Symmetries of exotic negatively curved manifolds

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Abstract

Let N be a smooth manifold that is homeomorphic but not diffeomorphic to a closed hyperbolic manifold M. In this paper, we study the extent to which N admits as much symmetry as M. Our main results are examples of N that exhibit two extremes of behavior. On the one hand, we find N with maximal symmetry, i.e. $\operatorname{Isom}(M)$ acts on N by isometries with respect to some negatively curved metric on N. For these examples, $\operatorname{Isom}(M)$ can be made arbitrarily large. On the other hand, we find N with little symmetry, i.e. no subgroup of $\operatorname{Isom}(M)$ of "small" index acts by diffeomorphisms of N. The construction of these examples incorporates a variety of techniques including smoothing theory and the Belolipetsky–Lubotzky method for constructing hyperbolic manifolds with a prescribed isometry group.

1 Introduction

Throughout this paper, $M = \mathbb{H}^n/\pi$ denotes a closed hyperbolic manifold with fundamental group π, and N denotes an *exotic smooth structure* (on M), i.e. a smooth manifold that is homeomorphic but not diffeomorphic to M. Define the *symmetry constant* of N as the supremum

$$s(N) = \sup_\rho \frac{|\operatorname{Isom}(N,\rho)|}{|\operatorname{Isom}(M)|},$$

over all Riemannian metrics ρ on N. In this paper we study the possible values of this invariant. There is an "easy" bound

$$\frac{1}{|\operatorname{Isom}(M)|} \leq s(N) \leq 1 \tag{1}$$

that follows from Mostow rigidity and a theorem of Borel (explained below). Our main results follow:

Theorem A (maximal symmetry constant)**.** *Fix n such that the group Θ_n of exotic spheres is nontrivial. For every $d > 0$, there exists a closed hyperbolic manifold M^n and an exotic smooth structure N such that $|\operatorname{Isom}(M)| \geq d$ and $s(N) = 1$.*

Theorem B (arbitrarily small symmetry constant)**.** *Fix n such that $\Theta_{n-1} \neq 0$. For every $d > 1$, there exists a closed hyperbolic manifold M^n and an exotic smooth structure N such that $s(N) \leq \frac{1}{d}$.*

The hypothesis $\Theta_n \neq 0$ is frequently true, e.g. $\Theta_{4k+3} \neq 0$ for every $k \geq 1$ and Θ_{4k+1} is nontrivial for any positive $k \notin \{1, 3, 7, 15, 31\}$. See [KM63, §7], [MS74, Appx. B], and [HHR16, Thm. 1.3].

The problem of computing $s(N)$ is related to two different problems in the study of transformation groups:

- *Degree of symmetry.* The degree of symmetry $\delta(W)$ of a manifold W is defined as the largest dimension of a compact Lie group with a smooth, effective action on W [HH69].

  When $W = \Sigma$ is an exotic sphere, computing $\delta(\Sigma)$ is equivalent to computing the supremum
  $$s(\Sigma) := \sup_{\rho} \frac{\dim \operatorname{Isom}(\Sigma, \rho)}{\dim \operatorname{Isom}(S^n)},$$
  over all Riemannian metrics ρ. Again there is a bound $\frac{1}{\dim \mathrm{SO}(n+1)} \leq s(\Sigma) \leq 1$, but the upper bound is not optimal. For example, Hsiang–Hsiang [Hsi67, HH65] prove that if $\Sigma \neq S^n$ has dimension $n \geq 40$, then $s(\Sigma) < \frac{n^2+8}{4(n^2+n)} < 1/4$.

  When W is an aspherical manifold and $\pi_1(W)$ is centerless, then $\delta(W) = 0$, i.e. W does not admit a nontrivial action of a connected Lie group [Bor83]. In this case it's fitting to define $\delta(W)$ as the largest order of a finite group that acts effectively on W. With this definition, for $W = N$ an exotic smooth structure on a hyperbolic manifold, $\delta(N)$ is closely related to $s(N)$; see equation (2) below.

- *Propagating group actions* [AD02]. One says that an F-action on Y *propagates across* a map $f : X \to Y$ if there is an F-action on X and an equivariant map $X \to Y$ that is homotopic to f. In particular, for an exotic smooth structure N on a hyperbolic manifold M, and for a subgroup $F < \operatorname{Isom}(M)$, one can ask whether or not the action of F propagates across some homeomorphism $N \to M$. This problem, and its relation to harmonic maps, is discussed in Farrell–Jones [FJ90]. Theorems A and B can be viewed as positive and negative results about propagating group actions, and give partial answers the question of [FJ90, pg. 487].

Remark. One could consider refinements of the symmetry constant such as $s_{<0}(N) = \sup_\rho \frac{|\operatorname{Isom}(N,\rho)|}{|\operatorname{Isom}(M)|}$, where the supremum is over all metrics with sectional curvature $K < 0$. In general, $s_{<0}(N) \leq s(N)$, but computing $s_{<0}(N)$ is more difficult (e.g. it does not reduce to a Nielsen realization problem; see below). We improve upon Theorem A by giving examples for which $s_{<0}(N) = s(N) = 1$.

Theorem C (maximal symmetry, achieved by negatively-curved metric)**.** *Fix n, and assume that either n is even or $|\Theta_n|$ is not a power of 2. Given $d > 0$, there exists a closed hyperbolic manifold M^n and an exotic smooth structure N such that $|\operatorname{Isom}(M)| \geq d$ and N admits a Riemannian metric ρ with negative sectional curvature so that $\operatorname{Isom}(N,\rho) \simeq \operatorname{Isom}(M)$.*

If $n = 4k+3$, then $|\Theta_n|$ is divisible by $2^{2k+1} - 1$; see [MS74, Appx. B].

1.1 Techniques

The problem of determining $s(N)$ is related to a *Nielsen realization problem*, which will be our main point of view. By Borel [Bor83] any compact Lie group that acts effectively on N is finite; furthermore, any finite subgroup of $\operatorname{Diff}(N)$ acts faithfully on $\pi = \pi_1(N)$. Consequently, for every ρ, the isometry group $\operatorname{Isom}(N,\rho)$ is a subgroup of $\operatorname{Out}(\pi) = \operatorname{Aut}(\pi)/\pi$. Furthermore, if $\dim M \geq 3$, then $\operatorname{Out}(\pi) \simeq \operatorname{Isom}(M)$ by Mostow rigidity. This explains the upper bound in (1). A subgroup $F < \operatorname{Out}(\pi)$ is said to be *realized by diffeomorphisms* when can we solve the lifting problem (commonly called the Nielsen realization problem — see e.g. [BW08] and [MT18]):

$$\begin{array}{ccc} & & \operatorname{Diff}(N) \\ & \nearrow & \big\downarrow \Psi_N \\ F & \hookrightarrow & \operatorname{Out}(\pi) \end{array}$$

If $F < \operatorname{Out}(\pi)$ and $F \simeq \operatorname{Isom}(N,\rho)$ for some ρ, then group F is a fortiori realized by diffeomorphisms. Conversely, if $F < \operatorname{Out}(\pi)$ is realized by diffeomorphisms, then by averaging a metric, we find ρ with $F < \operatorname{Isom}(N,\rho)$. Therefore,

$$s(N) = \max_F \frac{|F|}{|\operatorname{Out}(\pi)|}, \qquad (2)$$

where the maximum is over the subgroups $F < \operatorname{Out}(\pi)$ that are realized by diffeomorphisms. Note that $s(N) \leq \frac{|\operatorname{Im}\Psi_N|}{|\operatorname{Out}(\pi)|}$.

Farrell–Jones [FJ90] studied the Nielsen realization problem for $N = M\#\Sigma$, where M^n is a closed, oriented hyperbolic manifold and $\Sigma \in \Theta_n$ is a nontrivial exotic sphere. The main result of [FJ90] states that if M is stably parallelizable, $2\Sigma \neq 0$

in Θ_n, and M admits an orientation-reversing isometry, then $\operatorname{Im}\Psi_N < \operatorname{Out}(\pi)$ has index at least 2. In particular, $s(N) \leq 1/2$ for these examples.

Symmetric exotic smooth structures. Here we discuss the main components in the proof of Theorems A and C. We find our examples with $s(N) = 1$ among the manifolds $N = M\#\Sigma$ studied by Farrell–Jones. Using (2), observe that $s(N) = 1$ if and only if $\operatorname{Out}(\pi)$ is realized by diffeomorphisms of N. In particular, we must find examples where Ψ_N is surjective. The following results refine [FJ90, Thm. 1].

Theorem 1. *Let M^n be a closed, oriented hyperbolic manifold, let $\Sigma \in \Theta_n$ be a nontrivial exotic sphere, and let $N = M\#\Sigma$. Denote by $\operatorname{Out}^+(\pi) < \operatorname{Out}(\pi)$ the subgroup that acts trivially on $H_n(N) \simeq \mathbb{Z}$.*

- *(a) The image $\operatorname{Im}\Psi_N$ contains $\operatorname{Out}^+(\pi)$.*
- *(b) Fix $\alpha \in \operatorname{Out}(\pi) \setminus \operatorname{Out}^+(\pi)$. If $2\Sigma = 0$ in Θ_n, then $\alpha \in \operatorname{Im}\Psi_N$. The converse is true if M is stably parallelizable.*

Every closed hyperbolic manifold has a finite cover that is stably parallelizable [Sul79, pg. 553]. As a consequence of Theorem 1, if $2\Sigma = 0$, then Ψ_N is surjective, and if $2\Sigma \neq 0$, then $\operatorname{Im}\Psi_N = \operatorname{Out}^+(\pi)$. In any case, if M does not admit an orientation-reversing isometry, then Ψ_N is surjective. Farrell–Jones [FJ89a] show (implicitly) that reversing orientation is an obstruction to belonging to $\operatorname{Im}\Psi_N$ when $2\Sigma \neq 0$. According to Theorem 1, this is the only obstruction.

Having identified $\operatorname{Im}\Psi_N < \operatorname{Out}(\pi)$, we would like to know if this subgroup is realized by diffeomorphisms.

Theorem 2. *Fix $N = M\#\Sigma$ as in Theorem 1. Set $d = |\operatorname{Isom}^+(M)|$ and let $m \in \mathbb{N}$ be the size of the largest cyclic subgroup of Θ_n that contains Σ. Assume that $\gcd(d, m)$ divides $\frac{m}{|\Sigma|}$. Then $\operatorname{Out}^+(\pi)$ is realized by diffeomorphisms.*

The assumption $\gcd(d, m) \mid \frac{m}{|\Sigma|}$ guarantees that $\Sigma \in \Theta_n$ has a d-th root. This condition is satisfied, for example, whenever $|\operatorname{Isom}^+(M)|$ and $|\Sigma|$ are relatively prime.

If $\operatorname{Out}^+(\pi)$ is realized by diffeomorphisms of N, then $s(N) \geq 1/2$. By Theorems 1 and 2, if M is stably parallelizable and $2\Sigma \neq 0$, then $s(M\#\Sigma)$ is equal to 1/2 or 1, according to whether or not M admits an orientation-reversing isometry. This completely solves the Nielsen realization problem in these cases.

Theorem A reduces to Theorem 2. Fixing $\Sigma \neq S^n$, it's possible to find M so that $|\operatorname{Isom}^+(M)|$ and $|\Sigma|$ are relatively prime, and $|\operatorname{Isom}^+(M)|$ can be made arbitrarily large. This is a consequence of a result of Belolipetsky–Lubotzky [BL05]: for any finite group F, there exists a closed hyperbolic M^n with $\operatorname{Isom}(M) = F$. For their examples $\operatorname{Isom}(M) = \operatorname{Isom}^+(M)$. In particular, one can find examples where $\Psi_N : \operatorname{Diff}(N) \to \operatorname{Out}(\pi)$ is a split surjection with $|\operatorname{Out}(\pi)|$ arbitrarily large.

To prove Theorem C, one would like to promote the action of $\mathrm{Out}^+(\pi)$ on $N = M\#\Sigma$ produced in Theorem 2 to an action by isometries with respect to some negatively curved metric on N. Using a warped-metric construction of Farrell–Jones [FJ89a], it suffices to find an M that is stably parallelizable, has large injectivity radius, and such that $\mathrm{Isom}^+(M)$ acts freely on M. Arranging all of these conditions simultaneously becomes delicate, especially arranging that M is stably parallelizable (which is desired because it guarantees that $M\#\Sigma$ is not diffeomorphic to M). Because of this difficulty we take a less direct approach when $\dim M$ is odd — see Theorem 6.

Asymmetric exotic smooth structures. We explain the main ideas for proving Theorem B. For this, we consider exotic smooth structures $N = M_{c,\phi}$ obtained by removing a tubular neighborhood $S^1 \times D^{n-1} \hookrightarrow M$ of a geodesic $c \subset M$ and gluing in $S^1 \times D^{n-1}$ by a diffeomorphism $\mathbb{1} \times \phi$ of $S^1 \times S^{n-2}$, where $\phi \in \mathrm{Diff}(S^{n-2})$ is not isotopic to the identity. Farrell–Jones [FJ93] prove that $M_{c,\phi}$ is often an exotic smooth structure on M.

The strategy for proving Theorem B is to find $N = M_{c,\phi}$ and $F \simeq \mathbb{Z}/d\mathbb{Z}$ in $\mathrm{Out}(\pi)$ so that $\mathrm{Im}\,\Psi_N \cap F = 1$. This condition implies that the index of $\mathrm{Im}\,\Psi_N < \mathrm{Out}(\pi)$ is at least $|F|$, so $s(N) \leq \frac{1}{|F|}$. To show $F \cap \mathrm{Im}\,\Psi_N = 1$, we study how the smooth structure on $M_{c,\phi}$ changes if we choose a different geodesic c. This is complementary to [FJ93, Thm. 1.1], which studies how the smooth structure changes when the geodesic is fixed and the isotopy class $[\phi] \in \pi_0 \mathrm{Diff}(S^{n-2}) \simeq \Theta_{n-1}$ is changed. In Theorem 8 we give a criterion to guarantees that $M_{c_1,\phi}$ and $M_{c_2,\phi}$ are not *concordant*, i.e. there is no smooth structure on $M \times [0,1]$ that restricts to $M_{c_1,\phi} \sqcup M_{c_2,\phi}$ on the boundary. This is one of the main technical ingredients in the proof of Theorem B.

The proof of Theorem B works equally well when M is nonuniform, but we won't discuss this further.

Theorem B proves that $s(N)$ may be arbitrarily close to 0, as N varies over exotic smooth structures on all hyperbolic n-manifolds (when $\Theta_{n-1} \neq 0$), but if we fix the homeomorphism type, we know that $s(N) \geq \frac{1}{|\mathrm{Isom}(M)|}$. It would be interesting to know if there are examples where this lower bound is achieved. Of course if $\mathrm{Isom}(M) = 1$, then $s(N) = 1 = \frac{1}{|\mathrm{Isom}(M)|}$, so to make this interesting one should ask for examples such that $\mathrm{Isom}(M)$ is large.

Question 3. Does there exist n so that for every $d > 0$, there exists a hyperbolic manifold M^n and an exotic smooth structure N such that $|\mathrm{Isom}(M)| \geq d$ and $s(N) = \frac{1}{|\mathrm{Isom}(M)|}$?

Note that $s(N) = \frac{1}{|\mathrm{Isom}(M)|}$ if and only if $\Psi_N : \mathrm{Diff}(N) \to \mathrm{Out}(\pi)$ is trivial. Equivalently, $\mathrm{Isom}(N,\rho) = 1$ for every Riemannian metric ρ.

Section outline. In §2 we prove Theorems 1 and 2 and discuss some related questions of interest. In §3 we discuss the work of Belolipetsky–Lubotzky and use it to prove Theorem C. Finally, in §4 we prove Theorem B; specifically, we study when two smooth structures $M_{c_1,\phi}$ and $M_{c_2,\phi}$ are concordant, which we use as an obstruction to Nielsen realization.

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2 Symmetry constant for $N = M\#\Sigma$

In this section we prove Theorems 1 and 2.

2.1 The image of $\Psi_N : \mathrm{Diff}(N) \to \mathrm{Out}(\pi)$

Proof of Theorem 1. Let $N = M\#\Sigma$ as in the theorem. It will be convenient to fix $p \in M$ and a small metric ball $B = B_r(p)$ where the connected sum is performed.

First we prove (a). For this we fix $\alpha \in \mathrm{Out}^+(\pi) \simeq \mathrm{Isom}^+(M)$ and define $f \in \mathrm{Diff}(N)$ so that $\Psi_N(f) = \alpha$. View α as an isometry of M, and choose an isotopy $\alpha_t \in \mathrm{Diff}(M)$ so that $\alpha_0 = \alpha$ and $\alpha_1(B) = B$ and $\alpha_1\big|_B \in O(n)$ is an isometry of the ball; for example, if the radius r is sufficiently small, then we can isotope $\alpha(B)$ to B in M through isometric embeddings, and then extend the isotopy of B to an ambient isotopy. Since α is orientation-preserving, $\alpha_1\big|_B$ belongs to the identity component $SO(n) \subset O(n)$, and it is easy to see then that α_1 induces a diffeomorphism $f : N \to N$; for example, isotope $\alpha_1\big|_B$ further so that $\alpha_1\big|_{B_{r/2}(p)}$ is the identity and perform the connected sum along $B_{r/2}(p)$ instead of $B_r(p)$. This proves part (1).

To prove (b), assume that $\alpha \in \mathrm{Out}(\pi) \setminus \mathrm{Out}^+(\pi)$. Viewing α as an orientation-reversing isometry of M, the argument above defines an orientation-reversing diffeomorphism $h : M\#\Sigma \to M\#\overline{\Sigma}$ that induces α (recall that for $A\#B$, if the identification of the attaching disk is changed by an orientation-reversing involution, then the result is $A\#\overline{B}$, where $\overline{B}$ is B with the opposite orientation). If $2\Sigma = 0$ in Θ_n, then $\Sigma = \overline{\Sigma}$ (because $\overline{\Sigma} = -\Sigma$ in Θ_n), so $h \in \mathrm{Diff}(N)$ and $\Psi_N(h) = \alpha$. This proves the first statement of (b). The converse is already to contained in [FJ90, Thm. 1]. In short, if $\Psi_N(f) = \alpha$ for some $f \in \mathrm{Diff}(N)$, then $h \circ f$ is an orientation-preserving diffeomorphism $M\#\Sigma \to M\#\overline{\Sigma}$. When M is stably parallelizable, this implies that $2\Sigma = 0$ by [FJ89a, §2]. $\square$

2.2 Sections of $\Psi_N : \operatorname{Diff}(N) \to \operatorname{Im}\Psi$

Proof of Theorem 2. Since M is hyperbolic, $\operatorname{Out}(\pi)$ is realized by isometries of M (by Mostow rigidity). Set $F = \operatorname{Isom}^+(M)$. Since F is finite, there exists $p \in M$ whose stabilizer in F is trivial. Choose a ball B around p whose F-translates are disjoint. By assumption, $\gcd(|F|, m)$ divides $\frac{m}{|\Sigma|}$, which implies that there exists $\Sigma' \in \Theta_n$ so that $\Sigma = |F| \cdot \Sigma'$. Then $N = M\#\Sigma$ is diffeomorphic to $M\#\Sigma'\# \cdots \#\Sigma'$, where Σ' appears $|F|$ times. If we form the connected sum along the union of balls $F.B$, then we can extend the action of F on $M \setminus F.B$ to a smooth F-action on $N = M\#\Sigma'\# \cdots \#\Sigma'$ by rigidly permuting the exotic spheres. □

Remark. One might think that the above argument could be used to define an action of $\operatorname{Out}(\pi)$ on N under a similar constraint on $|\operatorname{Out}(\pi)|$ and $|\Sigma|$. This would contradict the fact that Ψ_N is frequently not surjective when M admits an orientation-reversing isometry. In the argument above, when M admits an orientation-reversing isometry, one obtains an action of $\operatorname{Out}(\pi)$ on $M\#k\Sigma'\#k\overline{\Sigma'}$, where $k = |\operatorname{Out}(\pi)|/2$. But $M\#k\Sigma'\#k\overline{\Sigma'}$ is diffeomorphic to M, not N.

It would be interesting to know if $\operatorname{Out}^+(\pi)$ ever acts on $N = M\#\Sigma$ when N has no "obvious" symmetry:

Question 4. Is Theorem 2 ever true without the assumption $\gcd(d, m) \mid \frac{m}{|\Sigma|}$? For example, fix $\alpha \in \operatorname{Isom}^+(M)$ of order d, and assume that α acts freely. Choose $\Sigma \in \Theta_n$ that does not admit a d-th root. Prove or disprove that the subgroup $\langle\alpha\rangle \simeq \mathbb{Z}/d\mathbb{Z}$ in $\operatorname{Out}^+(\pi)$ is realized by diffeomorphisms of $N = M\#\Sigma$.

In this direction, it would be interesting to know how the choice of Σ affects the answer to Question 4. For instance, in the study of the symmetry constant of $\Sigma \in \Theta_n$, there is a marked difference between (1) the standard sphere $\Sigma = S^n$, (2) the nontrivial exotic spheres that bound a parallelizable manifold $\Sigma \in bP_{n+1} \setminus \{S^n\}$, and (3) the remaining exotic spheres $\Sigma \in \Theta_n \setminus bP_{n+1}$. See [HH69]. Does this distinction play a role in Question 4?

Note that the subtlety in Question 4 disappears in the topological category: if W is an aspherical manifold with $\pi_1(W) \simeq \pi$, then $\operatorname{Homeo}(W) \to \operatorname{Out}(\pi)$ is a split surjection because W and M are homeomorphic by the solution of Farrell–Jones to the Borel conjecture in this case; see [Far02, Cor. 3 in §5].

We mention another problem related to Question 4. For this, let W^n be an exotic smooth structure on the torus T^n. There is a surjective homomorphism $\operatorname{Diff}^+(W) \to \operatorname{Out}^+(\pi_1(W)) \simeq \operatorname{SL}_n(\mathbb{Z})$, and whether or not this homomorphism splits is unknown. One approach to this question is to focus on maximal abelian subgroups of $\operatorname{SL}_n(\mathbb{Z})$ and try to use the dynamics of Anosov diffeomorphisms; see [FKS13, Question 1.4] and also [BRHW17]. Alternatively, an obstruction to realizing finite subgroups

$F < \mathrm{SL}_n(\mathbb{Z})$ as in Question 4 could provide an approach to the splitting problem for certain $W = T^n \# \Sigma$.

3 Realization by isometries

In this section, we prove Theorem C. The starting point of our argument is the following result from [BL05, Thm. 1.1 and §6.3].

Theorem 5 (Belolipetsky–Lubotzky)**.** *For every $n \geq 2$ and every finite group F, there exists infinitely many compact n-dimensional hyperbolic manifolds M with $\mathrm{Isom}(M) = \mathrm{Isom}^+(M) \simeq F$.*

The main result we prove here is as follows.

Theorem 6. *Fix a finite group F and fix $R > 0$. Among the hyperbolic manifolds M^n with $\mathrm{Isom}(M) = \mathrm{Isom}^+(M) \simeq F$, there exists M such that*

- *(a) the group F acts freely on M,*
- *(b) there is a cover $\widehat{M} \to M$ of degree $\ell \in \{1, 2, 4\}$ so that $\widehat{M}$ is stably parallelizable, and*
- *(c) $\mathrm{InjRad}(M) > R$.*

Furthermore, for (b), if n is even, then we can take $\ell = 1$.

Next we deduce Theorem C from Theorem 6.

Proof of Theorem C. Fix $d > 0$. If n is even, take any nontrivial $\Sigma \in \Theta_n$ and let F be a group with $|F| \geq d$ and $\gcd(|F|, |\Sigma|) = 1$. If $|\Theta_n| \neq 2^i$, take $\Sigma \in \Theta_n$ nontrivial of odd order and let F be a 2-primary group with $|F| \geq d$. In either case, there exists $\Sigma' \in \Theta_n$ with $\Sigma = |F| \cdot \Sigma'$. By Belolipetsky–Lubotzky and Theorem 2, for every M with $\mathrm{Isom}(M) \simeq \mathrm{Isom}^+(M) \simeq F$, the group F acts by diffeomorphisms of $N = M \# \Sigma \simeq M \# \Sigma' \# \cdots \# \Sigma'$. We need to show we can choose M and a negatively-curved metric ρ on N so that $F = \mathrm{Isom}(N, \rho)$ in $\mathrm{Diff}(N)$.

According to [FJ89a, Prop. 1.3], there is a constant $\tau_n > 0$ so that if M^n has injectivity radius $\mathrm{InjRad}(M) > \tau_n$, then $N = M \# \Sigma$ admits a negatively curved metric. This metric agrees with the hyperbolic metric on M away from the disk where the connected sum is performed, and on that disk, the metric is radially symmetric. Choose M satisfying Theorem 6 with $R = |F| \cdot \tau_n$ and such that F acts freely on M, so the quotient $\overline{M} = M/F$ is a hyperbolic manifold. Furthermore,

$$\mathrm{InjRad}(\overline{M}) \geq \mathrm{InjRad}(M)/|F| > \tau_n. \tag{3}$$

We prove this below. Now fix r with $\tau_n < r < \mathrm{InjRad}(\overline{M})$. From (3) it follows that for any ball $B = B_r(p)$ in M, the F-translates of B are disjoint. Fix such a ball B. As in the proof of Theorem 2, write $\Sigma = |F| \cdot \Sigma'$ and consider $M_0 = M \setminus F.B$. The manifold N is obtained by gluing $\mathbb{D}^n$ to each boundary component of M_0 by a fixed diffeomorphism $f \in \mathrm{Diff}(S^{n-1})$. Using the technique in [FJ89a], we give N a Riemannian metric ρ that agrees with the hyperbolic metric on M_0 and is a warped-product metric on each $\mathbb{D}^n$. Since $r > \tau_n$, [FJ89a, §3] guarantees that the resulting metric has negative curvature. The group F acts on N as in Theorem 2, and by construction it acts by isometries for the metric ρ.

Now we explain the inequality (3). To see the first inequality, note that $2\,\mathrm{InjRad}(M) = \mathrm{sys}(M)$, where $\mathrm{sys}(M)$ is *systole*, i.e. the length of the shortest geodesic. Under a d-fold isometric cover $M \to \overline{M}$, if $\overline{\gamma}$ is a closed geodesic of $\overline{M}$ and $\gamma \subset M$ is a connected component of its preimage, then $\mathrm{length}(\gamma) \le d \cdot \mathrm{length}(\overline{\gamma})$. It follows that $\mathrm{sys}(M) \le d \cdot \mathrm{sys}(\overline{M})$.

It remains is to show that N is not diffeomorphic to M. When n is even, then by Theorem 6 we can assume that M is stably parallelizable and so M is not diffeomorphic to $M \# \Sigma$ by Farrell–Jones [FJ89a]. In the general case, M has a stably parallelizable cover of degree 2 or 4. Suppose for a contradiction that $M \# \Sigma$ is diffeomorphic to M. Lifting to the cover $\widehat{M} \to M$, we find that $\widehat{M} \# \ell\Sigma$ is diffeomorphic to $\widehat{M}$. Note that $\ell\Sigma \neq 0$ in Θ_n since Σ has odd order and $\ell \in \{2, 4\}$. Since $\widehat{M}$ is stably parallelizable, by [FJ89a, Prop. 1.2], we conclude that $\widehat{M} \# \ell\Sigma$ is not diffeomorphic to $\widehat{M}$. This is a contradiction, so N is not diffeomorphic to M as desired. This completes the proof. □

Next we prove Theorem 6. Fix a finite group F. In what follows $M = \mathbb{H}^n/\pi$ will always denote one of the Belolipetsky–Lubotzky manifolds with $\mathrm{Isom}(M) = \mathrm{Isom}^+(M) \simeq F$. We have to explain why M can be chosen to satisfy (a), (b), and (c). We will see that [BL05, Thm. 2.1] already shows that (a) can be arranged, and that (b) can be arranged by modifying the proof of [BL05, Prop. 2.2]. Part (c) requires a different, separate argument. All of these arguments involve passing to certain congruence covers, so once we explain why (a), (b), and (c) can be arranged *individually*, it will be evident that they can be arranged *simultaneously*.

Recollection of Belolipetsky–Lubotzky [BL05]. Here we summarize the main results of [BL05], especially the aspects needed for our proof. Let Γ be a finitely generated group. Assume that $\Delta \lhd \Gamma$ is finite-index, normal, and that Δ surjects to a finite-rank free group:

$$1 \to K \to \Delta \to F_r \to 1$$

for some $r \ge 2$. The conjugation action of $N_\Gamma(K)$ on Δ preserves K, so $N_\Gamma(K)$ acts on F_r by automorphisms. Let $D < N_\Gamma(K)$ be the subgroup that acts on F_r by inner automorphisms. With this setup, the main algebraic construction of [BL05, Thm.

2.1] asserts that for any finite group F, there exists a finite-index subgroup $\pi < D$ with $N_\Gamma(\pi)/\pi \simeq F$ (in their notation, they use M instead of K and B instead of π).

In the application to hyperbolic manifolds, define Γ as the commensurator $\mathrm{Comm}(\Lambda)$ of a Gromov–Piatetski-Shapiro [GPS88] non-arithmetic lattice $\Lambda < \mathrm{SO}(n,1)$. By work of Mostow and Margulis, $\mathrm{Comm}(\Lambda)$ is a maximal discrete subgroup of $\mathrm{Isom}(\mathbb{H}^n)$, so for any $\pi < \Gamma$,

$$N_\Gamma(\pi)/\pi \simeq N_{\mathrm{Isom}(\mathbb{H}^n)}(\pi)/\pi \simeq \mathrm{Isom}(\mathbb{H}^n/\pi).$$

Hence to find $M = \mathbb{H}^n/\pi$ with $\mathrm{Isom}(M) \simeq F$, it suffices to find $\pi < \Gamma$ with $N_\Gamma(\pi)/\pi \simeq F$.

To define Δ, denote $G = O(n,1)$ and let $\mathcal{O}_S$ be ring of definition of Γ, so $\Gamma < G(\mathcal{O}_S)$. Let $\mathfrak{p} \subset \mathcal{O}_S$ be a prime ideal and denote $p \in \mathbb{N}$ the prime with $(p) = \mathfrak{p} \cap \mathbb{Z}$. We only deal with prime ideals $\mathfrak{p}$ where $\mathcal{O}_S/\mathfrak{p} \simeq \mathbb{F}_p$. Equivalently, p splits completely in $\mathcal{O}_S$; there are infinitely many such $\mathfrak{p}$ by Chebotarev's theorem. Reduction mod $\mathfrak{p}$ defines a map $\alpha_\mathfrak{p} : \Gamma \to G(\mathcal{O}_S/\mathfrak{p}) \simeq O_{n+1}(p)$ to an orthogonal group over $\mathbb{F}_p$. Define $\Gamma(\mathfrak{p}) = \ker(\overline{\alpha}_\mathfrak{p})$, where $\overline{\alpha}_\mathfrak{p} : \Gamma \to O_{n+1}(p) \to \mathrm{PO}_{n+1}(p)$. The group Δ is defined as $\Lambda \cap \Gamma(\mathfrak{p})$.

To ensure $\Delta \lhd \Gamma$, we want $\Lambda \lhd \Gamma$. In order to arrange this, after we've defined Γ, we replace Λ with a finite-index subgroup (still denoted Λ) so that $\Lambda \lhd \Gamma$ (note that this replacement does not change $\mathrm{Comm}(\Lambda)$). The group Δ surjects to a free group: By the cut-and-paste nature of the construction of [GPS88], Λ is either an amalgamated product or an HNN extension. For definiteness assume $\Lambda = \Lambda_1 *_{\Lambda_3} \Lambda_2$. Denoting $\Omega_{n+1}(p) = [O_{n+1}(p), O_{n+1}(p)]$, by strong approximation, for all but finitely many $\mathfrak{p}$, the image of $\overline{\alpha}_\mathfrak{p} : \Lambda \to \mathrm{PO}_{n+1}(p)$ contains $Q_p := \mathrm{P}\Omega_{n+1}(p)$, and the same is true for the restriction to Λ_1, Λ_2. Without loss of generality, we may assume $\mathrm{Im}(\overline{\alpha}_\mathfrak{p}) = Q_p$ (replace Λ by the intersection of all index-2 subgroups of Λ). Denoting $T_p = \overline{\alpha}_p(\Lambda_3)$, the map $\overline{\alpha}_p$ factors through surjective maps $\Lambda \xrightarrow{s} Q_p *_{T_p} Q_p \xrightarrow{t} Q_p$. Then $\Delta = \ker(t \circ s)$ surjects onto $\ker t$, which is a free group of rank $r \geq 2$ [BL05, Prop. 3.4].

Proof of Theorem 6. Fix a finite group F. We use the setup of the proceeding paragraphs. In particular, $\pi < D$ will always denote a subgroup with $N_\Gamma(\pi)/\pi \simeq F$, and our aim is to show that π can be chosen in such a way that $M = \mathbb{H}^n/\pi$ has properties (a), (b), and (c).

Part (a). By [BL05, pg. 465] the group $N_\Gamma(\pi)$ is contained in $D = \ker\big[N_\Gamma(K) \to \mathrm{Out}(F_r)\big]$, and [BL05, §5] shows that D is contained in $\Gamma(\mathfrak{p})$, which is torsion-free for p large. It follows that $\mathrm{Isom}(M) \simeq N_\Gamma(\pi)/\pi$ acts freely on M: if $x \in M$ is fixed by $g \neq 1 \in \mathrm{Isom}(M)$, then g lifts to $\tilde{g} \in N_\Gamma(\pi)$ that acts on $\mathbb{H}^n$ with a fixed point, but this contradicts the fact that $N_\Gamma(\pi)$ is torsion-free.

Part (b). As mentioned in part (a), we can arrange that $\pi < \Gamma(\mathfrak{p})$. Our main task for part (b) will be to show that we can also arrange that $\pi < \Gamma(\mathfrak{p}) \cap \Gamma(\mathfrak{q})$, where $\mathfrak{p}, \mathfrak{q} \subset \mathcal{O}_S$ are prime ideas with $\mathcal{O}_S/\mathfrak{p} \simeq \mathbb{F}_p$ and $\mathcal{O}_S/\mathfrak{q} \simeq \mathbb{F}_q$ for distinct primes p, q. Before we do this, we explain why this is enough to conclude that $M = \mathbb{H}^n/\pi$ has the desired stably parallelizable cover.

Suppose that $M = \mathbb{H}^n/\pi$ with $\pi < \Gamma(\mathfrak{p}) \cap \Gamma(\mathfrak{q})$. We will show that there is a cover $\widehat{M} \to M$ of degree 1, 2, or 4 so that $\widehat{M}$ has a tangential map $\widehat{M} \to S^n$, and hence $\widehat{M}$ is stably parallelizable. The group π is a subgroup of the identity component $\mathrm{SO}_0(n,1) < \mathrm{SO}(n,1)$. The inclusions $\pi \hookrightarrow \mathrm{SO}_0(n,1) \hookrightarrow \mathrm{SO}_{n+1}(\mathbb{C})$ define flat bundles over M. By Deligne–Sullivan [DS75], there is a particular cover $\widehat{M} \to M$ so that the map $\widehat{M} \to M \to B\,\mathrm{SO}_{n+1}(\mathbb{C})$ is homotopically trivial. This cover is the one corresponding to the subgroup $\widehat{\pi} = \pi \cap \ker(\alpha_\mathfrak{p}) \cap \ker(\alpha_\mathfrak{q})$ of π. Note that the index $[\pi : \widehat{\pi}]$ is 1, 2, or 4 because $\ker(\alpha_\mathfrak{p})$ has index 2 in $\ker(\overline{\alpha}_\mathfrak{p})$. Furthermore, if n is even, then $\mathrm{SO}_{n+1}(p) < O_{n+1}(p)$ has trivial center, so $\mathrm{SO}_{n+1}(p) \simeq \mathrm{PSO}_{n+1}(p)$, which implies that $\widehat{\pi} = \pi$.

Since there is a fibration

$$\mathrm{SO}_{n+1}(\mathbb{C})/\,\mathrm{SO}_0(n,1) \to B\,\mathrm{SO}_0(n,1) \to B\,\mathrm{SO}_{n+1}(\mathbb{C})$$

and $\widehat{M} \to B\,\mathrm{SO}_0(n,1) \to B\,\mathrm{SO}_{n+1}(\mathbb{C})$ is trivial, the map $\widehat{M} \to B\,\mathrm{SO}_0(n,1)$ lifts to $\mathrm{SO}_{n+1}(\mathbb{C})/\,\mathrm{SO}_0(n,1)$, which is homotopy equivalent to $\mathrm{SO}(n+1)/\,\mathrm{SO}(n) \simeq S^n$. This map $\widehat{M} \to S^n$ is a tangential map by Okun [Oku01, §5]. This completes the construction of the stably parallelizable cover.

Now we show we can find M with isometry group F and fundamental group $\pi < \Gamma(\mathfrak{p}) \cap \Gamma(\mathfrak{q})$. As above, fix $\mathfrak{p} \subset \mathcal{O}_S$ such that $\alpha_p : \Lambda \to Q_p$ is surjective and also $\alpha(\Lambda_1) = \alpha(\Lambda_2) = Q_p$.

Observation. Fix a prime ideal $\mathfrak{q} \subset \mathcal{O}_S$ and denote $q \in \mathbb{N}$ the prime with $(q) = \mathfrak{q} \cap \mathbb{Z}$. If the image of $\overline{\alpha}_\mathfrak{q} : \Lambda(p) \to \mathrm{PO}_{n+1}(q)$ contains Q_q, then the image of $\overline{\alpha}_{\mathfrak{p},\mathfrak{q}} : \Lambda \to \mathrm{PO}_{n+1}(p) \times \mathrm{PO}_{n+1}(q)$ defined by

$$\overline{\alpha}_{\mathfrak{p},\mathfrak{q}}(g) = (\overline{\alpha}_\mathfrak{p}(g), \overline{\alpha}_\mathfrak{q}(g))$$

contains $Q_p \times Q_q$. Indeed, if $(x, y) \in Q := Q_p \times Q_q$, then one has that $\overline{\alpha}_\mathfrak{p}(g) = x$ for some $g \in \Lambda$ and also $\overline{\alpha}_\mathfrak{q}(h) = \overline{\alpha}_\mathfrak{q}(g)^{-1}y$ for some $h \in \Lambda(\mathfrak{p})$. Thus $\overline{\alpha}_{\mathfrak{p},\mathfrak{q}}(gh) = (x, y)$.

We use the observation together with the strong approximation theorem to conclude that for all but finitely many of the infinitely many primes q that split completely, the image of each of Λ, Λ_1, and Λ_2 in $\mathrm{PO}_{n+1}(p) \times \mathrm{PO}_{n+1}(q)$ contains $Q_p \times Q_q$. As before, we may assume (by replacing Λ with a finite-index subgroup) that $\overline{\alpha}_{\mathfrak{p},\mathfrak{q}}(\Lambda) = Q_p \times Q_q$.

Set $T = \overline{\alpha}_{\mathfrak{p},\mathfrak{q}}(\Lambda_3)$. The subgroup $T < Q$ has the property that there are no nontrivial $N \lhd Q$ such that $1 \le N \le T$ (compare [BL05, §3.2]). This holds essentially for the same reasons it holds for $T_p < Q_p$ (see [BL05, §5]). In our case, we only need to notice that $T \le \mathrm{PO}_n(p) \times \mathrm{PO}_n(q)$, while the only nontrivial proper normal subgroups of Q are $Q_p \times 1$ and $1 \times Q_q$ (the latter fact holds because Q_p and Q_q are simple if p, q are sufficiently large and $Q_p \not\simeq Q_q$).

Setting $\Delta = \ker(\overline{\alpha}_{\mathfrak{p},\mathfrak{q}}) = \Lambda \cap \Gamma(\mathfrak{p}) \cap \Gamma(\mathfrak{q})$, we may repeat the argument of [BL05, §5] to conclude that $\pi < D$ is contained in $\Gamma(\mathfrak{p}) \cap \Gamma(\mathfrak{q})$. This finishes part (b).

Part (c). We explain why we can arrange for M to have isometry group F and arbitrarily large injectivity radius. This will follow (using Proposition 7 below) from the fact that π is a subgroup of matrices $\mathrm{SL}_m(\mathcal{O}_S)$ with coefficients in the ring $\mathcal{O}_S$ of S-integers in a number field L. Before proving Proposition 7 we recall a few facts about $\mathcal{O}_S$. Here $\mathcal{O}$ is the ring of integers in L, and S is a finite set of places (i.e. an equivalence class of absolute value on L) that includes all of the Archimedean places, and $\mathcal{O}_S = \{x \in L : t(x) \leq 1 \text{ for all places } t \notin S\}$.

For our proof of Proposition 7, we recall the description of the set of all places of L. This is the content of Ostrowski's theorem [Jan96, Ch. II]. The Archimedean places all come from embeddings of L into $\mathbb{R}$ or $\mathbb{C}$. The non-Archimedean places come from prime ideals $\mathfrak{q} \subset \mathcal{O}$ as follows. Given $\mathfrak{q}$, for $a \in \mathcal{O}$ define $\nu_{\mathfrak{q}}(a) \in \mathbb{Z}_{\geq 0}$ as the multiplicity of $\mathfrak{q}$ appearing in the prime factorization of the ideal $(a) \subset \mathcal{O}$; this is extended to $x = \frac{a}{b} \in L$ by $\nu_{\mathfrak{q}}(x) = \nu_{\mathfrak{q}}(a) - \nu_{\mathfrak{q}}(b)$. Denoting the norm $N(\mathfrak{q}) = |\mathcal{O}/\mathfrak{q}|$, the function $t_{\mathfrak{q}}(x) = N(\mathfrak{q})^{-\nu_{\mathfrak{q}}(x)}$ defines a place of L. The set of all places (normalized in the way we have described) satisfies the *product formula* $\prod t(x) = 1$ for any $x \in L^\times$ [Jan96, Ch. II, §6]. For future reference, observe that if $a \in \mathcal{O}$ and $\mathfrak{q} \nmid a$, then $t_{\mathfrak{q}}(a) = 1$, so only finitely many terms in the product $\prod t(x)$ differ from 1. Note also that if $(a) = \mathfrak{q}_1^{n_1} \cdots \mathfrak{q}_f^{n_f}$ is the prime factorization, then $N(a) = N(\mathfrak{q}_1)^{n_1} \cdots N(\mathfrak{q}_f)^{n_f}$, so by the product formula, $N(a)$ is also equal to the product $\prod_{t|\infty} t(a)$ over Archimedean places of L.

Proposition 7 (Injectivity radius growth in congruence covers)**.** *Let V be a closed aspherical Riemannian manifold with fundamental group π. Suppose there exists an injection $\pi \hookrightarrow \mathrm{SL}_m(\mathcal{O}_S)$, where $\mathcal{O}_S$ is the ring of S-integers in a number field L. For an ideal $\mathfrak{k} \subset \mathcal{O}$, denote*

$$\mathrm{SL}_m(\mathfrak{k}) = \ker\left[\,\mathrm{SL}_m(\mathcal{O}_S) \to \mathrm{SL}_m(\mathcal{O}_S/\mathfrak{k}\mathcal{O}_S)\right]$$

and let $V_{\mathfrak{k}}$ be the cover of V with fundamental group $\pi(\mathfrak{k}) := \pi \cap \mathrm{SL}_m(\mathfrak{k})$. Then there are constants C, D (depending only on V, m, and K, but not $\mathfrak{k}$) so that $\mathrm{InjRad}(V_{\mathfrak{k}}) \geq C \log k + D$*, where $(k) = \mathfrak{k} \cap \mathbb{Z}$.*

This statement is similar to the "Elementary Lemma" of [Gro96, §3.C.6]. The proof below is based on, and has some overlap with, the argument in [GL14, §4].

Proof of Proposition 7. Let $\widetilde{V}$ be the universal cover of V.

Fix the ideal $\mathfrak{k}$, and set $R = \mathrm{InjRad}(V_{\mathfrak{k}})$. By definition of InjRad, there exists $y, z \in \widetilde{V}$ and $\eta \in \pi(\mathfrak{k})$ so that $y, \eta y$ are both contained in the ball $B_{2R}(z)$. Then

$d(y, \eta y) \le 4R$; equivalently

$$R \ge \frac{1}{4} d(y, \eta y).$$

To prove the proposition, we will give a lower bound on $d(y, \eta y)$.

Since V is compact, π is finitely generated. Consider the generating set associated to the Dirichlet fundamental domain $\mathcal{D}$ centered at y for the action of π on $\widetilde{V}$ (generators are those $g \in \pi$ for which $g(\mathcal{D}) \cap \mathcal{D} \neq \varnothing$). For the word length $w : \pi \to \mathbb{Z}_{\geq 0}$ associated to this generating set, there is a bound $w(\eta) \le c_1 \cdot \big[d(y, \eta y) + 1\big]$, obtained as follows. Take a geodesic γ connecting $y, \eta y$, and cover it by $\lfloor d(y, \eta y) \rfloor + 1$ balls of radius 1. There is $c_1 > 0$ so that each ball intersects at most c_1 translates of $\mathcal{D}$, so γ intersects at most $c_1 \cdot \big[d(y, \eta y) + 1\big]$ translates of $\mathcal{D}$. This proves the aforementioned bound, which is equivalent to

$$d(y, \eta y) \ge (1/c_1) \cdot w(\eta) - 1.$$

To finish the proof, we prove

$$w(\eta) \ge c_2 \log k + c_3 \tag{4}$$

for some constants c_2, c_3. Now we use the assumptions that $\pi < \mathrm{SL}_m(\mathcal{O}_S)$ and $\eta \in \mathrm{SL}_m(\mathfrak{k})$. For $X = (x_{ij}) \in \mathrm{SL}_m(L)$ and $s \in S$, define

$$|X|_s = \max_{i,j} s(x_{ij}) \quad \text{and} \quad |X|_S = \sum_{s \in S} |X|_s.$$

By the formula for matrix multiplication $|XY|_S \le m|X|_S|Y|_S$. Write $\eta = X_1 \cdots X_{w(\eta)}$ with $X_i \in \mathrm{SL}_m(\mathcal{O}_S)$ belonging to our chosen generating set of π. Then $|\eta|_S \le m^{w(\eta)-1} \cdot M^{w(\eta)}$, where M is the maximum value of $|\cdot|_S$ on generators of π. On the other hand, we will show that $|\eta|_S \ge \ell \cdot k^{1/\ell} - \ell$, where $(k) = \mathfrak{k} \cap \mathbb{Z}$ and $\ell = |S|$. Then altogether we have

$$\ell \cdot k^{1/\ell} - \ell \le |\eta|_S \le m^{w(\eta)-1} \cdot M^{w(\eta)},$$

which gives a bound as in (4) after taking log. Note that $\log(k^{1/\ell} - 1) = \log(k^{1/\ell}) + \log(1 - k^{-1/\ell})$ and $\log(1 - k^{-1/\ell})$ is bounded below by the constant $\log(1 - 2^{-1/\ell})$.

Now we prove $|\eta|_S \ge \ell \cdot k^{1/\ell} - \ell$. Since $\eta \neq \mathrm{Id}$, some entry η_{ij} has the form $1 + x$ or x, where $x \in \mathfrak{k}\mathcal{O}_S$ is nonzero. Write $x = \frac{a}{b} \cdot x_1$, where $x_1 \in \mathfrak{k}$ and the only primes dividing a, b are primes in S. By the product formula

$$\prod_{s \in S} s(a/b) = 1 \quad \text{and} \quad \prod_{s \in S} s(x_1) = N(x_1).$$

Furthermore, $N(x_1) \ge N(\mathfrak{k}) \ge k$ because $(x_1) \subset \mathfrak{k}$ and $\mathbb{Z}/k\mathbb{Z} \subset \mathcal{O}/\mathfrak{k}$. Therefore, $\prod_{s \in S} s(x) \ge k$.

Next we show that $\prod_{s\in S} s(x) \geq k$ implies that $|x|_S := \sum_{s\in S} s(x) \geq \ell k^{1/\ell}$. This follows from some calculus: we want to minimize the function $\phi(x_1,\ldots,x_\ell) = x_1 + \cdots + x_\ell$ under the constraint $x_1\cdots x_\ell \geq k$. Since ϕ has no critical points, the minimum is achieved on the set $x_1\cdots x_\ell = k$. Using Lagrange multipliers, one finds that ϕ has a unique minimum at $x = (k^{1/\ell},\ldots,k^{1/\ell})$ and the minimum value is $\phi(x) = \ell\cdot k^{1/\ell}$.

Since η_{ij} is either x or $1+x$, in either case $|\eta_{ij}|_S \geq \sum_{s\in S}[s(x)-1] \geq \ell\cdot k^{1/\ell} - \ell$. Combining everything we conclude that

$$|\eta|_S \geq |\eta_{ij}|_S \geq \ell\cdot k^{1/\ell} - \ell.$$

This completes the proof. □

4 Symmetry constant for $N = M_{c,\phi}$

In this section we prove Theorem B. As mentioned in the introduction, the goal is to find smooth structures N and large subgroups $F < \mathrm{Out}(\pi)$ so that $\mathrm{Im}\,\Psi_N \cap F = 1$. To this end, we consider the exotic smooth structures $N = M_{c,\phi}$ studied in [FJ93]. Here M is hyperbolic, c is a simple closed geodesic, and $\phi \in \mathrm{Diff}(S^{n-2})$. Choosing a framing $\iota : S^1\times D^{n-1} \to M$ of c, the manifold $M_{c,\phi}$ is defined as the quotient of

$$S^1\times D^{n-1} \coprod M \setminus \iota(S^1\times \ \mathrm{int}(D^{n-1}))$$

by the identification $(x,v) \leftrightarrow \iota(x,\phi(v))$ for $(x,v) \in S^1\times S^{n-2}$.

We prove Theorem B in 3 steps.

4.1 Non-concordant smooth structures (Step 1)

Our mechanism for constructing $\alpha \in \mathrm{Out}(\pi)$ such that $\alpha \notin \mathrm{Im}\,\Psi_N$ is Theorem 8 below. Before we state it, recall some facts about smooth structures that will be used here and in the next subsection.

Smoothings of topological manifolds. By a smooth manifold N we mean a topological manifold with a smooth atlas of charts $\mathbb{R}^n \supset U_\alpha \to N$ (which we call a *smooth structure*). If N (resp. M) is a smooth (resp. topological) manifold and $h : N \to M$ is a homeomorphism, then we obtain a smooth structure on M by pushforward. The map h is called a *marking*. Two markings $h_0 : N_0 \to M$ and $h_1 : N_1 \to M$ determine the same smooth structure on M if there is a diffeomorphism $g : N_0 \to N_1$ so that $h_1 g = h_0$.

Two smooth structures N_0, N_1 on M are *concordant* if there exists a smooth structure on $M\times[0,1]$ whose restriction to $M\times\{i\}$ is N_i for $i = 0,1$. The main fact about concordances that we use is that classifying concordance classes reduces to homotopy theory: there is a bijection between the set of concordance classes of smooth structures on M and the set of based homotopy classes of maps $[M, \operatorname{Top}/O]$.

As remarked in [FJ93, §1], the concordance class of the smooth structure $M_{c,\phi}$ is independent of the choice of framing and is also independent of the choice of representative of the isotopy class $[\phi] \in \pi_0 \operatorname{Diff}(S^{n-2})$.

Theorem 8 (non-concordant smooth structures)**.** *Let M be a smooth closed manifold. Assume M is stably parallelizable. Let $c_1, \ldots, c_\ell$ be disjoint closed curves in M. Assume that there exists a homomorphism $\Delta : \pi_1(M) \to \mathbb{Z}^\ell$ such that $\Delta(c_1), \ldots, \Delta(c_\ell)$ generate $\mathbb{Z}^\ell$. For any nontrivial isotopy class $[\phi] \in \pi_0 \operatorname{Diff}(S^{n-2})$, no two of the smooth structures $M_{c_1,\phi}, \ldots, M_{c_\ell,\phi}$ are concordant.*

Proof. Given a codimension-0 embedding $\lambda : X \to Y$ of open manifolds, we denote λ' the induced map of 1-point compactifications, obtained by collapsing $Y \setminus X$ to a point. Also X_+ denotes the space X with a disjoint basepoint.

Let $\iota_1, \ldots, \iota_\ell : S^1 \times D^{n-1} \hookrightarrow M$ be framings of $c_1, \ldots, c_\ell$. Use $\iota_1, \ldots, \iota_\ell$ to define an embedding $\iota : \coprod_\ell S^1 \times D^{n-1} \hookrightarrow M$. The induced collapse map has the form $\iota' : M \to \bigvee_\ell \Sigma^{n-1}(S^1_+)$. Consider the composition

$$\hat{\iota} : M_+ \to M \xrightarrow{\iota'} \bigvee_\ell \Sigma^{n-1}(S^1_+) \to \bigvee_\ell S^{n-1},$$

where the last map is induced from the obvious maps $\Sigma^{n-1}(S^1_+) \simeq S^n \vee S^{n-1} \to S^{n-1}$. It suffices to show that the induced map

$$\hat{\iota}^* : \big[\bigvee_\ell S^{n-1},\ \operatorname{Top}/O\big] \to \big[M_+,\ \operatorname{Top}/O\big]$$

is injective. This is because, under the bijection between concordance classes of smooth structures on M and $[M, \operatorname{Top}/O]$, the concordance class of $M_{c_j,\phi}$ corresponds to the map

$$M \xrightarrow{\hat{\iota}} \bigvee_\ell S^{n-1} \xrightarrow{\pi_j} S^{n-1} \xrightarrow{\hat{\phi}} \operatorname{Top}/O,$$

where π_j collapses every sphere other than the j-th sphere to the basepoint, and $\hat{\phi}$ corresponds to $[\phi] \in \pi_0 \operatorname{Diff}(S^{n-2})$ under the bijections $[S^{n-1}, \operatorname{Top}/O] \simeq \Theta_{n-1} \simeq \pi_0 \operatorname{Diff}(S^{n-2})$.

To show that $\hat{\iota}^*$ is injective, we use that Top/O is an infinite loop space. In particular, there exists a space Y such that $\Omega^{n+\ell} Y \simeq \operatorname{Top}/O$, and for any space A, there are natural bijections $[A, \operatorname{Top}/O] \simeq [A, \Omega^{n+\ell} Y] \simeq [\Sigma^{n+\ell} A, Y]$. This allows us to view $\hat{\iota}^*$ as map

$$\big[\bigvee_\ell S^{2n+\ell-1}, Y\big] \to \big[\Sigma^{n+\ell}(M_+), Y\big].$$

This map can also be obtained by considering the embedding $\iota\times 1 : \big(\coprod_\ell S^1 \times D^{n-1}\big)\times D^{n+\ell} \hookrightarrow M \times D^{n+\ell}$ and the composition $\widehat{\iota\times 1} : \Sigma^{n+\ell}(M_+) \xrightarrow{(\iota\times 1)'} \bigvee_\ell \Sigma^{2n+\ell}(S^1_+) \to \bigvee_\ell S^{2n+\ell-1}$, similar to before.

The homomorphism Δ is induced by a map $\delta : M \to T^\ell$ to the torus, and we can assume δ is smooth. Take a Whitney embedding $\epsilon : M \to D^{2n}$, and consider the induced embedding $\delta \times \epsilon : M \to T^\ell \times D^{2n}$. Since M is a stably parallelizable, $M \subset T^\ell \times D^{2n}$ has trivial normal bundle $\nu_M \simeq \epsilon^{n+\ell}$. (To see this, observe that $TM \oplus \nu_M \simeq \epsilon^{2n+\ell}$. Since M is stably parallelizable, $TM \oplus \epsilon \simeq \epsilon^{n+1}$, which implies that $\epsilon^{n+1} \oplus \nu_M \simeq \epsilon^{2n+\ell+1}$. Since $\mathrm{rank}(\nu_M) > \dim M$, this implies that ν_M is the trivial bundle by [KM63, Lem. 3.5].) Then there is an embedding $\kappa : M \times D^{n+\ell} \to T^\ell \times D^{2n}$.

Consider now the composition

$$p : \Sigma^{2n}(T^\ell_+) \xrightarrow{\kappa'} \Sigma^{n+\ell}(M_+) \xrightarrow{\widehat{\iota\times 1}} \bigvee_\ell S^{2n+\ell-1}.$$

To prove the theorem, we show that the induced map

$$p^* : \big[\bigvee_\ell S^{2n+\ell-1}, Y\big] \to \big[\Sigma^{2n}(T^\ell_+), Y\big]$$

is injective. First observe the homotopy equivalence $\Sigma^{2n}(T^\ell_+) \sim \bigvee_{i=0}^{\ell} \binom{\ell}{i} S^{2n+i}$. This follows from general homotopy equivalences $\Sigma(A_+) \sim \Sigma A \vee S^1$ and $\Sigma(A \times B) \sim \Sigma A \vee \Sigma B \vee \Sigma(A \wedge B)$. Since $\Delta(c_1), \ldots, \Delta(c_\ell)$ generate $\pi_1(T^\ell)$, the inclusion $\ell\, S^{2n+\ell-1} \subset \bigvee_{i=0}^{\ell} \binom{\ell}{i} S^{2n+i}$ is a right inverse to p, up to homotopy. This implies that p^* is injective. □

4.2 Outer automorphisms not realized by diffeomorphisms (Step 2)

Next we apply Theorem 8 to give a criterion that guarantees that $\alpha \in \mathrm{Out}(\pi)$ is not in the image of $\Psi_N : \mathrm{Diff}(N) \to \mathrm{Out}(\pi)$.

Theorem 9 (obstruction to Nielsen realization)**.** *Let M be a hyperbolic manifold and fix a simple closed geodesic c in M. Let $N = M_{c,\phi}$ be an exotic smooth structure. Assume that $\alpha \in \mathrm{Isom}(M) \simeq \mathrm{Out}(\pi)$ is such that $M_{c,\phi}$ and $M_{\alpha(c),\phi}$ are not concordant. Then $\alpha \notin \mathrm{Im}\,\Psi_N$.*

Proof. Suppose for a contradiction that there is a diffeomorphism $f : N \to N$ such that $\Psi_N(f) = \alpha$.

Set $N_0 = N$ and $N_1 = M_{\alpha(c),\phi}$, and observe that $\alpha : M \to M$ induces a diffeomorphism $g_1 : N_0 \to N_1$. Define $g_2 = g_1 \circ f^{-1}$. Denoting $h_i : N_i \to M$ be the obvious homeomorphisms, the composition

$$M \xrightarrow{h_0^{-1}} N_0 \xrightarrow{g_2} N_1 \xrightarrow{h_1} M$$

induces the identity on π and is therefore homotopic to the identity. From this homotopy, we obtain a homotopy equivalence $H_0 : M \times [0,1] \to M \times [0,1]$, which restricts to a homeomorphism on the boundary. By [FJ89b, Cor. 10.6], H_0 is homotopic rel boundary to a homeomorphism H. Then the composition

$$N_0 \times [0,1] \xrightarrow{h_0 \times \mathrm{id}} M \times [0,1] \xrightarrow{H} M \times [0,1]$$

defines a smooth structure on $M \times [0,1]$ whose restriction to $M \times \{i\}$ is N_i for $i = 0,1$, i.e. N_0 and N_1 are concordant. This contradicts our assumption, so $\alpha \notin \mathrm{Im}\,\Psi_N$. $\square$

4.3 Examples (Step 3)

To complete the proof of Theorem B, we explain how to obtain examples of stably parallelizable M that satisfy the assumptions of Theorems 8 and 9. This is the content of the following proposition.

Proposition 10. *Fix $n \geq 2$. For any $d \geq 2$, there exists a stably parallelizable hyperbolic manifold M^n, a geodesic c, a subgroup $F < \mathrm{Isom}(M)$ isomorphic to $\mathbb{Z}/d\mathbb{Z} = \langle \alpha \rangle$, and $\rho \in H^1(M) \simeq \mathrm{Hom}(H_1(M), \mathbb{Z})$ such that*

$$\rho(\alpha^j c) = \begin{cases} 1 & j = 0 \\ 0 & 1 \leq j \leq d-1. \end{cases} \tag{5}$$

Consequently, the homomorphism $\Delta : H_1(M) \to \mathbb{Z}^d$ whose i-th coordinate is $\rho \circ \alpha^{-i}$ has the property that $\Delta(c), \ldots, \Delta(\alpha^{d-1}c)$ generate $\mathbb{Z}^d$.

In [Lub96], Lubotzky gave examples of hyperbolic M (both arithmetic and non-arithmetic) with a surjection $\pi_1(M) \twoheadrightarrow F_r$ to a free group of rank $r \geq 2$. By passing to a cover, we can assume that M is stably parallelizable [Sul79, pg. 553]. Proposition 10 is proved by passing to a further cover, using the general procedure of the following lemma.

Lemma 11. *Let X be a CW-complex, and let F_r denote a free group of rank $r \geq 2$. Assume there is a surjection $\pi_1(X) \twoheadrightarrow F_r$. Then for any $d \geq 2$, there exists a regular cover $Y \to X$ with deck group $\mathbb{Z}/d\mathbb{Z} = \langle \alpha \rangle$ and $c \in \pi_1(Y)$ and $\rho \in H^1(Y)$ satisfying (5).*

Proof. Take F_r with generators $a_1, \ldots, a_r$. Consider $F_r \twoheadrightarrow \mathbb{Z}/d\mathbb{Z}$ defined by $a_1 \mapsto 1$ and $a_i \mapsto 0$ for $2 \leq i \leq r$. Then $\ker[F_r \twoheadrightarrow \mathbb{Z}/d\mathbb{Z}] \simeq F_k$ with $k = 1 + d(r-1)$. It's easy to compute $H_1(F_k)$ as a $F = \mathbb{Z}/d\mathbb{Z}$-module:

$$H_1(F_k) \simeq \mathbb{Z}\{b_1\} \oplus \mathbb{Z}F\{b_2, \ldots, b_k\}.$$

(For example, realize $1 \to F_k \to F_r \to \mathbb{Z}/d\mathbb{Z} \to 0$ as a $(\mathbb{Z}/d\mathbb{Z})$-covering of graphs.) Then also $H^1(F_k) \simeq \mathbb{Z}\{\beta_1\} \oplus \mathbb{Z}F\{\beta_2, \ldots, \beta_k\}$, where β_i is dual to b_i.

Let $Y \to X$ be the cover such that $\pi_1(Y) = \ker\big[\pi_1(X) \twoheadrightarrow F_r \twoheadrightarrow \mathbb{Z}/d\mathbb{Z}\big]$. Then $\pi_1(Y) \twoheadrightarrow F_k$, and $H_1(Y) \to H_1(F_k)$ is $(\mathbb{Z}/d\mathbb{Z})$-equivariant. Choose $c \in \pi_1(Y)$ so that $c \mapsto b_2$ under $\pi_1(Y) \twoheadrightarrow F_k$, and define $\rho : \pi_1(Y) \twoheadrightarrow F_k \xrightarrow{\beta_2} \mathbb{Z}$. It's easy to verify that ρ satisfies (5). This proves the lemma. □

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