
$$\tfrac{1}{2}\big(\pm(J_1 + J_2) \pm J_3 \pm J_4\big)$$

We can identify these roots with the roots of $\mathfrak{su}_2 \times \mathfrak{sp}_3$ inside $\mathfrak{h}^*(\mathfrak{su}_2 \times \mathfrak{sp}_3)$ by the identification

$$\begin{aligned} J_1 - J_2 &\leftrightarrow 2L_1 \\ J_1 + J_2 &\leftrightarrow 2L_2 \\ J_3 + J_4 &\leftrightarrow 2L_3 \\ J_3 - J_4 &\leftrightarrow 2L_4 \end{aligned}$$

Under this identification, the weights of the isotropy representation $\mathfrak{p} = W \otimes \lambda_2^1$ will be the roots of $\mathfrak{f}_4$ that are not roots of $\mathfrak{su}_2 \times \mathfrak{sp}_3$. Then the weights of the isotropy representation are $\pm L_1 \pm L_i$ for $i = 2, 3, 4$ and $\pm L_1 \pm L_2 \pm L_3 \pm L_4$ (for all 16 sign choices).

Let $S \subset K$ be a maximal torus. As an element of $H^*(BS) \simeq \mathbb{Q}[y_1] \otimes \mathbb{Q}[y_2, y_3, y_4]$, the total Chern class for the isotropy representation is

$$c(\iota_{\mathbb{C}}) = \prod_{i=2}^{4} \big(1-(y_1+y_i)^2\big)\big(1-(y_1-y_i)^2\big) \prod \big(1-(y_1+\epsilon_2 y_2+\epsilon_3 y_3+\epsilon_4 y_4)^2\big) \tag{4.21}$$

where the second product is over all tuples $(\epsilon_2, \epsilon_3, \epsilon_4) \in \{\pm 1\}^3$.

$\underline{F_{4(-20)}}$ The maximal compact subgroup $K \subset F_{4(-20)}$ has Lie algebra $\mathfrak{so}_9$. Following [1] (pg. 51), $\mathfrak{f}_4$ decomposes as a representation of K as

$$\mathfrak{f}_4 = \mathfrak{so}_9 \oplus \Delta$$

where Δ is the spin representation of K. The isotropy representation $\mathfrak{p} = \Delta$ has weights $\frac{1}{2}(\pm L_1 \pm \cdots \pm L_4)$.

Let $S \subset K$ be the maximal torus. As an element of $H^*(BS) \simeq \mathbb{Q}[y_1, \ldots, y_4]$, the total Chern class of the isotropy representation is

$$c(\iota_{\mathbb{C}}) = \prod_{(\epsilon_1,\epsilon_2,\epsilon_3)\in\{\pm 1\}^3} \left(1 - \frac{1}{4}(\epsilon_1 y_1 + \epsilon_2 y_2 + \epsilon_3 y_3 + y_4)^2\right). \tag{4.22}$$

4.12. **Isotropy representation for real forms of G_2.** The maximal compact subgroup of $G_{2(2)}$ is $K = \mathrm{SU}_2 \times \mathrm{SU}_2$. As a K-representation, $\mathfrak{g}_2$ decomposes

$$\mathfrak{g}_2 = \big(\mathfrak{su}_2 \times \mathfrak{su}_2\big) + \mathrm{Sym}^2(V) \otimes V$$

where V is the standard representation of SU_2, and $\mathrm{Sym}^2(V)$ is the 2nd symmetric power. The weights of $\mathfrak{p} = \mathrm{Sym}^2(V) \otimes V$ are $\pm L_1 \pm L_2$ and $\pm 3L_1 \pm L_2$. (Compare with Lecture 22 of [6] or pg. 393 of [7].)

Let $S \subset \mathrm{SU}_2 \times \mathrm{SU}_2$ be a maximal torus. As an element of $H^*(BS) \simeq \mathbb{Q}[y_1, y_2]$, the total Chern class is

$$c(\iota_{\mathbb{C}}) = (1 - (y_2 + 3y_1)^2)(1 - (y_2 + y_1)^2)(1 - (y_2 - y_1)^2)(1 - (y_2 - 3y_1)^2). \tag{4.23}$$

5. Computing $\alpha_2^* : H^*(BG) \to H^*(BG^\delta)$

The following theorem allows us to compute $\alpha_2 : H^*(BG) \to H^*(BG^\delta)$ for G a real, semisimple Lie group. See Milnor [14], Theorem 2.

Theorem 5.1 (Milnor)**.** *Let G be a real, simple, connected Lie group. Assume that its complexification $G_{\mathbb{C}}$ is simple. Then sequence*

$$H^*(BG_{\mathbb{C}}; \mathbb{Q}) \xrightarrow{i^*} H^*(BG; \mathbb{Q}) \xrightarrow{\alpha_2^*} H^*(BG^\delta; \mathbb{Q})$$

induced by the maps $G^\delta \to G \to G_{\mathbb{C}}$ is "exact" in the sense that the kernel of α_2^* is the ideal generated by the image of $i^* : H^k(BG_{\mathbb{C}}) \to H^k(BG)$ for $k > 0$.

Theorem 5.1 applies to all the groups G in Table 2. If G is one of the complex Lie groups from Table 1, then $G_{\mathbb{C}} \simeq G \times G$ is not simple, so Theorem 5.1 does not apply. In this case, we have the following theorem, whose proof comes from Chern-Weil theory (see Milnor [14] Lemma 11).

Theorem 5.2. *Let G be a complex, simple Lie group with finitely many components. Then $\alpha_2^* : H^i(BG; \mathbb{Q}) \to H^i(BG^\delta; \mathbb{Q})$ is zero for $i > 0$.*

By Theorem 5.2, if G is one of the groups from Table 1, then α_2^* is zero in positive degrees and so $p_i(\Gamma\backslash G/K) = 0$ for $i > 0$. This proves Theorem 1.2 for the groups in Table 1.

In the remainder of this section we use Theorem 5.1 to compute α_2^* for all the G in Table 2. Let G be one of the groups in Table 2 with maximal compact subgroup K. Let $G_{\mathbb{C}}$ be the complexification of G and let U be the maximal compact subgroup of $G_{\mathbb{C}}$. Let $S \subset K$ and $S' \subset U$ be maximal tori such that complexification $G \to G_{\mathbb{C}}$ sends S into S'. We have the following diagram of inclusions, and an induced diagram on cohomology.

$$\begin{array}{ccc} G & \longrightarrow & G_{\mathbb{C}} \\ \uparrow & & \uparrow \\ K & \longrightarrow & U \\ \uparrow & & \uparrow \\ S & \longrightarrow & S' \end{array} \qquad \begin{array}{ccc} H^*(BG) & \xleftarrow{i^*} & H^*(BG_{\mathbb{C}}) \\ \downarrow \simeq & & \downarrow \simeq \\ H^*(BK) & \longleftarrow & H^*(BU) \\ \downarrow \simeq & & \downarrow \simeq \\ H^*(BS)^W & \xleftarrow{j^*} & H^*(BS')^{W'} \\ \downarrow & & \downarrow \\ H^*(BS) & \longleftarrow & H^*(BS') \end{array}$$

In the diagram on the right, the top two vertical arrows are isomorphisms because $K \hookrightarrow G$ and $U \hookrightarrow G_{\mathbb{C}}$ are homotopy equivalences. The middle two vertical arrows are isomorphisms by Theorem 3.1. Thus to compute the map i^* it is enough to compute j^*. We do this in the remainder of this section.

5.1. **Computing α_2^* for $G = \mathbf{SL}_n(\mathbb{R})$.** Here $K = \mathrm{SO}_n$ and $U = \mathrm{SU}_n$. For $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, let

$$\underline{\theta} = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}.$$

Let $k = [n/2]$. The map $S \to S'$ sends

$$\begin{pmatrix} \underline{\theta}_1 & & & \\ & \ddots & & \\ & & \underline{\theta}_k & \\ & & & (1) \end{pmatrix} \mapsto \begin{pmatrix} D & & \\ & D^{-1} & \\ & & (1) \end{pmatrix} \tag{5.1}$$

where

$$D = \begin{pmatrix} e^{i\theta_1} & & \\ & \ddots & \\ & & e^{i\theta_k} \end{pmatrix},$$

and the 1 in (5.1) appears only if n is odd. Identify

$$H^*(BS')^{W'} \simeq \frac{\mathrm{Sym}(x_1 \ldots, x_n)}{(x_1 + \cdots + x_n)}$$

and

$$H^*(BS)^W \simeq \begin{cases} \mathrm{Sym}(y_1^2, \ldots, y_k^2) & n = 2k+1 \\ \langle \mathrm{Sym}(y_1^2, \ldots, y_k^2), y_1 \cdots y_k \rangle & n = 2k \end{cases}$$

From (5.1) we conclude that the image of $j^* : H^*(BS')^{W'} \to H^*(BS)^W$ is $\mathrm{Sym}(y_1^2, \ldots, y_k^2)$.

5.2. **Computing α_2^* for $G = \mathbf{SU}_{p,q}$.** Here $K = S(U_p \times U_q)$ and $U = \mathrm{SU}_{p+q}$, and the inclusion $S \hookrightarrow S'$ is an isomorphism. Identify

$$H^*(BS')^{W'} \simeq \frac{\mathrm{Sym}(x_1 \ldots, x_{p+q})}{(x_1 + \cdots + x_{p+q})}$$

and

$$H^*(BS)^W \simeq \frac{\mathrm{Sym}(y_1 \ldots, y_p) \otimes \mathrm{Sym}(z_1, \ldots, z_q)}{(y_1 + \cdots + y_p + z_1 + \cdots + z_q)}.$$

The image of $j^* : H^*(BS')^{W'} \to H^*(BS)^W$ is $\mathrm{Sym}(y_1, \ldots, y_p, z_1, \ldots, z_q)$.

Corollary 5.3. *Let $p, q \geq 2$ and let $G = \mathrm{SU}_{p,q}$. Then $\alpha_2^* : H^*(BG) \to H^*(BG^\delta)$ is injective on the linear subgroup generated by $\sum_{i<j} y_i y_j$ and $\left(\sum y_i\right)^2$.*

Proof. For $1 \leq i \leq p$ (resp. $1 \leq i \leq q$), let σ_i^y (resp. σ_i^z) denote the i-th elementary symmetric polynomial in $\{y_1, \ldots, y_p\}$ (resp. $\{z_1, \ldots, z_q\}$), viewed as elements of $H^*(BS)^W \simeq H^*(BG)$. Similarly, let $\sigma_i^{y,z}$ denote the i-th elementary symmetric polynomial in $\{y_1, \ldots, y_p, z_1, \ldots, z_q\}$. Let $\mathcal{I}$ denote the ideal

$$\mathcal{I} = (\sigma_1^{y,z}, \ldots, \sigma_{p+q}^{y,z}).$$

The following relations are easy to verify.

$$\sigma_1^{y,z} = \sigma_1^y + \sigma_1^z, \quad \text{and} \quad \sigma_2^{y,z} = \sigma_2^y + \sigma_1^y \sigma_1^z + \sigma_2^z. \tag{5.2}$$

Note that $(\sigma_1^y)^2 = \big(\sum y_i\big)^2$ and $\sigma_2^y = \sum_{i<j} y_i y_j$. By Theorem 5.1, to prove the corollary it is enough to show that no nontrivial linear combination $c\cdot\sigma_2^y + d\cdot(\sigma_1^y)^2$ belongs to $\mathcal{I}$. The elements of $\mathcal{I}$ that have total degree 2 all have the form

$$(a\sigma_1^y + b\sigma_1^z)\cdot\sigma_1^{y,z} + m\cdot\sigma_2^{y,z}$$

where a, b, m are scalars. By simple linear algebra, one checks that if

$$c\cdot\sigma_2^y + d\cdot(\sigma_1^y)^2 = (a\sigma_1^y + b\sigma_1^z)\cdot\sigma_1^{y,z} + m\cdot\sigma_2^{y,z},$$

then $m = a = b = 0$. This implies $c = d = 0$ because σ_2^y and $(\sigma_1^y)^2$ are linearly independent in $H^*(BS)^W$. Hence no nontrivial linear combination $c\cdot\sigma_2^y + d\cdot(\sigma_2^y)^2$ belongs to $\mathcal{I}$. □

Corollary 5.4. *Let $p \geq 2$ and let $G = \mathrm{SU}_{p,1}$. Then $\alpha_2^* : H^*(BG) \to H^*(BG^\delta)$ is nonzero on $\sum_{i<j} y_i y_j$, and*

$$\alpha_2^*\big(\sum y_i\big)^2 = \alpha_2^*\big(\sum_{i<j} y_i y_j\big).$$

Proof. We use the same notation as in the proof of Corollary 5.3. Since $q = 1$ in the present case, the relations in (5.2) simplify to the following relations.

$$\sigma_1^{y,z} = \sigma_1^y + \sigma_1^z, \quad \text{and} \quad \sigma_2^{y,z} = \sigma_2^y + \sigma_1^y\sigma_1^z. \tag{5.3}$$

Since the elements σ_i^y, σ_j^z are all linearly independent in $H^2(BS)^W$, it is a simple matter of linear algebra to show that σ_2^y is not in $\mathcal{I}$, and hence is not in the kernel of α_2^*.

To see that $\alpha_2^*\big((\sigma_1^y)^2\big) = \alpha_2^*(\sigma_2^y)$, note that the first equation in (5.3) implies that

$$\sigma_1^y\cdot\sigma_1^{y,z} = (\sigma_1^y)^2 + \sigma_1^y\sigma_1^z \in \mathcal{I}.$$

Then $\sigma_2^y - (\sigma_1^y)^2 = \sigma_2^{y,z} - \sigma_1^y\cdot\sigma_1^{y,z}$ belongs to $\mathcal{I}$ (the right hand side obviously does). This implies that σ_2^y and $(\sigma_1^y)^2$ have the same image under α_2^*. □

5.3. **Computing α_2^* for $G = \mathbf{Sp}_{2n}(\mathbb{R})$.** Here $K = U_n$ and $U = \mathrm{Sp}(n)$, and the inclusion $S \hookrightarrow S'$ is an isomorphism. Identify

$$H^*(BS')^{W'} \simeq \mathrm{Sym}(x_1^2, \dots, x_n^2) \quad \text{and} \quad H^*(BS)^W \simeq \mathrm{Sym}(y_1, \dots, y_n).$$

The image of $j^* : H^*(BS')^{W'} \to H^*(BS)^W$ is $\mathrm{Sym}(y_1^2, \dots, y_n^2)$.

Corollary 5.5. *Let $G = \mathrm{Sp}_{2n}(\mathbb{R})$. Then $\alpha_2^*\big(\sum y_i\big)^2 \neq 0$.*

Proof. Let σ_i (resp. ω_i) denote the i-th elementary symmetric polynomial in $\{y_1, \dots, y_n\}$ (resp. $\{y_1^2, \dots, y_n^2\}$). Let $\mathcal{I}$ denote the ideal $(\omega_1, \dots, \omega_n)$.

By Theorem 5.1, to show $\sigma_1^2 = \big(\sum y_i\big)^2$ is not in $\ker\alpha_2^*$, it is enough to show $\sigma_1^2 \notin \mathcal{I}$. Note that $\omega_1 = \sigma_1^2 - 2\sigma_2$ and that the only elements of $\mathcal{I}$ of degree 2 are scalar multiples of ω_1. Since σ_1^2 is obviously not a multiple of $\sigma_1^2 - 2\sigma_2$, we conclude $\sigma_1^2 \notin \mathcal{I}$. □

5.4. **Computing α_2^* for $G = \mathbf{SO}_{p,q}$.** Here $K = \mathrm{SO}_p \times \mathrm{SO}_q$ and $U = \mathrm{SO}_{p+q}$. Let $m = [(p+q)/2]$, let $a = [p/2]$, and let $b = [q/2]$. Identify

$$H^*(BS') \simeq \mathbb{Q}[x_1, \ldots, x_m] \quad \text{and} \quad H^*(BS) \simeq [y_1, \ldots, y_p, z_1, \ldots, z_q].$$

It is not hard to see that the image of $j^* : H^*(BS')^{W'} \to H^*(BS)^W$ contains $\mathrm{Sym}(y_1^2, \ldots, y_p^2, z_1^2, \ldots, z_q^2)$.

Corollary 5.6. *Let $p, q \geq 2$ and let $G = \mathrm{SO}_{p,q}$. Then $\alpha_2^*\big(\sum y_i^2\big) \neq 0$.*

Proof. The proof is the same as in Corollary 5.5. The image of

$$H^4(BS')^{W'} \simeq H^4(BG_{\mathbb{C}}) \to H^4(BG) \simeq H^4(BS)^W$$

is generated by multiples of $\sum y_i^2 + \sum z_j^2$, and $\sum y_i^2$ does not have this form. □

Corollary 5.7. *Let $p \geq 4$ and let $G = \mathrm{SO}_{p,p}$. Then $\alpha_2^* : H^*(BG) \to H^*(BG^\delta)$ is injective on the linear subgroup generated by $\sum_{i<j} y_i^2 y_j^2$ and $\big(\sum y_i^2\big)^2$.*

Proof. The proof is identical to the proof of Corollary 5.3 after replacing y_i and z_j by y_i^2 and z_j^2. □

5.5. **Computing α_2^* for $G = \mathbf{Sp}_{p,q}$.** Here $K = \mathrm{Sp}(p) \times \mathrm{Sp}(q)$ and $U = \mathrm{Sp}(p+q)$, and the inclusion $S \hookrightarrow S'$ is an isomorphism. Identify

$$H^*(BS')^{W'} \simeq \mathrm{Sym}(x_1^2, \ldots, x_{p+q}^2)$$

and

$$H^*(BS)^W \simeq \mathrm{Sym}(y_1^2, \ldots, y_p^2) \otimes \mathrm{Sym}(z_1^2, \ldots, z_q^2).$$

The image of $j^* : H^*(BS')^{W'} \to H^*(BS)^W$ is $\mathrm{Sym}(y_1^2, \ldots, y_p^2, z_1^2, \ldots, z_q^2)$. We have the following Corollaries 5.8 and 5.9 whose proofs are identical to the proofs for Corollaries 5.6 and 5.7, respectively.

Corollary 5.8. *Let $p, q \geq 1$ and let $G = \mathrm{Sp}_{p,q}$. Then $\alpha_2^*\big(\sum y_i^2\big) \neq 0$.*

Corollary 5.9. *Let $q \geq 1$ and $p \geq 2$, and let $G = \mathrm{Sp}_{p,q}$. Then α_2^2 is injective on $\big(\sum y_i\big)^2$. If $p, q \geq 2$, then α_2^* is injective on the linear subgroup generated by $\sum_{i<j} y_i^2 y_j^2$ and $\big(\sum y_i^2\big)^2$.*

5.6. **Computing α_2^* for $G = \mathbf{SO}_{2n}^*$.** Here $K = U_n$ and $U = \mathrm{SO}_{2n}$, and the inclusion $S \hookrightarrow S'$ is an isomorphism. Identify

$$H^*(BS') \simeq \mathbb{Q}[x_1, \ldots, x_n] \quad \text{and} \quad H^*(BS) \simeq \mathbb{Q}[y_1, \ldots, y_n].$$

The image of

$$j^* : H^*(BS')^{W'} \to H^*(BS)^W = \mathrm{Sym}(y_1, \ldots, y_n)$$

contains $\mathrm{Sym}(y_1^2, \ldots, y_n^2)$. We have the following corollary, whose proof is identical to the proof of Corollary 5.5.

Corollary 5.10. *Let $G = \mathrm{SO}_{2n}^*$. Then $\alpha_2^*\big(\sum y_i\big)^2 \neq 0$.*

5.7. **Computing α_2^* for $G = \mathbf{SU}_{2n}^*$.** Here $K = \mathrm{Sp}_n$ and $U = \mathrm{SU}_{2n}$. Identify

$$H^*(BS')^{W'} \simeq \frac{\mathrm{Sym}(x_1, \dots, x_{2n})}{(x_1 + \cdots + x_{2n})}$$

and

$$H^*(BS)^W \simeq \mathrm{Sym}(y_1^2, \dots, y_n^2).$$

It is not hard to see that $j^* : H^*(BS')^{W'} \to H^*(BS)^W$ is surjective.

Corollary 5.11. *Let $G = \mathrm{SU}_{2n}^*$. For $i > 0$, the map $\alpha_2^* : H^i(BG) \to H^i(BG^\delta)$ is zero.*

5.8. **Computing α_2^* for G a real form of E_8.**

$\underline{E_{8(8)}}$ Here $\mathrm{Lie}(K) = \mathfrak{so}_{16}$ and $U = E_8$, and the inclusion $S \hookrightarrow S'$ is an isomorphism. As in Section 3 we identify

$$H^1(S') \simeq \langle J_1, \dots, J_8 \rangle \quad \text{and} \quad H^1(S) \simeq \langle L_1, \dots, L_8 \rangle.$$

Under $H^1(S') \to H^1(S)$ we have $J_i \mapsto L_i$. Let $z_i = \tau(J_i) \in H^2(BS')$ and let $y_i = \tau(L_i) \in H^2(BS)$. In Section 3, we explained that

$$H^*(BS')^{W'} \simeq \mathbb{Q}[I_2, I_8, I_{12}, I_{14}, I_{18}, I_{20}, I_{24}, I_{30}].$$

Under $j^* : H^4(BS') \to H^4(BS)$ the polynomial I_2 maps to a multiple of $y_1^2 + \cdots + y_8^2$. By Theorem 5.1 the elements of $H^8(BS)^W$ in the kernel α_2^* are multiples of

$$\left(y_1^2 + \cdots + y_8^2\right)^2 = \left(\sum_{i=1}^8 y_i^4\right) + 2\left(\sum_{1 \le i < j \le 8} y_i^2 y_j^2\right)$$

This implies the following corollary.

Corollary 5.12. *Let $G = E_{8(8)}$. With the notation above, $\alpha_2^*\big(\sum_{i=1}^8 y_i^4\big) \neq 0$.*

$\underline{E_{8(-24)}}$ Here $K = E_7 \times \mathrm{SU}_2$ and $U = E_8$, and the inclusion $S \hookrightarrow S'$ is an isomorphism. Identify $H^1(S') \simeq \langle J_1, \dots, J_8 \rangle$ and

$$H^1(S) \simeq \langle J_1, \dots, J_6, J_7 + J_8 \rangle \oplus \langle J_7 - J_8 \rangle.$$

Let $z_i = \tau(L_i) \in H^2(BS')$ for $1 \le i \le 8$. Let $y_i = \tau(L_i) \in H^2(BS)$ for $1 \le i \le 6$, let $y_7 = \tau(J_7 + J_8)$ and $y_8 = \tau(L_7 - L_8)$. In Section 3 we explained that

$$H^*(BS')^{W'} \simeq \mathbb{Q}[I_2, I_8, I_{12}, I_{14}, I_{18}, I_{20}, I_{24}, I_{30}],$$

and $I_2 = 30(z_1^2 + \cdots + z_8^2)$. Under $j^* : H^2(BS') \to H^2(BS)$, $z_i \mapsto y_i$ for $1 \le i \le 6$, $z_7 \mapsto \frac{y_7 + y_8}{2}$, and $z_8 \mapsto \frac{y_7 - y_8}{2}$, so

$$j^*(I_2) = 30(y_1^2 + \cdots + y_6^2) + 15(y_7^2 + y_8^2).$$

By Theorem 5.1, every element of the kernel $H^4(BE_{8(-24)}) \to H^4\big(B(E_{8(-24)})^\delta\big)$ is a scalar multiple of $j^*(I_2)$. This implies the following corollary.

Corollary 5.13. *Let $G = E_{8(-24)}$. With the notation above, $\alpha_2^*(y_8^2) \neq 0$.*

5.9. **Computing α_2^* for G a real form of E_7.**

$\underline{E_{7(7)}}$ Here $\mathrm{Lie}(K) = \mathfrak{su}_8$ and $U = E_7$, and the inclusion $S \hookrightarrow S'$ is an isomorphism. We identify $H^1(S')$ as the subspace of $\langle J_1, \ldots, J_8\rangle$ orthogonal to $J_1 + \cdots + J_8$, and we identify $H^1(S)$ as the subspace of $\langle L_1, \ldots, L_8\rangle$ orthogonal to $L_1 + \cdots + L_8$. Then $H^1(S') \to H^1(S)$ is the obvious map $J_i \mapsto L_i$. Let $z_i = \tau(J_i) \in H^2(BS')$ and $y_i = \tau(L_i) \in H^2(BS)$. In Section 3 we explained that

$$H^*(BS')^{W'} \simeq \mathbb{Q}[I_2, I_6, I_8, I_{10}, I_{12}, I_{14}, I_{18}],$$

and computed $I_2 = \frac{3}{4}\left[7\left(\sum z_i^2\right) + 2\left(\sum z_i z_j\right)\right]$ for this copy of $E_7 \subset E_8$. Under $j^* : H^*(BS') \to H^*(BS)$

$$I_2 \mapsto \frac{3}{4}\left[7(\sum y_i^2) + 2(\sum y_i y_j)\right] = -9\left(\sum y_i y_j\right).$$

The equality follows because of the relation $\sum y_i^2 = -2\sum y_i y_j$ in $H^*(BS)^W$. By Theorem 5.1, every element of the kernel of

$$\alpha_2^* : H^8\big(BE_{7(7)}\big) \to H^8\big(B(E_{7(7)})^\delta\big)$$

is a scalar multiple of $\left(\sum y_i^2\right)^2$. This implies the following corollary.

Corollary 5.14. *Let $G = E_{7(7)}$. With the notation above, $\alpha_2^*\big(\sum y_i^4\big) \neq 0$.*

$\underline{E_{7(-5)}}$ Here $\mathrm{Lie}(K) = \mathfrak{so}_{12} \times \mathfrak{su}_2$ and $U = E_7$, and the inclusion $S \hookrightarrow S'$ is an isomorphism. As in Section 3, we identify

$$H^1(S') \simeq \langle J_1, \ldots, J_6, J_7 + J_8\rangle \quad \text{and} \quad H^1(S) \simeq \langle L_1, \ldots, L_6\rangle \oplus \langle L_7\rangle.$$

Under $H^1(S') \to H^1(S)$ we have

$$\begin{cases} J_i & \mapsto & L_i & 1 \le i \le 6 \\ J_7 + J_8 & \mapsto & 2L_7 & \end{cases}$$

Let $z_i = \tau(J_i) \in H^2(BS')$ for $1 \le i \le 6$ and let $z_7 = \tau(J_7 + J_8)$. Let $y_i = \tau(L_i) \in H^2(BS)$. In Section 3 we explained that

$$H^*(BS')^{W'} \simeq \mathbb{Q}[I_2, I_6, I_8, I_{10}, I_{12}, I_{14}, I_{18}].$$

Under $j^* : H^4(BS') \to H^4(BS)$,

$$I_2 \mapsto 6(y_1^2 + y_2^2 + y_3^2 + y_4^2 + y_5^2 + y_6^2 + 2y_7^2).$$

In particular, by Theorem 5.1 every element of the kernel of

$$\alpha_2^* : H^4(BE_{7(-5)}) \to H^4\big(B(E_{7(-5)})^\delta\big)$$

is a scalar multiple of $j^*(I_2)$. This implies the following corollary.

Corollary 5.15. *Let $G = E_{7(5)}$. With the notation above, $\alpha_2^*(y_7^2) \neq 0$.*

$\underline{E_{7(-25)}}$ Here $\mathrm{Lie}(K) = \mathfrak{e}_6 \times \mathfrak{so}_2$ and $U = E_7$, and the inclusion $S \hookrightarrow S'$ is an isomorphism. Identify $H^1(S') \simeq \langle J_1, \ldots, J_6, J_7 + J_8\rangle$ and

$$H^1(S) \simeq \langle J_1, \ldots, J_5, J_6 + J_7 + J_8\rangle \oplus \langle 2J_6 - J_7 - J_8\rangle.$$

Let $z_i = \tau(J_i) \in H^2(BS')$ for $1 \le i \le, 6$, and let $z_7 = \tau(J_7 + J_8)$. For $1 \le i \le 5$, let $y_i = \tau(J_i) \in H^2(BS)$, and let $y_6 = \tau(J_6 + J_7 + J_8)$ and $y_7 = \tau(2J_6 - J_7 - J_8)$.

In Section 3 we explained that

$$H^*(BS')^{W'} \simeq \mathbb{Q}[I_2, I_6, I_8, I_{10}, I_{12}, I_{14}, I_{18}],$$

where $I_2 = 6(z_1^2 + \cdots + z_6^2) + 3z_7^2$. Under $j^* : H^2(BS') \to H^2(BS)$, $z_i \mapsto y_i$ for $1 \leq i \leq 5$, $z_6 \mapsto \frac{y_6 - y_7}{3}$, $z_7 \mapsto \frac{2y_6 + y_7}{3}$, so

$$j^*(I_2) = 6(y_1^2 + y_2^2 + y_3^2 + y_4^2 + y_5^2) + 2y_6^2 + y_7^2.$$

By Theorem 5.1, $j^*(I_2)$ generates the kernel of

$$\alpha_2^* : H^4\big(BE_{7(-25)}\big) \to H^4\big(B(E_{7(-25)})^\delta\big).$$

In particular, we have the following corollary.

Corollary 5.16. *Let $G = E_{7(-25)}$. With the notation above, $\alpha_2^*(y_7^2) \neq 0$.*

5.10. **Computing α_2^* for G a real form of E_6.**

$\underline{E_{6(6)}}$ Here $\mathrm{Lie}(K) = \mathfrak{sp}_4$ and $U = E_6$. Identify $H^1(S) \simeq \langle L_1, \ldots, L_4 \rangle$ and

$$H^1(S') \subset \langle J_1, \ldots, J_6, J_7 - J_8 \rangle$$

as the subspace orthogonal to the roots in (3.4). Under $\mathfrak{sp}_4 \hookrightarrow \mathfrak{e}_6 \hookrightarrow \mathfrak{e}_8$ the Cartan subalgebra $\mathfrak{h}(\mathfrak{sp}_4) \subset \mathfrak{sp}_4$ is contained in the image of the Cartan subalgebra for $\mathfrak{su}_6 \times \mathfrak{su}_2 \subset \mathfrak{su}_8$. Under $\mathfrak{h}^*(\mathfrak{e}_8) \to \mathfrak{h}^*(\mathfrak{sp}_4)$, we have

$$\begin{array}{rcrl} J_i & \mapsto & L_i & 1 \leq i \leq 3 \\ J_{i+3} & \mapsto & -L_i & 1 \leq i \leq 3 \\ J_7 & \mapsto & L_4 & \\ J_8 & \mapsto & -L_4 & \end{array}$$

It is then easy to determine the restriction of this map to $\mathfrak{h}^*(\mathfrak{e}_6)$.

Let $z_i = \tau(J_i)$ for $1 \leq i \leq 6$ and let $z_7 = \tau(J_7 - J_8)$ in $H^2(BS')$. Let $y_i = \tau(L_i) \in H^2(BS)$ for $1 \leq i \leq 4$. From Section 3, $H^*(BS')^{W'} \simeq \mathbb{Q}[I_2, I_5, I_6, I_8, I_9, I_{12}]$, where

$$I_2 = 5(z_1^2 + \cdots + z_6^2) + 3z_7^2 - 2\left(\sum_{1 \leq i < j \leq 6} z_i z_j\right).$$

Under $j^* : H^*(BS') \to H^*(BS)$,

$$I_2 \mapsto 12(y_1^2 + y_2^2 + y_3^2 + y_4^2).$$

By Theorem 5.1, $j^*(I_2)$ and $j^*(I_2)^2$ generate (as a vector space) the kernel of

$$\alpha_2^* : H^k\big(BE_{6(6)}\big) \to H^k\big(B(E_{6(6)})^\delta\big)$$

for $k = 4, 8$. In particular, we have the following corollary.

Corollary 5.17. *Let $G = E_{6(6)}$. With the notation above, $\alpha_2^*\big(\sum y_i^4\big) \neq 0$.*

$\underline{E_{6(2)}}$ Here $\mathrm{Lie}(K) = \mathfrak{su}_6 \times \mathfrak{su}_2$ and $U = E_6$, and the inclusion $S \hookrightarrow S'$ is an isomorphism. We identify $H^1(S') \subset \langle J_1, \ldots, J_6, J_7 - J_8 \rangle$ as the subspace of $\langle J_1, \ldots, J_8 \rangle$ orthogonal to the roots in (3.4). We identify $H^1(S)$ as the subspace of $\langle L_1, \ldots, L_6 \rangle \oplus \langle L_7 \rangle$ orthogonal to $L_1 + \cdots + L_6$. The map $H^1(S') \to H^1(S)$ is given by $J_i \mapsto L_i$ for $1 \leq i \leq 6$ and $J_7 - J_8 \mapsto 2L_i$.

Let $z_i = \tau(J_i) \in H^2(BS')$ for $1 \leq i \leq 6$ and let $z_7 = \tau(J_7 - J_8)$. Let $y_i = \tau(L_i) \in H^2(BS)$ for $1 \leq i \leq 7$. In Section 3, we explained that

$$H^*(BS')^{W'} \simeq \mathbb{Q}[I_2, I_5, I_6, I_8, I_9, I_{12}],$$

where

$$I_2 = 5(z_1^2 + \cdots + z_6^2) + 3\, z_7^2 - 2\left(\sum_{1\le i<j\le 6} z_i z_j\right).$$

Under $j^* : H^*(BS') \to H^*(BS)$,

$$I_2 \mapsto 5(y_1^2 + \cdots + y_6^2) + 12\, y_7^2 - 2\left(\sum_{1\le i<j\le 6} y_i y_j\right).$$

Since $\sum y_i^2 = -2\sum y_i y_j$ in $H^*(B\mathrm{SU}_6)$, we can re-write this polynomial

$$j^*(I_2) = 6\big(y_1^2 + \cdots + y_6^2 + 2y_7^2\big).$$

By Theorem 5.1, the kernel of

$$\alpha_2^* : H^4(BE_{6(2)}) \to H^4\big(B(E_{6(2)})^\delta\big)$$

is generated by scalar multiples of $j^*(I_2)$. In particular, we have the following corollary.

Corollary 5.18. *Let $G = E_{6(2)}$. With the notation above, $\alpha_2^*(y_7^2) \neq 0$.*

$\underline{E_{6(-14)}}$ Here $\mathrm{Lie}(K) = \mathfrak{so}_{10} \times \mathfrak{so}_2$ and $U = E_6$, and the inclusion $S \hookrightarrow S'$ is an isomorphism. As in Section 3 we identify

$$H^1(S') \simeq \langle J_1, \ldots, J_5, J_6 + J_7 + J_8\rangle \quad \text{and} \quad H^1(S) \simeq \langle L_1, \ldots, L_5\rangle \oplus \langle L_6\rangle.$$

Under $H^1(S') \to H^1(S)$ we have $J_i \mapsto L_i$ for $1 \le i \le 5$ and $J_6 + J_7 + J_8 \mapsto L_6$. Let $z_i = \tau(J_i) \in H^2(BS')$ for $1 \le i \le 5$ and let $w = \tau(J_6 + J_7 + J_8)$. Let $y_i = \tau(L_i) \in H^2(BS)$. In Section 3, we explained that

$$H^*(BS')^{W'} \simeq \mathbb{Q}[I_2, I_5, I_6, I_8, I_9, I_{12}].$$

Under $H^4(BS') \to H^4(BS)$,

$$I_2 \mapsto 6(y_1^2 + y_2^2 + y_3^2 + y_4^2 + y_5^2) + 3\, y_6^2.$$

In particular, by Theorem 5.1 every element of the kernel of

$$\alpha_2^* : H^4(BE_{6(-14)}) \to H^4\big(B(E_{6(-14)})^\delta\big)$$

is a scalar multiple of $j^*(I_2)$. This implies the following corollary.

Corollary 5.19. *Let $G = E_{6(-14)}$. With the notation above, $\alpha_2^*(y_6^2) \neq 0$.*

$\underline{E_{6(-26)}}$ Here $K = F_4$ and $U = E_6$. Identify

$$H^1(S) \simeq \langle L_1, \ldots, L_4\rangle \quad \text{and} \quad H^1(S') = \langle J_1, \ldots, J_5, J_6 + J_7 + J_8\rangle.$$

Let $z_i = \tau(J_i) \in H^2(BS')$ for $1 \le i \le 5$ and let $z_6 = \tau(J_6 + J_7 + J_8)$. For $1 \le i \le 4$, let $y_i = \tau(L_i) \in H^2(BS)$.

Under $H^*(BS') \to H^*(BS)$ we have $z_5, z_6 \mapsto 0$ and $z_i \mapsto y_i$ for $i = 1, \ldots, 4$. By the Equations (3.5) and (3.3), it is easy to see that j^* is surjective.

Corollary 5.20. *Let $G = E_{6(-26)}$. For $i > 0$, the map $\alpha_2^* : H^i\big(BG\big) \to H^i\big(BG^\delta\big)$ is zero.*

5.11. **Computing α_2^* for G a real form of F_4.**

$\underline{F_{4(4)}}$. Here $\mathrm{Lie}(K) = \mathfrak{su}_2 \times \mathfrak{sp}_3$ and $U = F_4$, and the inclusion $S \hookrightarrow S'$ is an isomorphism. As in Section 3 we identify $H^1(S') \simeq \langle J_1, \dots, J_4 \rangle$ and we identify $H^1(S) \simeq \langle L_1 \rangle \oplus \langle L_2, L_3, L_4 \rangle$. Let $z_i = \tau(J_i) \in H^*(BS')$ and $y_i = \tau(L_i) \in H^*(BS)$ for $i = 1, \dots, 4$. In Section 4.11, we described the isomorphism $H^*(S') \to H^*(S)$, and so under $j^* : H^*(BS') \to H^*(BS)$,

$$j : \begin{cases} z_1 & \mapsto & y_1 + y_2 \\ z_2 & \mapsto & y_2 - y_1 \\ z_3 & \mapsto & y_3 + y_4 \\ z_4 & \mapsto & y_3 - y_4 \end{cases}$$

In Section 3 we explained that $H^*(BS')^{W'} \simeq \mathbb{Q}[I_2, I_6, I_8, I_{12}]$, where

$$I_2 = 3(z_1^2 + z_2^2 + z_3^2 + z_4^2).$$

Under $j^* : H^*(BS') \to H^*(BS)$,

$$\begin{aligned} I_2/3 &\mapsto (y_1 + y_2)^2 + (y_2 - y_1)^2 + (y_3 + y_4)^2 + (y_3 - y_4)^2 \\ &= 2(y_1^2 + y_2^2 + y_3^2 + y_4^2). \end{aligned}$$

By Theorem 5.1 the elements of $H^4(BS)^W$ that are in the kernel of α_2^* are all scalar multiples of $y_1^2 + \cdots + y_4^2$. The following corollary will be used in Section 6.

Corollary 5.21. *Let $G = F_{4(4)}$. With the notation above, $\alpha_2^*(y_1^2) \neq 0$.*

$\underline{F_{4(-20)}}$. Here $\mathrm{Lie}(K) = \mathfrak{so}_9$ and $U = F_4$, and the inclusion $S \hookrightarrow S'$ is an isomorphism. As in Section 3,

$$H^1(S') \simeq \langle J_1, \dots, J_4 \rangle \quad \text{and} \quad H^1(S) \simeq \langle L_1, \dots, L_4 \rangle.$$

Under $H^1(S') \to H^1(S)$ we have $J_i \mapsto L_i$. Let $z_i = \tau(J_i) \in H^2(BS')$ and $y_i = \tau(L_i) \in H^2(BS)$ for $i = 1, \dots, 4$. In Section 3 we explained that $H^*(BS')^{W'} \simeq \mathbb{Q}[I_2, I_6, I_8, I_{12}]$, where $I_2 = 3(z_1^2 + z_2^2 + z_3^2 + z_4^2)$. By Theorem 5.1, the elements of $H^8(BS)^W$ that are in the kernel of α_2^* are all scalar multiples of

$$j^*(I_2)^2 = 9 \cdot \big[\big(\sum_{i=1}^4 y_i^4 \big) + 2 \big(\sum_{1 \leq i < j \leq 4} y_i^2 y_j^2 \big) \big].$$

This implies the following corollary.

Corollary 5.22. *Let $G = F_{4(-20)}$. With the notation above, $\alpha_2^*\big(\sum_{1 \leq i < j \leq 4} y_i^2 y_j^2 \big) \neq 0$.*

5.12. **Computing α_2^* for G a real form of G_2.**

$\underline{G_{2(2)}}$ Here $K = \mathrm{SU}_2 \times \mathrm{SU}_2$ and $U = G_2$, and the inclusion $S \hookrightarrow S'$ is an isomorphism. $H^1(S') \simeq \langle J_1, J_2 \rangle$ and $H^1(S) \simeq \langle L_1, L_2 \rangle$, and under $H^1(S') \to H^1(S)$

$$\begin{cases} J_1 & \mapsto & 2L_1 \\ J_2 & \mapsto & -3L_1 + L_2 \end{cases}$$

Let $z_i = \tau(J_i)$ in $H^2(BS')$ for $i = 1, 2$ and $y_i = \tau(L_i)$ in $H^2(BS)$ for $j = 1, 2$.

In Section 3 we explained that $H^*(BS')^{W'} \simeq \mathbb{Q}[I_2, I_6]$, where

$$I_2 = 2(3z_1^2 + 3z_1z_2 + z_2^2).$$

Under $j^* : H^*(BS') \to H^*(BS)$, $I_2 \mapsto 2(3y_1^2 + y_2^2)$. By Theorem 5.1, $3y_1^2 + y_2^2$ generates the kernel of $\alpha_2^* : H^4(BG_{2(2)}) \to H^4\big(B(G_{2(2)})^\delta\big)$. In particular, we have the following corollary.

Corollary 5.23. *Let $G = G_{2(2)}$. With the notation above, $\alpha_2^*(y_1^2) \neq 0$.*

6. Conclusion

Recall that G is a simple Lie group from Table 2 and $\Gamma \subset G$ is a cocompact lattice. Our goal in this section is to find nonzero elements in the image of the composition

$$H^*(B\mathrm{Homeo}(S^{n-1})) \xrightarrow{\alpha_1^*} H^*(BG) \xrightarrow{\alpha_2^*} H^*(BG^\delta) \xrightarrow{\alpha_3^*} H^*(B\Gamma).$$

In Section 4 we showed that $\alpha_1^*(q_i) = p_i(\iota) = (-1)^i c_{2i}(\iota_{\mathbb{C}})$, and we computed the polynomial $c(\iota_{\mathbb{C}})$. The kernel of α_2^* is generated as an ideal by the image of $i^* : H^{>0}(BG_{\mathbb{C}}) \to H^{>0}(BG)$ by Theorem 5.1, and i^* was computed in Section 5. Finally, α_3^* is injective by the following proposition.

Proposition 6.1. *Let $\Gamma \subset G$ be a cocompact lattice. Then the image of $H^*(BG) \to H^*(BG^\delta)$ injects into $H^*(B\Gamma)$.*

The proof of Proposition 6.1 uses a transfer argument and can be found in Section 2.3 of [5]. We can thus combine the computations from Sections 4 and 5 to determine if $p_i(\Gamma\backslash G/K) \neq 0$.

6.1. Pontryagin classes for $\mathbf{SL}_n(\mathbb{R})$-manifolds.

Theorem 6.2. *Let $G = \mathrm{SL}_n(\mathbb{R})$ and let $\Gamma \subset G$ be a cocompact lattice. Then $p_i(\Gamma\backslash G/K) = 0$ for $i > 0$.*

Proof. In Section 4.1 we computed the total Chern class $c(\iota_{\mathbb{C}})$ of the isotropy representation. By Equations (4.1) and (4.2), $c(\iota_{\mathbb{C}})$ is a symmetric polynomial in $\{y_1^2, \dots, y_k^2\}$, where $k = [n/2]$. This implies that $c(\iota_{\mathbb{C}})$ is in the image of

$$H^*(B\mathrm{SL}_n(\mathbb{C})) \to H^*(B\mathrm{SL}_n(\mathbb{R})),$$

by Section 5.1. Then $\alpha_2^*\big(c(\iota_{\mathbb{C}})\big) = 0$ by Theorem 5.1. Hence

$$p_i(M) = \alpha_3^*\alpha_2^*\alpha_1^*(q_i) = (-1)^i\, \alpha_3^*\alpha_2^*\big(c(\iota_{\mathbb{C}})\big) = 0. \qquad \square$$

6.2. Pontryagin classes for $\mathbf{SU}_{p,q}$-manifolds.

Theorem 6.3. *Let $p, q \geq 1$ and $(p, q) \neq (1, 1)$. Let $G = \mathrm{SU}_{p,q}$ and let $\Gamma \subset G$ be a cocompact lattice. Then $p_1(\Gamma\backslash G/K) \neq 0$.*

Note that $\mathrm{SU}_{1,1} \simeq \mathrm{SO}_{2,1}$. This case is covered in Theorem 6.7.

Proof of Theorem 6.3. From Equation (4.3), one computes

$$p_1(\iota) = q\left(\sum y_i^2\right) + p\left(\sum z_j^2\right) - 2\sum y_i z_j.$$

Using the relation $\sum y_i + \sum z_j = 0$ in $H^*(BS)^W \simeq H^*(BG)$, and the relation

$$\alpha_2^*\left(\sum y_i^2 + \sum z_j^2\right) = 0$$

from Section 5.2, it follows that

$$\alpha_2^*\, p_1(\iota) = (q-p+2)\, \alpha_2^* \left(\sum y_i \right)^2 + 2(p-q)\, \alpha_2^* \left(\sum y_i y_j \right).$$

If $q = 1$, then $\alpha_2^*\big(\sum y_i \big)^2 = \alpha_2^*\big(\sum y_i y_j \big)$ by Corollary 5.4 and so

$$\alpha_2^*\, p_1(\iota) = (p+1)\, \alpha_2^*\big(\sum y_i y_j \big),$$

which is nonzero (again, see Corollary 5.4).

If $q \geq 2$, then $\alpha_2^*\big(\sum y_i \big)^2$ and $\alpha_2^*\big(\sum y_i y_j \big)$ are linearly independent by Corollary 5.3, and it follows that $\alpha_2^*\, p_1(\iota) \neq 0$. Hence $p_1(M) = \alpha_3^* \alpha_2^*\, p_1(\iota) \neq 0$ by Proposition 6.1. □

6.3. **Pontryagin classes for $\mathbf{Sp}_{2n}(\mathbb{R})$-manifolds.**

Theorem 6.4. *Fix $n \geq 2$. Let $G = \mathrm{Sp}_{2n}(\mathbb{R})$ and let $\Gamma \subset G$ be a cocompact lattice. Then $p_1(\Gamma\backslash G/K) \neq 0$.*

Proof. From Equation (4.4), one computes

$$p_1(\iota) = (n+3) \sum y_i^2 + 2 \sum y_i y_j.$$

By Section 5.3, $\alpha_2^*\big(\sum y_i^2 \big) = 0$. Using the relation $\big(\sum y_i \big)^2 = \sum y_i^2 + 2 \sum y_i y_j$,

$$\alpha_2^*\, p_1(\iota) = 2\, \alpha_2^*\big(\sum y_i y_j \big) = \alpha_2^*\big(\sum y_i \big)^2,$$

which is nonzero by Corollary 5.5. Hence $p_1(M) = \alpha_3^* \alpha_2^*\, p_1(\iota) \neq 0$ by Proposition 6.1. □

6.4. **Pontryagin classes for $\mathbf{SO}_{p,q}$-manifolds.**

Theorem 6.5. *Let $p, q \geq 2$. Let $G = \mathrm{SO}_{p,q}$ and let $\Gamma \subset G$ be a cocompact lattice. Then $p_1(\Gamma\backslash G/K) \neq 0$ if and only if $p \neq q$.*

Theorem 6.6. *Fix $p \geq 4$. Let $G = \mathrm{SO}_{p,p}$ and let $\Gamma \subset G$ be a cocompact lattice. Then $p_2(\Gamma\backslash G/K) \neq 0$.*

Remark. $\mathrm{SO}_{2,2} \simeq \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$ is not simple and $\mathrm{SO}_{3,3} \simeq \mathrm{SL}_4(\mathbb{R})$ is covered by Theorem 6.2.

Theorem 6.7. *Let $G = \mathrm{SO}_{p,1}$. Let $\Gamma \subset G$ be a cocompact lattice. Then $p_i(\Gamma\backslash G/K) = 0$ for $i > 0$.*

Proof of Theorem 6.5. Let $a = [p/2]$ and $b = [q/2]$.

Case 1. Assume $p = 2a$ and $q = 2b$ are both even. From Equation (4.5) one computes

$$p_1(\iota) = 2b\big(\sum y_i^2 \big) + 2a\big(\sum z_j^2 \big).$$

By Section 5.4, $\alpha_2^*\big(\sum y_i^2 \big) + \alpha_2^*\big(\sum z_j^2 \big) = 0$, and so

$$\alpha_2^*\, p_1(\iota) = 2(b-a)\, \alpha_2^*\big(\sum y_i^2 \big).$$

By Corollary 5.6, $\alpha_2^*\big(\sum y_i^2 \big) \neq 0$. The assumptions that $p \neq q$ and that p, q are both even imply that $\alpha_2^*\, p_1(\iota) \neq 0$. Hence $p_1(M) = \alpha_3^* \alpha_2^*\, p_1(\iota) \neq 0$ by Proposition 6.1.

Case 2. Assume $p = 2a$ and $q = 2b + 1$. From Equation (4.6) one computes

$$p_1(\iota) = (2b+1)\big(\sum y_i^2\big) + 2a\big(\sum z_j^2\big).$$

Similar to Case 1,

$$\alpha_2^*\, p_1(\iota) = (2b - 2a + 1)\, \alpha_2^*\big(\sum y_i^2\big),$$

which is nonzero and so $p_1(M) = \alpha_3^*\alpha_2^*\, p_1(\iota) \neq 0$.

Case 3. Assume $p = 2a + 1$ and $q = 2b$. Using Equation (4.7), in an entirely similar fashion to Case 2,

$$\alpha_2^*\, p_1(\iota) = (2b - 2a - 1)\, \alpha_2^*\big(\sum y_i^2\big),$$

which is nonzero, and this implies $p_1(M) \neq 0$.

Case 4. Assume $p = 2a + 1$ and $q = 2b + 1$. From Equation (4.8) one computes

$$p_1(\iota) = (2b+1)\big(\sum y_i^2\big) + (2a+1)\big(\sum z_j^2\big).$$

Then

$$\alpha_2^*\, p_1(\iota) = 2(b - a)\, \alpha_2^*\big(\sum y_i^2\big),$$

which is nonzero since $p \neq q$ and p and q are both odd. Then $p_1(M) \neq 0$ in this case. □

Proof of Theorem 6.6. We separate the cases when p is even and when p is odd.

Case 1. Assume $p = 2a$ is even. From Equation (4.5) one computes

$$p_2(\iota) = \binom{2a}{2}\sum y_i^4 + \binom{2a}{2}\sum z_j^4 + (2a)^2\sum y_i^2y_k^2 + (2a)^2\sum z_j^2z_\ell^2 + (4a^2-6)\sum y_i^2z_j^2$$

By the computation in Section 5.4,

$$\alpha_2^*\big(\sum y_i^4 + \sum z_j^4\big) = 0 \quad \text{and} \quad \alpha_2^*\big(\sum y_i^2y_k^2 + \sum z_j^2z_\ell^2 + \sum y_i^2z_j^2\big) = 0.$$

From these relations, it follows that

$$\alpha_2^*\, p_2(\iota) = -6\, \alpha_2^*\big(\sum y_i^2z_j^2\big). \tag{6.1}$$

From the fact that $\alpha_2^*\big(\sum y_i^2 + \sum z_j^2\big)^2 = 0$, it follows that

$$\alpha_2^*\big(\sum y_i^2z_j^2\big) = \alpha_2^*\big(\sum y_i^2\big)^2. \tag{6.2}$$

By Corollary 5.7, $\alpha_2^*\big(\sum y_i^2\big)^2 \neq 0$ for $p \geq 4$, so we conclude from (6.1) and (6.2) that $\alpha_2^*\, p_2(\iota) \neq 0$. Then $p_2(M) = \alpha_3^*\alpha_2^*\, p_2(\iota) \neq 0$ by Proposition 6.1.

Case 2. Assume $p = 2a + 1$ is odd. The total Chern class of the isotropy representation is given in Equation (4.8). Let $A = \prod_{i=1}^a(1 - y_i^2)\prod_{j=1}^a(1 - z_j^2)$. From the computation in Section 5.4 (combined with Theorem 5.1), it is immediate that $\alpha_2^*(A) = 1$. Then the computation of $\alpha_2^*\, p_2(\iota)$ is the exact same is in Case 1. Again we conclude $p_2(M) \neq 0$. □

Proof of Theorem 6.7. If $G = \mathrm{SO}_{p,1}$, then $b = 0$ and the total Chern class is a symmetric polynomial in $\{y_1^2, \ldots, y_a^2\}$. In Section 5.4, we saw that all such polynomials are in the image of $H^*(BG_{\mathbb{C}}) \to H^*(BG)$, and hence in the kernel of α_2^* by Theorem 5.1. This implies $p_i(M) = 0$ for all $i \geq 1$. □

6.5. **Pontryagin classes for $\mathrm{Sp}_{p,q}$-manifolds.** It is worth noting the similarity between $\mathrm{Sp}_{p,q}$ and $\mathrm{SO}_{2p,2q}$. On the level of cohomology,

$$H^*(B\mathrm{Sp}_{p,q}) \simeq \mathrm{Sym}(y_1^2, \dots, y_p^2) \otimes \mathrm{Sym}(z_1^2, \dots, z_q^2),$$

while

$$H^*(B\mathrm{SO}_{2p,2q}) \simeq \langle \mathrm{Sym}(y_1^2, \dots, y_p^2), y_1 \cdots y_p \rangle \otimes \langle \mathrm{Sym}(z_1^2, \dots, z_q^2), z_1 \cdots z_q \rangle.$$

Note also from Sections 4.4 and 4.5, the weights of the isotropy representation for both are $\pm y_i \pm z_j$ for $i = 1, \dots, p$ and $j = 1, \dots q$. Then in both cases the total Chern class of the isotropy representation is

$$c(\iota_{\mathbb{C}}) = \prod (1 + y_i + z_j)(1 + y_i - z_j)(1 - y_i + z_j)(1 - y_i - z_j).$$

Finally, by Sections 5.4 and 5.5, for $G = \mathrm{Sp}_{p,q}$ or $G = \mathrm{SO}_{2p,2q}$,

$$\mathrm{Im}\,[j^* : H^*(BG_{\mathbb{C}}) \to H^*(BG)] = \mathrm{Sym}(y_1^2, \dots, y_p^2, z_1^2, \dots, z_q^2),$$

so the relations used to compute $\alpha_2^* p_i(\iota)$ for $\mathrm{SO}_{2p,2q}$ also compute $\alpha_2^* p_i(\iota)$ for $\mathrm{Sp}_{p,q}$. From these observations and from the proofs of Theorems 6.5 and 6.6, we have the following two theorems. (One should replace the use of Corollaries 5.6 and 5.7 in the proofs of Theorems 6.5 and 6.6 by Corollaries 5.8 and 5.9.)

Theorem 6.8. *Let $p, q \geq 1$ such that $(p, q) \neq (1, 1)$. Let $G = \mathrm{Sp}_{p,q}$ and let $\Gamma \subset G$ be a cocompact lattice. Then $p_1(\Gamma\backslash G/K) \neq 0$ if and only if $p \neq q$.*

Note that $\mathrm{Sp}_{1,1} \simeq \mathrm{SO}_{4,1}$, which is treated in Theorem 6.7.

Theorem 6.9. *Fix $p \geq 2$. Let $G = \mathrm{Sp}_{p,p}$ and let $\Gamma \subset G$ be a cocompact lattice. Then $p_2(\Gamma\backslash G/K) \neq 0$.*

6.6. **Pontryagin classes for $\mathbf{SO}_{2n}^*$-manifolds.**

Theorem 6.10. *Fix $n \geq 3$. Let $G = \mathrm{SO}_{2n}^*$ and let $\Gamma \subset G$ be a cocompact lattice. Then $p_1(\Gamma\backslash G/K) \neq 0$.*

Remark 6.11. We only consider $n \geq 3$ because $\mathrm{SO}_2^* \simeq \mathbb{C}$ and SO_4^* is not simple.

Proof. From Equation (4.10), one computes

$$p_1(\iota) = (n-1) \sum y_i^2 + 2 \sum y_i y_j.$$

By the computation in Section 5.6, $\alpha_2^*\big(\sum y_i^2\big) = 0$. Using the relation $\big(\sum y_i\big)^2 = \big(\sum y_i^2\big) + 2\sum y_i y_j$, it follows that

$$\alpha_2^*\, p_1(\iota) = 2\, \alpha_2^*\big(\sum y_i y_j\big) = \alpha_2^*\big(\sum y_i\big)^2,$$

which nonzero by Corollary 5.10. Then $p_1(M) = \alpha_3^* \alpha_2^*\, p_1(\iota) \neq 0$ by Proposition 6.1. □

6.7. **Pontryagin classes for $\mathbf{SU}_{2n}^*$-manifolds.**

Theorem 6.12. *Let $G = \mathrm{SU}_{2n}^*$ and let $\Gamma \subset G$ be a cocompact lattice. Then $p_i(\Gamma\backslash G/K) = 0$ for $i > 0$.*

Proof. By Corollary 5.11, α_2^* is zero in positive degrees. □

6.8. **Pontryagin classes for E_8-manifolds.**

Theorem 6.13. *Let $G = E_{8(8)}$ and let $\Gamma \subset G$ be a cocompact lattice. Then $p_1(\Gamma\backslash G/K) = 0$ and $p_2(\Gamma\backslash G/K) \neq 0$.*

Proof. From Equation (4.12), one computes

$$p_1(\iota) = 16(y_1^2 + \cdots + y_8^2)$$

and

$$p_2(\iota) = 126\left(\sum_{i=1}^{8} y_i^4\right) + 244\left(\sum_{1\leq i<j\leq 8} y_i^2 y_j^2\right).$$

By Theorem 5.1 and the computation of Section 5.8, $\sum_{i=1}^8 y_i^2$ and $\left(\sum_{i=1}^8 y_i^2\right)^2$ generate the kernel of

$$\alpha_2^* : H^k(BE_{8(8)}) \to H^k\big(B(E_{8(8)})^\delta\big)$$

for $k = 4, 8$. Then $\alpha_2^*\, p_1(\iota) = 0$ and

$$\alpha_2^*\, p_2(\iota) = 4\,\alpha_2^*\left(\sum y_i^4\right) + 122\,\underbrace{\alpha_2^*\left[\left(\sum y_i^2\right)^2\right]}_{=0}$$

By Corollary 5.12, $\alpha_2^*\, p_2(\iota) = 4\,\alpha_2^*\left(\sum y_i^4\right) \neq 0$, and so $p_2(M) = \alpha_3^*\alpha_2^*\, p_2(\iota) \neq 0$ by Proposition 6.1. □

Theorem 6.14. *Let $G = E_{8(-24)}$ and let $\Gamma \subset G$ be a cocompact lattice. Then $p_1(\Gamma\backslash G/K) \neq 0$.*

Proof. From Equation (4.13), one computes

$$p_1(\iota) = 12(y_1^2 + \cdots + y_6^2) + 6y_7^2 + 14y_8^2.$$

By Theorem 5.1 and the computation of Section 5.8, $2(y_1^2 + \cdots + y_6^2) + y_7^2$ generates the kernel of

$$\alpha_2^* : H^4(BE_{8(-24)}) \to H^4\big(B(E_{8(-24)})^\delta\big).$$

Then

$$\alpha_2^*\, p_1(\iota) = 14\,\alpha_2^*(y_8^2)$$

is nonzero by Corollary 5.13, and so $p_1(M) = \alpha_3^*\alpha_2^* p_1(\iota) \neq 0$ by Proposition 6.1. □

6.9. **Pontryagin classes for E_7-manifolds.**

Theorem 6.15. *Let $G = E_{7(7)}$ and let $\Gamma \subset G$ be a cocompact lattice. Then $p_1(\Gamma\backslash G/K) = 0$ and $p_2(\Gamma\backslash G/K) \neq 0$.*

Proof. From Equation (4.14), one computes

$$p_1(\iota) = 595\left(\sum y_i^2\right) + 1210\left(\sum y_i y_j\right),$$

and

$$\begin{aligned} p_2(\iota) = \;& 52\,360\left(\sum y_i^4\right) + 220\,660\left(\sum y_i^3 y_j\right) + 336\,790\left(\sum y_i^2 y_j^2\right) \\ & + 684\,810\left(\sum y_i^2 y_j y_k\right) + 1\,392\,444\left(\sum y_i y_j y_k y_\ell\right). \end{aligned}$$

By Theorem 5.1 and the computation of Section 5.9, for $k = 4, 8$, the kernel of

$$\alpha_2^* : H^*(BS)^W \simeq H^k(BE_{7(7)}) \to H^k\big(B(E_{7(7)})^\delta\big)$$

is generated by $\big(\sum y_i^2\big)$ and $\big(\sum y_i^2\big)^2$. Since $\sum y_i^2$ and $\sum y_i y_j$ are linearly dependent in $H^*(B\mathrm{SU}_8)$, it follows that $\alpha_2^*\, p_1(\iota) = 0$. Using relations among the symmetric polynomials and that $\alpha_2^*\big(\sum y_i^2\big)^2 = 0$, we find that

$$\alpha_2^*\, p_2(\iota) = -348\,109\, \alpha_2^*\big(\sum y_i^4\big),$$

which is nonzero by Corollary 5.14. Then $p_2(M) = \alpha_3^*\alpha_2^*\, p_2(\iota) \neq 0$ by Proposition 6.1. □

Theorem 6.16. *Let $G = E_{7(-5)}$ and let $\Gamma \subset G$ be a cocompact lattice. Then $p_1(\Gamma\backslash G/K) \neq 0$.*

Proof. From Equation (4.15), one computes

$$p_1(\iota) = 8\big(\sum_{i=1}^{7} y_i^2\big).$$

By Theorem 5.1 and the computation of Section 5.9, the kernel of

$$\alpha_2^* : H^4(BE_{7(-5)}) \to H^4\big(B(E_{7(-5)})^\delta\big)$$

consists of scalar multiples of $I = y_1^2 + \cdots + y_6^2 + 2y_7^2$, and so

$$\alpha_2^*\, p_1(\iota) = 8\, \alpha_2^*(I - y_7^2) = -8\, \alpha_2^*(y_7^2)$$

which is nonzero by Corollary 5.15. Then $p_1(M) = \alpha_3^*\alpha_2^*\, p_1(\iota) \neq 0$ by Proposition 6.1. □

Theorem 6.17. *Let $G = E_{7(-25)}$ and let $\Gamma \subset G$ be a cocompact lattice. Then $p_1(\Gamma\backslash G/K) \neq 0$.*

Proof. From Equation (4.16), one computes

$$p_1(\iota) = 6(y_1^2 + \cdots + y_5^2) + 2y_6^2 + 3y_7^2.$$

From Section 5.9, the kernel of

$$\alpha_2^* : H^4\big(BE_{7(-25)}\big) \to H^4\big(B(E_{7(-25)})^\delta\big)$$

is generated by

$$6(y_1^2 + y_2^2 + y_3^2 + y_4^2 + y_5^2) + 2y_6^2 + y_7^2.$$

Then

$$\alpha_2^*\, p_1(\iota) = 2\, \alpha_2^*(y_7^2).$$

By Corollary 5.16, $\alpha_2^*(y_7^2) \neq 0$, and so $p_1(M) = \alpha_3^*\alpha_2^*\, p_1(\iota) \neq 0$ by Proposition 6.1. □

6.10. **Pontryagin classes for E_6-manifolds.**

Theorem 6.18. *Let $G = E_{6(6)}$ and let $\Gamma \subset G$ be a cocompact lattice. Then $p_1(\Gamma\backslash G/K) = 0$ and $p_2(\Gamma\backslash G/K) \neq 0$.*

Proof. From Equation (4.17), one computes

$$p_1(\iota) = 14(y_1^2 + y_2^2 + y_3^2 + y_4^2)$$

and

$$p_2(\iota) = 91\left(\sum y_i^4\right) + 166\left(\sum y_i^2 y_j^2\right).$$

From Section 5.10, $I = y_1^2 + y_2^2 + y_3^2 + y_4^2$ and I^2 generate the kernel of

$$\alpha_2^* : H^k\big(BE_{6(6)}\big) \to H^k\big(B(E_{6(-14)})^\delta\big)$$

for $k = 4, 8$. Then $\alpha_2^*\, p_1(\iota) = 0$ and

$$\alpha_2^*\, p_2(\iota) = 8\, \alpha_2^* \left(\sum y_i^4 \right) + 83\, \underbrace{\alpha_2^* \left(\sum y_i^4 + 2 \sum y_i^2 y_j^2 \right)}_{=0}.$$

By Corollary 5.17, $\alpha_2^* \left(\sum y_i^4 \right) \neq 0$, and so $p_2(M) = \alpha_3^* \alpha_2^*\, p_2(\iota) \neq 0$ by Proposition 6.1. □

Theorem 6.19. *Let $G = E_{6(2)}$ and let $\Gamma \subset G$ be a cocompact lattice. Then $p_1(\Gamma \backslash G/K) \neq 0$.*

Proof. From Equation (4.18), one computes

$$p_1(\iota) = 10\big(\sum_{i=1}^{6} y_i^2 \big) + 20\, y_7^2 + 8\big(\sum_{1 \leq i < j \leq 6} y_i y_j \big) = 6\big(\sum_{i=1}^{6} y_i^2 \big) + 20\, y_7^2.$$

In Section 5.10, we computed that

$$I = \sum_{i=1}^{6} y_i^2 + 2 y_7^2$$

generates (as a vector space) the kernel of

$$\alpha_2^* : H^4\big(BE_{6(2)}\big) \to H^4\big(B(E_{6(2)})^\delta\big).$$

Then

$$\alpha_2^*\, p_1(\iota) = \alpha_2^*(6I + 8\, y_7^2) = 8\, \alpha_2^*(y_7^2).$$

By Corollary 5.18, $\alpha_2^*(y_7^2) \neq 0$. Then $p_1(M) = \alpha_3^* \alpha_2^*\, p_1(M) \neq 0$ by Proposition 6.1. □

Theorem 6.20. *Let $G = E_{6(-14)}$ and let $\Gamma \subset G$ be a cocompact lattice. Then $p_1(\Gamma \backslash G/K) \neq 0$.*

Proof. From Equation (4.19), one computes

$$p_1(\iota) = 4(y_1^2 + y_2^2 + y_3^2 + y_4^2 + y_5^2) + 144\, y_6^2.$$

By Theorem 5.1 and the computation of Section 5.10,

$$I = 3(y_1^2 + \cdots + y_5^2) + y_6^2$$

generates the kernel of

$$\alpha_2^* : H^4\big(BE_{6(-14)}\big) \to H^4\big(B(E_{6(-14)})^\delta\big).$$

Since $p_1(\iota) \neq c \cdot I$ for any scalar c, we conclude that $p_1(\iota)$ is not in $\ker \alpha_2^*$. Hence $p_1(M) = \alpha_3^* \alpha_2^*\, p_1(\iota) \neq 0$ by Proposition 6.1. □

Theorem 6.21. *Let $G = E_{6(-26)}$ and let $\Gamma \subset G$ be a cocompact lattice. Then $p_i(\Gamma \backslash G/K) = 0$ for $i > 0$.*

Proof. By Corollary 5.20, α_2^* is zero in positive degrees. □

6.11. **Pontryagin classes for F_4-manifolds.**

Theorem 6.22. *Let $G = F_{4(4)}$ and let $\Gamma \subset G$ be a cocompact lattice. Then $p_1(\Gamma\backslash G/K) \neq 0$.*

Proof. From Equation (4.21), one computes

$$p_1(\iota) = 14y_1^2 + 10y_2^2 + 10y_3^2 + 10y_4^2.$$

By Theorem 5.1 and the computation in Section 5.11, $\sum_{i=1}^4 y_i^2$ generates the kernel of

$$\alpha_2^* : H^4(BF_{4(-20)}) \to H^4(BF_{4(-20)}^\delta).$$

Then

$$\alpha_2^*\, p_1(\iota) = 4\, \alpha_2^*(y_1^2) + 10\, \underbrace{\alpha_2^*(y_1^2 + y_2^2 + y_3^2 + y_4^2)}_{=0}.$$

By Corollary 5.21, $\alpha_2^*\, p_1(\iota) = 4\alpha_2^*(y_1^2) \neq 0$, so $p_1(M) = \alpha_3^*\alpha_2^*\, p_1(\iota) \neq 0$ by Proposition 6.1. □

Theorem 6.23. *Let $G = F_{4(-20)}$ and let $\Gamma \subset G$ be a cocompact lattice. Then $p_1(\Gamma\backslash G/K) = 0$ and $p_2(\Gamma\backslash G/K) \neq 0$.*

Proof. From Equation (4.22), one computes

$$p_1(\iota) = 2(y_1^2 + \cdots + y_4^2) \quad \text{and} \quad p_2(\iota) = \frac{7}{4}\left(\sum y_i^4\right) + \frac{5}{2}\left(\sum_{1\leq i<j\leq 4} y_i^2 y_j^2\right).$$

By Theorem 5.1 and the computation in Section 5.11, $\sum_{i=1}^4 y_i^2$ and $\left(\sum_{i=1}^4 y_i^2\right)^2 = \sum y_i^4 + 2\sum y_i^2 y_j^2$ generate the kernel of

$$\alpha_2^* : H^k(BF_{4(-20)}) \to H^k(BF_{4(-20)}^\delta)$$

for $k = 4, 8$, respectively. Then $\alpha_2^*\, p_1(\iota) = 0$ and

$$\alpha_2^*\, p_2(\iota) = \frac{7}{4}\underbrace{\alpha_2^*\left(\sum y_i^4 + 2\sum y_i^2 y_j^2\right)}_{=0} - \frac{2}{2}\alpha_2^*\left(\sum y_i^2 y_j^2\right) = -\alpha_2^*\left(\sum y_i^2 y_j^2\right).$$

Then $\alpha_2^* p_2(\iota) \neq 0$ by Corollary 5.22, and so $p_2(M) = \alpha_3^*\alpha_2^* p_2(\iota) \neq 0$ by Proposition 6.1. □

6.12. **Pontryagin classes for G_2-manifolds.**

Theorem 6.24. *Let $G = G_{2(2)}$, and let $\Gamma \subset G$ be a cocompact lattice. Then $p_1(\Gamma\backslash G/K) \neq 0$.*

Proof. From Equation (4.23), one computes

$$\alpha_2^* : H^4(BG_{2(2)}) \to H^4(BG_{2(2)}^\delta)$$

Then

$$\alpha_2^*\, p_1(\iota) = \alpha_2^*\big(8y_1^2 + 12y_1^2 + 4y_2^2\big) = 8\, \alpha_2^*(y_1^2) + 4\, \underbrace{\alpha_2^*(3y_1^2 + y_2^2)}_{=0} = 8\, \alpha_2^*(y_1^2).$$

By Corollary 5.23, $\alpha_2^*(y_1^2) \neq 0$, and so $p_1(M) = \alpha_3^*\alpha_2^*\, p_1(\iota) \neq 0$ by Proposition 6.1. □

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