SURFACE MAPPING CLASS GROUP ACTIONS ON 3-MANIFOLDS

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ABSTRACT. For each circle bundle $S^1 \to X \to \Sigma_g$ over a surface with genus $g \geq 2$, there is a natural surjection $\pi : \mathrm{Homeo}^+(X) \to \mathrm{Mod}(\Sigma_g)$. When X is the unit tangent bundle $U\Sigma_g$, it is well-known that π splits. On the other hand π does not split when the Euler number $e(X)$ is not divisible by the Euler characteristic $\chi(\Sigma_g)$ by [CT23]. In this paper we show that this homomorphism does not split in many cases where $\chi(\Sigma_g)$ divides $e(X)$.

1. Introduction

Let Σ_g be a closed oriented surface of genus $g \geq 2$, and let $X_{g,e}$ denote the oriented S^1-bundle over Σ_g with Euler number e. Let $\mathrm{Homeo}^+(X_{g,e})$ be the group of orientation-preserving homeomorphisms of $X_{g,e}$ that act trivially on the center of $\pi_1(X_{g,e})$, and let $\mathrm{Mod}(X_{g,e}) := \pi_0\big(\mathrm{Homeo}^+(X_{g,e})\big)$ denote the mapping class group.

The (generalized) Nielsen realization problem for $X_{g,e}$ asks whether the surjective homomorphism

$$\mathrm{Homeo}^+(X_{g,e}) \to \mathrm{Mod}(X_{g,e})$$

splits over subgroups of $\mathrm{Mod}(X_{g,e})$. In this paper we study a closely related problem. For each g, e there is a surjection $\mathrm{Mod}(X_{g,e}) \to \mathrm{Mod}(\Sigma_g)$. Consider the composition

$$\pi_{g,e} : \mathrm{Homeo}^+(X_{g,e}) \twoheadrightarrow \mathrm{Mod}(X_{g,e}) \twoheadrightarrow \mathrm{Mod}(\Sigma_g).$$

Problem 1.1. Does $\pi_{g,e} : \mathrm{Homeo}^+(X_{g,e}) \twoheadrightarrow \mathrm{Mod}(\Sigma_g)$ spilt?

If $e = \pm(2g-2)$, then $X_{g,e}$ is the unit (co)tangent bundle, and $\pi_{g,e}$ does split; see [Sou10, §1]. On the other hand, if e is not divisible by $2g-2$, then the surjection $\mathrm{Mod}(X_{g,e}) \to \mathrm{Mod}(\Sigma_g)$ does not split by work of the second two authors [CT23], so $\pi_{g,e}$ also does not split in these cases. Given this, it remains to study the case when e is divisible by $2g-2$ and $e \neq \pm(2g-2)$. In these cases $\mathrm{Mod}(X_{g,e}) \to \mathrm{Mod}(\Sigma_g)$ does split [CT23], but we prove $\pi_{g,e}$ does not split in many cases.

Theorem A. *Fix a surface Σ_g of genus g and $e \in \mathbb{Z}$. Assume that $g = 4k-1$ where $k \geq 3$ and k is not a power of 2, and assume that e is divisible by $(2g-2)2p$ where p is an odd prime dividing k. Then the natural surjective homomorphism $\pi_{g,e} : \mathrm{Homeo}^+(X_{g,e}) \to \mathrm{Mod}(\Sigma_g)$ does not split.*

Date: November 26, 2023.

For example, if $e = 0$, we find that $\pi_{g,e} : \mathrm{Homeo}^+(\Sigma_g \times S^1) \to \mathrm{Mod}(\Sigma_g)$ does not split when $g = 11, 19, 23, 27, 35, 39, 43, 47, \ldots$.

Theorem A solves the Nielsen realization problem for $\mathrm{Mod}(\Sigma_g)$ subgroups of $\mathrm{Mod}(X_{g,e})$ in the cases of the theorem. Specifically, if e is divisible by $2g-2$, then $\mathrm{Mod}(X_{g,e}) \cong H^1(\Sigma_g; \mathbb{Z}) \rtimes \mathrm{Mod}(\Sigma_g)$ [CT23], and every $\mathrm{Mod}(\Sigma_g)$ subgroup of $\mathrm{Mod}(X_{g,e})$ is the image of a splitting of $\mathrm{Mod}(X_{g,e}) \to \mathrm{Mod}(\Sigma_g)$. By Theorem A, $\mathrm{Homeo}^+(X_{g,e}) \to \mathrm{Mod}(X_{g,e})$ does not split over any of these $\mathrm{Mod}(\Sigma_g)$ subgroups.

Theorem A has the following topological consequence. When $2g-2$ divides e, there is a "tautological" $X_{g,e}$-bundle $E_{g,e}^{\mathrm{taut}} \to B\,\mathrm{Homeo}(\Sigma_g)$ whose monodromy

$$\mathrm{Mod}(\Sigma_g) \cong \pi_1\big(B\,\mathrm{Homeo}(\Sigma_g)\big) \to \mathrm{Mod}(X_{g,e})$$

splits the surjection $\mathrm{Mod}(X_{g,e}) \to \mathrm{Mod}(\Sigma_g)$ (c.f. [CT23, §1]). One can ask whether or not the bundle $E_{g,e}^{\mathrm{taut}} \to B\,\mathrm{Homeo}(\Sigma_g)$ is flat. Recall that an X-bundle $E \to B$ is *flat* if there is a homomorphism $\rho : \pi_1(B) \to \mathrm{Homeo}(X)$ and an X-bundle isomorphism $E \cong X \rtimes_\rho B$. Such bundles are characterized by the existence of a horizontal foliation on E, or, equivalently, by the property that their monodromy $\pi_1(B) \to \mathrm{Mod}(X)$ lifts to $\mathrm{Homeo}(X)$. When $e = 2g-2$, the bundle $E_{g,e}^{\mathrm{taut}} \to B\,\mathrm{Homeo}(\Sigma_g)$ is flat because of the splitting of $\pi_{g,e}$ in this case. When $\pi_{g,e}$ does not split, we deduce that $E_{g,e}^{\mathrm{taut}} \to B\,\mathrm{Homeo}(S_g)$ is not flat.

Corollary 1.2. *Fix g, e as in the statement of Theorem A. Then the tautological $X_{g,e}$-bundle $E_{g,e}^{taut} \to B\,\mathrm{Homeo}(S_g)$ is not flat.*

Short proof sketch of Theorem A. The proof strategy is similar to an argument of Chen–Salter [CS22] that shows that $\mathrm{Homeo}^+(\Sigma_g) \to \mathrm{Mod}(\Sigma_g)$ does not split when $g \geq 2$. Theorem A is proved by contradiction: assuming the existence of a splitting $\mathrm{Mod}(\Sigma_g) \to \mathrm{Homeo}^+(X_{g,e})$, first we obtain, by lifting, an action of the based mapping class group $\mathrm{Mod}(\Sigma_g, *)$ on the cover $\widehat{X}_{g,e} \cong \mathbb{R}^2 \times S^1$ corresponding to the center of $\pi_1(X_{g,e})$. The conditions on g and e in Theorem A guarantee the existence of a $\mathbb{Z}/2p\mathbb{Z}$ subgroup of $\mathrm{Mod}(\Sigma_g, *)$ for which we can show the action on $\widehat{X}_{g,e}$ has a fixed circle. Denoting a generator of $\mathbb{Z}/2p\mathbb{Z}$ by α, we show that $\mathrm{Mod}(\Sigma_g, *)$ is generated by the centralizers of α^2 and α^p. This shows that the entire group $\mathrm{Mod}(\Sigma_g, *)$ acts on $\widehat{X}_{g,e}$ with a fixed circle, which contradicts the fact that the point-pushing subgroup $\pi_1(\Sigma_g) < \mathrm{Mod}(\Sigma_g, *)$ acts freely (by deck transformations) on $\widehat{X}_{g,e}$.

Other questions. Related to the $\mathrm{Mod}(\Sigma_g)$ action on the unit tangent bundle $U\Sigma_g$, we pose the following question.

Question 1.3. Do either of the following surjections split?

$$\mathrm{Diff}^+(U\Sigma_g) \to \mathrm{Mod}(\Sigma_g) \quad \text{or} \quad \mathrm{Homeo}(U\Sigma_g) \to \mathrm{Mod}(U\Sigma_g)$$

If one includes orientation-reversing diffeomorphisms and mapping classes, then if $g \geq 12$, then $\mathrm{Diff}(U\Sigma_g) \to \mathrm{Mod}^{\pm}(\Sigma_g)$ does not split by Souto [Sou10, Thm. 1].

Acknowledgement. The authors LC and BT are supported by NSF grants DMS2203178, DMS-2104346, and DMS-2005409.

2. Proof of Theorem A

Fix $g = 4k - 1$ and e as in the theorem statement, and set $\Sigma = \Sigma_g$ and $X = X_{g,e}$. Suppose for a contradiction that there is a homomorphism

$$\sigma : \mathrm{Mod}(\Sigma) \to \mathrm{Homeo}(X)$$

whose composition with $\pi = \pi_{g,e} : \mathrm{Homeo}(X) \to \mathrm{Mod}(\Sigma)$ is the identity.

2.1. **Step 1: lifting argument.** Consider the covering space $\widehat{X} = \widetilde{\Sigma} \times S^1$ of X, where $\widetilde{\Sigma} \cong \mathbb{R}^2$ is the universal cover. This is the covering corresponding to the center ζ of $\pi_1(X)$. Given the action of $\mathrm{Mod}(\Sigma)$ on X, we consider the set of all lifts of homeomorphisms in this action to $\widehat{X}$. This is an action of the pointed mapping class group $\mathrm{Mod}(\Sigma, *)$ on $\widehat{X}$. To explain this, we start with the following general proposition.

Proposition 2.1. *Let Y be a closed manifold. Let $\zeta < \pi_1(Y)$ be the center of the fundamental group, and denote $\Delta = \pi_1(Y)/\zeta$. Let $\widehat{Y} \to Y$ be the covering space with $\pi_1(\widehat{Y}) = \zeta$. Fix a basepoint $* \in Y$. Assume that the evaluation map*

$$\mathrm{Homeo}(Y) \to Y, \quad f \mapsto f(*)$$

induces a surjection $\pi_1\big(\mathrm{Homeo}(Y)\big) \twoheadrightarrow \zeta < \pi_1(Y)$. Then there is a commutative diagram

(1)

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \Delta & \longrightarrow & \mathrm{Homeo}(\widehat{Y})^{\Delta} & \xrightarrow{q} & \mathrm{Homeo}(Y) & \longrightarrow & 1 \\
 & & \| & & \downarrow \widehat{p} & & \downarrow p & & \\
1 & \longrightarrow & \Delta & \longrightarrow & \mathrm{Mod}(Y,*) & \longrightarrow & \mathrm{Mod}(Y) & \longrightarrow & 1
\end{array}$$

whose rows are exact, where the bottom row is the (generalized) Birman exact sequence. Furthermore, this diagram is a pullback diagram.

A version of Proposition 2.1 when $Y = \Sigma_g$ (whose center is trivial) is used in [CS22].

We prove Proposition 2.1 after explaining how it gives the desired lifting. In our situation, the center of $\pi_1(X)$ is the kernel of $\pi_1(X) \to \pi_1(\Sigma)$ since

$\pi_1(\Sigma)$ has trivial center. Thus Proposition 2.1 gives us the following diagram.

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \pi_1(\Sigma) & \longrightarrow & \mathrm{Homeo}(\widehat{X})^\Delta & \xrightarrow{q} & \mathrm{Homeo}(X) & \longrightarrow & 1\\
 & & \| & & \downarrow \widehat{p} & & \downarrow p & & \\
1 & \longrightarrow & \pi_1(\Sigma) & \longrightarrow & \mathrm{Mod}(X,*) & \longrightarrow & \mathrm{Mod}(X) & \longrightarrow & 1\\
 & & \| & & \uparrow & & \uparrow & & \\
1 & \longrightarrow & \pi_1(\Sigma) & \longrightarrow & \mathrm{Mod}(\Sigma,*) & \longrightarrow & \mathrm{Mod}(\Sigma) & \longrightarrow & 1
\end{array}$$

The splitting σ defines a subgroup $\mathrm{Mod}(\Sigma) < \mathrm{Mod}(X)$ and a splitting of p over this subgroup. Since the top row is a pullback of the middle row, it follows that $\widehat{p}$ splits over $\mathrm{Mod}(\Sigma,*)$ (this uses only general facts about pullbacks). Denote this splitting by

$$\widehat{\sigma} : \mathrm{Mod}(\Sigma,*) \to \mathrm{Homeo}(\widehat{X})^\Delta.$$

Under this splitting the point-pushing subgroup $\pi_1(\Sigma)$ acts by deck transformations.

Remark 2.2. If $G < \mathrm{Mod}(\Sigma)$ and $\sigma(G)$ has a fixed point $*$, then after choosing a lift $\widehat{*}$ of $*$, one can lift canonically elements of $\sigma(G)$ to $\widehat{X}$ by choosing the unique lift that fixes $\widehat{*}$. This implies that $G < \mathrm{Mod}(\Sigma)$ can be lifted to $G < \mathrm{Mod}(\Sigma,*)$ so that $\widehat{\sigma}(G)$ has a fixed point.

Proof of Proposition 2.1. First we recall the construction of the bottom row of diagram (1). Evaluation at $* \in Y$ defines a fibration

$$\mathrm{Homeo}(Y,*) \to \mathrm{Homeo}(Y) \xrightarrow{\epsilon} Y.$$

The long exact sequence of homotopy groups gives an exact sequence

$$\pi_1\big(\mathrm{Homeo}(Y)\big) \xrightarrow{\epsilon_*} \pi_1(Y) \to \mathrm{Mod}(Y,*) \to \mathrm{Mod}(Y) \to 1.$$

In general the image of ϵ_* is contained in the center of $\pi_1(Y)$; see e.g. [Hat02, §1.1, Exer. 20]. By assumption, ϵ_* surjects onto the center, so we obtain the short exact sequence in the bottom row of (1). The homomorphism $\pi_1(Y) \to \mathrm{Mod}(Y,*)$ is the so-called "point-pushing" homomorphism. It sends $\eta \in \pi_1(Y)$ (basepoint$=*$) to the time-1 map of an isotopy that pushes $*$ around η in reverse (this follows directly from the definition of the connecting homomorphism in the long exact sequence; note that it makes sense for the reverse of η to appear in defining this homomorphism since concatenation of paths is left-to-right, while composition of functions is right-to-left).

Next we define $\widehat{p} : \mathrm{Homeo}(\widehat{Y})^\Delta \to \mathrm{Mod}(Y,*)$. Fix a point $\widehat{*} \in \widehat{Y}$ that covers the basepoint $* \in Y$. Given $f \in \mathrm{Homeo}(\widehat{Y})^\Delta$. Choose a path $[0,1] \to \widehat{Y}$ from $\widehat{*}$ to $f(\widehat{*})$ and let γ_f denote the composition $[0,1] \to \widehat{Y} \to Y$. By isotopy extension, there exists an isotopy $h_t : Y \to Y$ where $h_0 = \mathrm{id}_Y$ and $h_t(*) = \gamma_f(t)$ for each $t \in [0,1]$. Define

$$\widehat{p}(f) = [h_1 \circ q(f)].$$

The map $\widehat{p}$ is well-defined. The choice of γ_f is unique only up to an element of $\pi_1(\widehat{Y}) = \zeta$. This implies that the isotopy class $[h_1 \circ f]$ is only well-defined up to composition by a point-pushing mapping class by an element of ζ, but such a point-push is trivial by assumption.

It is a straightforward exercise to check that $\widehat{p}$ is a homomorphism. The right square in diagram (1) commutes because $q(f)$ and $h_1 \circ q(f)$ are isotopic by construction. It is easy to see that the left square in the diagram commutes by applying the definition of $\widehat{p}$ to deck transformations.

Finally, regarding the claim that the diagram is a pullback, we show that the map to the fibered product

$$\widehat{p} \times q : \mathrm{Homeo}(\widehat{Y})^{\Delta} \to \mathrm{Mod}(Y, *) \times_{\mathrm{Mod}(Y)} \mathrm{Homeo}(Y)$$

is an isomorphism. The codomain consists of pairs $(\phi, g) \in \mathrm{Mod}(Y, *) \times \mathrm{Homeo}(Y)$ such that g is isotopic to a representative of the isotopy class ϕ.

We define an inverse ι to $\widehat{p} \times q$. Given (ϕ, g) in the fibered product, choose an isotopy g_t from g to a homeomorphism representing ϕ. Lift g_t to an isotopy $\widetilde{g}_t$ such that $\widetilde{g}_1$ fixes $\widehat{*}$, and define $\iota(\phi, g) = \widetilde{g}_0$. The reader can check that the maps ι and $\widehat{p} \times q$ are inverses. □

2.2. **Step 2: finite group action rigidity.** Recall that $g = 4k - 1$ and $k \geq 3$ is not a power of 2; let p be an odd prime dividing k. From Step 1, we have homomorphism $\widehat{\sigma} : \mathrm{Mod}(\Sigma, *) \to \mathrm{Homeo}(\widehat{X})$ that descends to a splitting $\sigma : \mathrm{Mod}(\Sigma) \to \mathrm{Homeo}(X)$. In this section we describe the action of a particular finite subgroup of $\mathrm{Mod}(\Sigma, *)$ on $\widehat{X}$.

Proposition 2.3. *There exists an element $\alpha \in \mathrm{Mod}(\Sigma, *)$ of order $2p$ such that the fixed sets of $\widehat{\sigma}(\alpha)$, $\widehat{\sigma}(\alpha)^2$, and $\widehat{\sigma}(\alpha)^p$ coincide and are equal to an embedded circle $c \subset \widehat{X}$.*

It is worth noting that the fixed set of a finite-order, orientation-preserving homeomorphism of a 3-manifold can be wildly embedded [MZ54].

In order to prove Proposition 2.3 we first construct the specific element α. Then we prove (Proposition 2.5) a weaker version of Proposition 2.3 with the additional assumption that the action is smooth. Finally, we combine this with a result of Pardon [Par21] and Smith theory to prove Proposition 2.3.

Construction of $\boldsymbol{\alpha}$. We obtain α as an element in a dihedral subgroup D_{4k} of $\mathrm{Mod}(\Sigma)$, where D_{4k} denotes the dihedral group of order $8k$. The dihedral action $D_{4k} \curvearrowright \Sigma$ we use has quotient Σ/D_{4k} homeomorphic to T^2 and the quotient $\Sigma \to \Sigma/D_{4k}$ has a single branch point; it is determined by the homomorphism

$$\langle x, y\rangle = F_2 \cong \pi_1(T^2 \setminus \mathrm{pt}) \to D_{4k} = \langle a, b \mid a^{4k} = b^2 = 1, bab = a^{-1}\rangle$$

$$x \mapsto a, \quad y \mapsto b.$$

By Riemann–Hurwitz, the genus of Σ is $4k-1$. The orbifold $O = \Sigma/D_{4k}$ has fundamental group

$$\pi_1^{orb}(O) = \langle x, y, h \mid h^{2k} = 1, h = [x, y]\rangle,$$

and there is a short exact sequence

$$1 \to \pi_1(\Sigma) \to \pi_1^{orb}(O) \to D_{4k} \to 1. \tag{2}$$

This sequence induces a homomorphism $\pi_1^{orb}(O) \to \mathrm{Mod}(\Sigma, *)$. We take $\alpha = h^{k/p}$, where p, as defined above, is an odd prime dividing k, which exists by assumption. Then α is an element of order $2p$ in the subgroup $\langle h\rangle \cong \mathbb{Z}/2k\mathbb{Z}$ of $\pi_1^{orb}(O) < \mathrm{Mod}(\Sigma, *)$.

Remark 2.4. The argument that follows works equally well when Σ/D_{4k} is a genus-g surface and $\Sigma \to \Sigma/D_{4k}$ has a single branched point. This provides more values of g, e for which the conclusion of Theorem A holds.

Smooth case. Here we prove the following proposition.

Proposition 2.5. *Fix $D_{4k} < \mathrm{Mod}(\Sigma)$ as above. Suppose that $\sigma : D_{4k} \to \mathrm{Diff}^+(X)$ and is a splitting of $\pi : \mathrm{Homeo}^+(X) \to \mathrm{Mod}(\Sigma)$ over D_{4k}. Then $\widehat{\sigma}(\alpha)$ fixes a unique circle on $\widehat{X} = \mathbb{H}^2 \times S^1$. Consequently, the fixed set of $\sigma(a^{2k/p})$ is nonempty.*

The last part of the statement of Proposition 2.5 follows from the preceding statement because the image of α under $\pi_1^{orb}(O) \to D_{4k}$ is $a^{2k/p}$.

Proof of Proposition 2.5. First we reduce to a more geometric setting. By Meeks–Scott [MS86, Thm. 2.1], the smooth(!) action $\sigma(D_{4k}) \curvearrowright X$ preserves some geometric metric on X. There are two possibilities for the geometry: if $e(X) = 0$, then X has $\mathbb{H}^2 \times \mathbb{R}$-geometry, and if $e(X) \neq 0$, then X has $\widetilde{\mathrm{PSL}_2(\mathbb{R})}$-geometry. We treat these cases in parallel.

The universal cover $\widetilde{X}$ (with the induced geometric structure) is either $\mathbb{H}^2 \times \mathbb{R}$ or $\widetilde{\mathrm{PSL}_2(\mathbb{R})}$. In either case, $\widetilde{X}$ has an isometric foliation by lines whose leaf space is isometric to $\mathbb{H}^2$, and this foliation is preserved by $\mathrm{Isom}(\widetilde{X})$, so there is a homomorphism $\mathrm{Isom}(\widetilde{X}) \to \mathrm{Isom}(\mathbb{H}^2)$. Let $\mathrm{Isom}^+(\widetilde{X}) < \mathrm{Isom}(\widetilde{X})$ be the group whose action on the leaves and on the leaf space are both orientation preserving. There is an exact sequence

$$1 \to \mathbb{R} \to \mathrm{Isom}^+(\widetilde{X}) \xrightarrow{F} \mathrm{Isom}^+(\mathbb{H}^2) \to 1. \tag{3}$$

See also [Sco83, §4].

Next consider the group Λ of all lifts of elements of $\sigma(D_{4k}) < \mathrm{Isom}(X)$ to $\mathrm{Isom}^+(\widetilde{X})$. This yields an exact sequence

$$1 \to \pi_1(X) \to \Lambda \to D_{4k} \to 1.$$

The action of Λ on $\widetilde{X}$ induces an action of Λ/ζ on $\widetilde{X}/\zeta = \widehat{X} \cong \mathbb{H}^2 \times S^1$, where ζ is the center of $\pi_1(X)$. This action extends to an action of $\mathrm{Isom}^+(\widetilde{X})/\zeta$, and there is a homomorphism

$$\rho : \Lambda/\zeta \to \mathrm{Isom}^+(\widetilde{X})/\zeta \xrightarrow{\cong} \mathrm{Isom}(\widehat{X}).$$

The last map is an isomorphism by the general formula $\mathrm{Isom}(\widetilde{X}/\Lambda) = N_{\mathrm{Isom}(\widetilde{X})}(\Lambda)/\Lambda$ for discrete subgroups $\Lambda < \mathrm{Isom}(\widetilde{X})$.

To prove the proposition, we first identify Λ/ζ with $\pi_1^{orb}(O)$ (Claim 2.6). Then it is a formal consequence of our setup that $\rho(h^{k/p}) = \widehat{\sigma}(\alpha)$, and after showing $\mathrm{Isom}^+(\widetilde{X})/\zeta \cong \mathrm{Isom}^+(\mathbb{H}^2) \times \mathrm{SO}(2)$ (Claim 2.7), we show that $\rho(h^{k/p})$ fixes a unique circle in $\widehat{X}$ (Claim 2.8).

Claim 2.6. The restriction of the sequence (3) to Λ is a short exact sequence

$$1 \to \zeta \to \Lambda \to \pi_1^{orb}(O) \to 1$$

where ζ is the center of $\pi_1(X)$.

Proof of Claim 2.6. Recall the map $F : \mathrm{Isom}^+(\widetilde{X}) \to \mathrm{Isom}^+(\mathbb{H}^2)$ from (3). First we identify $F(\Lambda) < \mathrm{Isom}^+(\mathbb{H}^2)$ with $\pi_1^{orb}(O)$. For this, it suffices to show that $F(\Lambda)$ fits into a short exact sequence

$$1 \to \pi_1(\Sigma) \to F(\Lambda) \to D_{4k} \to 1, \tag{4}$$

where the "monodromy" $D_{4k} \to \mathrm{Out}^+\big(\pi_1(\Sigma)\big) \cong \mathrm{Mod}(\Sigma)$ has image the given subgroup $D_{4k} < \mathrm{Mod}(\Sigma)$. This implies that $F(\Lambda) \cong \pi_1^{orb}(O)$ because $\pi_1^{orb}(O)$ is an extension of the same form (see (2)), and extensions of $\pi_1(\Sigma)$ are determined by their monodromy [Bro82, §IV.3].

To construct the extension (4), first note that the restriction of (3) to $\pi_1(X)$ is the short exact sequence

$$1 \to \zeta \to \pi_1(X) \to \pi_1(\Sigma) \to 1.$$

The group $\pi_1(\Sigma) = F(\pi_1(X))$ is normal in $F(\Lambda)$ because $\pi_1(X)$ is normal in Λ. Furthermore, the surjection $\Lambda \to F(\Lambda)$ induces a surjection $D_{4k} = \Lambda/\pi_1(X) \to F(\Lambda)/\pi_1(\Sigma)$.

The quotient map $\widetilde{X} \to \mathbb{H}^2$, which is equivariant with respect to $\Lambda \to F(\Lambda)$ descends to a map $X = \widetilde{X}/\pi_1(X) \to \mathbb{H}^2/\pi_1(\Sigma) = \Sigma$ that's equivariant with respect to $D_{4k} = \Lambda/\pi_1(X) \to F(\Lambda)/\pi_1(\Sigma)$.

Since σ is a realization, the induced action of $\sigma(D_{4k})$ on Σ is a realization of the $D_{4k} < \mathrm{Mod}(\Sigma)$, and in particular the D_{4k} action on Σ is faithful. Therefore, $F(\Lambda)/\pi_1(\Sigma) \cong D_{4k}$, and the monodromy of the associated extension

$$1 \to \pi_1(\Sigma) \to F(\Lambda) \to D_{4k} \to 1,$$

is the given inclusion $D_{4k} < \mathrm{Mod}(\Sigma)$. This concludes the proof that $F(\Lambda)$ is isomorphic to $\pi_1^{orb}(O)$.

To finish the proof of Claim 2.6, it remains to show that the intersection of Λ with $\mathbb{R} = \ker(F)$ is ζ. We do this by showing (i) $\Lambda \cap \mathbb{R}$ is the center of

Λ, and (ii) the center of Λ is contained in $\pi_1(X)$. Together with the obvious containment $\zeta < \Lambda \cap \mathbb{R}$, (i) and (ii) imply $\Lambda \cap \mathbb{R} = \zeta$.

(i): First note that $\Lambda \cap \mathbb{R}$ is central because $\mathbb{R}$ is central in $\mathrm{Isom}(\widetilde{X})$. On the other hand, the center of Λ is contained in $\Lambda \cap \mathbb{R}$ because the center of $\Lambda/(\Lambda \cap \mathbb{R}) \cong \pi_1^{orb}(O)$ has trivial center.

(ii): To show the center of Λ is contained in $\pi_1(X)$, we show that the center of Λ projects trivially to $D_{4k} = \Lambda/\pi_1(X)$. This is true because $\Lambda \to D_{4k}$ factors through $\pi_1^{orb}(O)$, which has trivial center. □

We summarize the relation between the relevant groups in Diagram (5).

$$
\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \zeta & = & \zeta & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \to & \pi_1(X) & \to & \Lambda & \to & D_{4k} & \to & 1 \\
 & & \downarrow & & \downarrow & & \| & & \\
1 & \to & \pi_1(\Sigma) & \to & \pi_1^{orb}(O) & \to & D_{4k} & \to & 1 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1 & & & &
\end{array}
\tag{5}
$$

By Claim 2.6, $\Lambda/\zeta \cong \pi_1^{orb}(O)$, so ρ takes the form

$$\rho : \pi_1^{orb}(O) \to \mathrm{Isom}^+(\widetilde{X})/\zeta \cong \mathrm{Isom}(\widehat{X})$$

By construction, this homomorphism is the restriction of $\widehat{\sigma} : \mathrm{Mod}(\Sigma, *) \to \mathrm{Homeo}(\widehat{X})$ to $\pi_1^{orb}(O)$. Since $\alpha = h^{k/p}$, to show the fixed set of $\widehat{\sigma}(\alpha)$ is a circle, it suffices to show the same statement for $\rho(h^{k/p})$. To prove this, we first compute $\mathrm{Isom}(\widehat{X}) \cong \mathrm{Isom}^+(\widetilde{X})/\zeta$.

Claim 2.7. The group $\mathrm{Isom}^+(\widetilde{X})/\zeta$ is isomorphic to $\mathrm{Isom}^+(\mathbb{H}^2) \times \mathrm{SO}(2)$.

Proof of Claim 2.7. First note that there is an extension

$$1 \to \mathrm{SO}(2) \to \mathrm{Isom}^+(\widetilde{X})/\zeta \to \mathrm{Isom}^+(\mathbb{H}^2) \to 1$$

induced from (3). This sequence is obviously split when $\widetilde{X} = \mathbb{H}^2 \times \mathbb{R}$ since $\mathrm{Isom}^+(\widetilde{X}) \cong \mathrm{Isom}^+(\mathbb{H}^2) \times \mathbb{R}$ is a product.

Assume now that $\widetilde{X} = \widetilde{\mathrm{PSL}_2(\mathbb{R})}$, and write $e = (2g-2)n$ where n is a nonzero integer. Let K denote the kernel of the universal cover homomorphism $\widetilde{\mathrm{PSL}_2(\mathbb{R})} \to \mathrm{PSL}_2(\mathbb{R})$.

We claim that $\zeta = \frac{1}{n}K$. To see this, note that the extension

$$1 \to K \to \widetilde{\mathrm{PSL}_2(\mathbb{R})} \to \mathrm{PSL}_2(\mathbb{R}) \to 1$$

pulled back under a Fuchsian representation $\pi_1(\Sigma) \to \mathrm{PSL}_2(\mathbb{R})$ induces the extension of the unit tangent bundle group $\pi_1(U\Sigma)$, which has Euler number $2-2g$, and there is an n-fold fiberwise cover $U\Sigma \to X$, so the center of $\pi_1(U\Sigma) < \pi_1(X)$ is generated by the n-the power of the generator of the center of $\pi_1(X)$, i.e. $\zeta = \frac{1}{n}K$.

The inclusion of $\widetilde{\mathrm{PSL}_2(\mathbb{R})}$ in $\mathrm{Isom}\left(\widetilde{\mathrm{PSL}_2(\mathbb{R})}\right)$ (given by left-multiplication) descends to a homomorphism

$$\mathrm{PSL}_2(\mathbb{R}) = \widetilde{\mathrm{PSL}_2(\mathbb{R})}/K \to \mathrm{Isom}\left(\widetilde{\mathrm{PSL}_2(\mathbb{R})}\right)/K \twoheadrightarrow \mathrm{Isom}\left(\widetilde{\mathrm{PSL}_2(\mathbb{R})}\right)/\zeta$$

that defines a splitting of the sequence

$$1 \to \mathrm{SO}(2) \to \mathrm{Isom}(\widetilde{\mathrm{PSL}_2(\mathbb{R})})/\zeta \to \mathrm{PSL}_2(\mathbb{R}) \to 1. \qquad \square$$

The following Claim 2.8 is the last step in the proof of Proposition 2.5.

Claim 2.8. Let p be an odd prime dividing k. If e is divisible by $(2g-2)2p$, then the fixed set of $\rho(h^{k/p})$ is a circle.

Before proving the claim, we explain how the factors of $\mathrm{Isom}^+(\mathbb{H}^2) \times \mathrm{SO}(2) \cong \mathrm{Isom}^+(\widehat{X})$ act on $\widehat{X} = \widetilde{X}/\zeta$.

Remark 2.9. Consider the isomorphism $\mathrm{Isom}(\widehat{X}) \cong \mathrm{Isom}^+(\mathbb{H}^2) \times \mathrm{SO}(2)$ from Claim 2.7. In each case ($e = 0$ or $e \neq 0$) the action of $\mathrm{SO}(2)$ on $\widehat{X}$ covers the identity of $\mathbb{H}^2$ and acts freely by rotation on the circle fibers of $X \to \mathbb{H}^2$. For the $\mathrm{Isom}^+(\mathbb{H}^2)$ action, when $e = 0$, then $\widehat{X} \cong \mathbb{H}^2 \times S^1$ is a metric product, and the action of $\mathrm{Isom}^+(\mathbb{H}^2)$ is trivial on the S^1 factor and is the natural action on $\mathbb{H}^2$. If $e = (2g-2)n$ is nonzero, then

$$\widehat{X} \cong \widetilde{\mathrm{PSL}_2(\mathbb{R})}/\zeta \cong \mathrm{PSL}_2(\mathbb{R})/(\mathbb{Z}/n\mathbb{Z}),$$

and with respect to this isomorphism, the action of $\mathrm{Isom}^+(\mathbb{H}) \cong \mathrm{PSL}_2(\mathbb{R})$ on $\widehat{X}$ is induced from left multiplication of $\mathrm{PSL}_2(\mathbb{R})$ on $\mathrm{PSL}_2(\mathbb{R})$. Identifying $\mathrm{PSL}_2(\mathbb{R})$ with the unit tangent $U\mathbb{H}^2$, we can also view $\mathrm{PSL}_2(\mathbb{R})/(\mathbb{Z}/n\mathbb{Z})$ as the quotient of $U\mathbb{H}^2$ by the $\mathbb{Z}/n\mathbb{Z}$ action that covers the identity of $\mathbb{H}^2$ and rotates each fiber.

Proof of Claim 2.8. Write $e = (2g-2)2pm$ for some integer m.

First note that since $\rho(h)$ has finite order, the induced isometry of $\mathbb{H}^2$ has a unique fixed point, so $\rho(h)$ preserves a unique circle C of the fibering $\widehat{X} \to \mathbb{H}^2$. The same is true for $\rho(h^{k/p})$, and we will show that $\rho(h^{k/p})$ acts trivially on C.

Since $h = [x, y]$ is a commutator in $\pi_1^{orb}(O)$ and $\mathrm{SO}(2)$ is abelian, we find that the projection

$$\rho(h) \in \mathrm{Isom}^+(\widehat{X}) \cong \mathrm{Isom}^+(\mathbb{H}^2) \times \mathrm{SO}(2) \to \mathrm{SO}(2)$$

is trivial. Therefore, the action of $\rho(h)$ on $\widehat{X}$ factors through $\mathrm{Isom}^+(\mathbb{H}^2)$ acting on $\widehat{X}$. This action is described in Remark 2.9. If $e = 0$, since $\mathrm{Isom}^+(\mathbb{H}^2)$ acts trivially on the S^1 factor of $\widehat{X} \cong \mathbb{H}^2 \times S^1$, we conclude that

$\rho(h)$ acts trivially on C. If $e \neq 0$, then $\rho(h)$ acts as a a rotation by $2\pi(pm/k)$ on C, so $\rho(h^{k/p})$ acts as a rotation by $2\pi m$, which is trivial. □

This completes the proof of Proposition 2.5. □

Homeomorphism case. Here prove Proposition 2.3.

Proof of Proposition 2.3. By Pardon [Par21, Thm. 1.1], there is a sequence of smooth D_{4k} actions converging in $\operatorname{Hom}\big(D_{4k}, \operatorname{Homeo}(X)\big)$ to the given action of $\sigma(D_{4k})$ on X. Sufficiently close approximates also give a splitting of π over $D_{4k} < \operatorname{Mod}(\Sigma)$ because $\operatorname{Homeo}(X)$ is locally path connected [EK71].

For each of the smooth approximations of $\sigma(D_{4k})$, the fixed set of $a^{2k/p}$ is nonempty by Proposition 2.5. This implies that $\sigma(a^{2k/p})$ has a fixed point (a sequence of fixed points, one for each smooth action, sub-converges to a fixed point of the $\sigma(a^{2k/p})$ action). By Remark 2.2, there exists a lift of $a^{2k/p} \in D_{4k} < \operatorname{Mod}(\Sigma)$ to a finite order element $\alpha' \in \pi_1^{orb}(O) < \operatorname{Mod}(\Sigma, *)$ so that $\widehat{\sigma}(\alpha')$ has a fixed point. Since $\pi_1^{orb}(O)$ has a unique conjugacy class of finite subgroup of order $2p$, the subgroups $\langle\alpha'\rangle$ and $\langle\alpha\rangle$ are conjugate, so the fixed set of $\widehat{\sigma}(\alpha)$ is nonempty.

It remains to show the fixed set of $\widehat{\sigma}(\alpha)$ is a circle, and that this circle is the same as the fixed sets of $\widehat{\sigma}(\alpha)^2$ and $\widehat{\sigma}(\alpha)^p$.

First we show (using Smith theory) that both $\widehat{\sigma}(\alpha)^2$ and $\widehat{\sigma}(\alpha)^p$ have fixed set a single circle (we are not yet claiming/arguing that the fixed sets of $\widehat{\sigma}(\alpha)^2$ and $\widehat{\sigma}(\alpha)^p$ are the same). To see this, we focus on $\widehat{\sigma}(\alpha)^2$ for concreteness. Consider the group Λ_0 of all lifts of powers of $\widehat{\sigma}(\alpha)^2$ to the universal cover $\widetilde{X}$. This group is an extension

$$1 \to \mathbb{Z} \to \Lambda_0 \to \mathbb{Z}/p\mathbb{Z} \to 1,$$

which is central and split; hence $\Lambda_0 \cong \mathbb{Z}\times\mathbb{Z}/p\mathbb{Z}$. It is central because α acts orientation-preservingly on fibers of $X \to \Sigma$ (otherwise, the action of α on Σ would reverse orientation, contrary to the construction); it splits because $\widehat{\sigma}(\alpha)$ has a fixed point. The (lifted) action of $\widehat{\sigma}(\alpha)^2$ on $\widetilde{X}$ has fixed set a line (i.e. embedded copy of $\mathbb{R}$) by Smith theory and local Smith theory [Bre12, Theorem 20.1], and this line is preserved and acted properly by $\mathbb{Z} < \Lambda_0$; thus $\widehat{\sigma}(\alpha)^2$ acts on $\widehat{X}$ with a circle in its fixed set. Furthermore, each component of the fixed set of $\widehat{\sigma}(\alpha)^2$ acting on $\widehat{X}$ corresponds to a distinct conjugacy class of order-p subgroup of Λ_0. Since there is only one $\mathbb{Z}/p\mathbb{Z}$ subgroup of Λ_0, the fixed set of $\widehat{\sigma}(\alpha)^2$ is connected, i.e. a single circle. The same argument[1] works for $\widehat{\sigma}(\alpha)^p$.

Now we determine the fixed set of $\widehat{\sigma}(\alpha)$. First observe that $\widehat{\sigma}(\alpha)$ preserves the fixed set of $\widehat{\sigma}(\alpha)^2$ and has a fixed point there (the fixed set of $\widehat{\sigma}(\alpha)$ is nonempty and contained in the fixed set of $\widehat{\sigma}(\alpha)^2$). The only $\mathbb{Z}/2p\mathbb{Z}$ action on the circle with a fixed point is the trivial action, so in fact the fixed sets

[1]Smith theory applies to prime-order finite cyclic group actions, so we cannot apply this argument directly to $\widehat{\sigma}(\alpha)$.

of $\widehat{\sigma}(\alpha)$ and $\widehat{\sigma}(\alpha)^2$ are the same. The same argument applies to $\widehat{\sigma}(\alpha)$ and $\widehat{\sigma}(\alpha)^p$. This proves Proposition 2.3. □

2.3. **Step 3: centralizer argument.** Recall that we have defined α as an element of order $2p$ in $\pi_1^{orb}(O) < \mathrm{Mod}(\Sigma, *)$. In this step we prove that $\mathrm{Mod}(\Sigma, *)$ is generated by the centralizers of α^2 and α^p.

Proposition 2.10 (centralizer property)**.** *Let $\alpha \in \mathrm{Mod}(\Sigma, *)$ be the element of order $2p$ constructed above. Then*

$$\mathrm{Mod}(\Sigma, *) = \langle C(\alpha^2), C(\alpha^p)\rangle,$$

where $C(-)$ denotes the centralizer in $\mathrm{Mod}(\Sigma, *)$.

Strategy for proving Proposition 2.10. Set $\Gamma = \langle C(\alpha^2), C(\alpha^p)\rangle$. Our method for showing $\Gamma = \mathrm{Mod}(\Sigma, *)$, which is similar to the proof of [CS22, Thm. 1.1], is to inductively build subsurfaces

$$S_0 \subset S_1 \subset \cdots \subset S_N \subset \Sigma \setminus \{*\} \tag{6}$$

such that $\mathrm{Mod}(S_n) \subset \Gamma$ for each n and S_N fills $\Sigma \setminus \{*\}$ (i.e. each boundary component of S_N is inessential in $\Sigma \setminus \{*\}$). The fact that S_N fills implies that $\mathrm{Mod}(S_N) = \mathrm{Mod}(\Sigma, *)$, so then $\mathrm{Mod}(\Sigma, *) \subset \Gamma$ by the last step in the inductive argument.

In order to ensure that $\mathrm{Mod}(S_n) \subset \Gamma$, the subsurface S_n is obtained from S_{n-1} by an operation known as *subsurface stabilization.* If $S \subset \Sigma$ is a subsurface and $c \subset \Sigma$ is a simple closed curve that intersects S in a single arc, then the *stabilization of S along c* is the subsurface $S \cup N(c)$, where $N(c)$ is a regular neighborhood of c. It is easy to show that $\mathrm{Mod}(S \cup N(c))$ is generated by $\mathrm{Mod}(S)$ and the Dehn twist τ_c [CS22, Lem. 4.2], so if $\mathrm{Mod}(S) \subset \Gamma$ and $\tau_c \in \Gamma$, then $\mathrm{Mod}(S \cup N(c)) \subset \Gamma$. Therefore, for the proof, it suffices to find a sequence of subsurface stabilizations along curves whose Dehn twist belongs to $\Gamma = \langle C(\alpha^2), C(\alpha^p)\rangle$.

Model for the α action. Our proof of Proposition 2.10 makes use of an explicit model for Σ with its α action, which is pictured below in the case $k = 6$ and $p = 3$ (recall that $g = 4k - 1$ and p is an odd prime dividing k).

The surface Σ is built out of two copies of the standard action of $\mathbb{Z}/2p\mathbb{Z}$ on S^2 and one copy of a free action of $\mathbb{Z}/2p\mathbb{Z}$ on T^2. We glue each copy of S^2 to T^2 along k/p free orbits by an equivariant connected sum. In the figure, α acts by vertical translation. Note that the fixed points of α on S^2 are not pictured in the figure – they are at $\pm\infty$ along the x-axis.

To derive this model, recall that $D_{4k} = \langle a, b \mid a^{4k} = 1 = b^2, bab = a^{-1}\rangle$ has abelianization $D_{4k} \to (\mathbb{Z}/2\mathbb{Z})^2$ with kernel $\langle a^2\rangle \cong \mathbb{Z}/2k\mathbb{Z}$. Then there is a sequence of regular covers

$$\Sigma \xrightarrow{\langle a^2\rangle} \Sigma/\langle a^2\rangle \xrightarrow{(\mathbb{Z}/2\mathbb{Z})^2} \Sigma/D_{4k}.$$

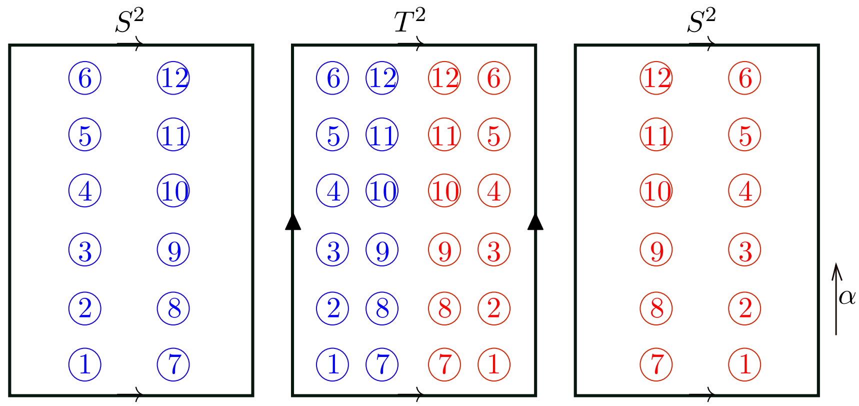

FIGURE 1. The model of the surface Σ_{23} where $p = 3$.

The cover $\Sigma/\langle a^2\rangle \to \Sigma/D_{4k}$ is unbranched and is the $\mathbb{Z}/2\mathbb{Z}$-homology cover of T^2 (in particular, $\Sigma/\langle a^2\rangle$ is also a torus). The cover $\Sigma \to \Sigma/\langle a^2\rangle$ is branched over four points; the local monodromy around the branched points is a^2 at two of the branched points and a^{-2} at the other two. Choosing branched cuts joining branched points in $a^{\pm 2}$ pairs gives a model for Σ, and one can check that this model is equivalent to the one described above. (The spheres in Figure 1 arise from pre-images under $\Sigma \to \Sigma/\langle a^2\rangle$ of neighborhoods of the branch cuts.)

By Remark 2.2, since $\sigma(a^{2k/p})$ has a fixed point, the subgroup $\langle\alpha\rangle \subset \mathrm{Mod}(\Sigma,*)$, which lifts $\langle a^{2k/p}\rangle$, has a fixed point. The different lifts of $\langle a^{2k/p}\rangle$ to a finite subgroup of $\mathrm{Mod}(\Sigma,*)$ are in one-to-one correspondence to fixed points of $a^{2k/p}$. Since these fixed points are permuted transitively by the action of D_{4k}, the different lifts of $\langle a^{2k/p}\rangle$ are conjugate. Consequently, for the purpose of our argument, we can choose $*$ to be any one of the four fixed points of α and prove Proposition 2.10 for this choice, without loss of generality. (It will also be evident from the argument that a similar argument applies if $*$ is changed to another fixed point.)

Remark 2.11. We do not known how generally the relation $\mathrm{Mod}(\Sigma,*) = \langle C(\alpha^2), C(\alpha^p)\rangle$ holds. For example, it may hold for every $\mathbb{Z}/2p\mathbb{Z}$ subgroup of $\mathrm{Mod}(\Sigma,*)$. We do not know a general (abstract) approach to this problem.

Proof of Proposition 2.10.

Symmetry breaking. In preparation for constructing a sequence of subsurface stabilizations, in this paragraph we find a suitable collection of Dehn twists that belong to Γ. The obvious way for τ_c to belong to Γ is if c is preserved by either α^2 or α^p. More generally, we use a process that we call *symmetry breaking* to show $\tau_c \in \Gamma$ for certain c. We formulate this in the following lemma, which is similar to [CS22, Lem. 3.2].

Lemma 2.12 (Symmetry breaking)**.** *Assume that $c, d \subset \Sigma$ are simple closed curves that intersect once and $\tau_d \in \Gamma$. Suppose that either* (i) *$\alpha^p(c)$ is* *disjoint from c and d or* (ii) *the curves $\alpha^2(c), \alpha^4(c) \dots, \alpha^{2p-2}(c)$ are disjoint from d and the curves $c, \alpha^2(c), \alpha^4(c) \dots, \alpha^{2p-2}(c)$ are pairwise disjoint. Then $\tau_c \in \Gamma$.*

Proof of Lemma 2.12. We prove case (i) of the statement; case (ii) is similar. Since Dehn twists about disjoint curves commute,

$$\big(\tau_c\tau_{\alpha^p(c)}\big)\tau_d\big(\tau_c\tau_{\alpha^p(c)}\big)^{-1} = \tau_c\tau_d\tau_c^{-1}.$$

The left hand side of the equation is in Γ because $\tau_d \in \Gamma$ by assumption and $\tau_c\tau_{\alpha^p(c)} \in \Gamma$ because the curves $c, \alpha^p(c)$ are permuted by α^p and are disjoint (so their twists commute), and thus $\tau_c\tau_{\alpha^p(c)} \in C(\alpha^p) \subset \Gamma$. Since c and d intersect once, the braid relation implies that $\tau_d\tau_c\tau_d^{-1} = \tau_c\tau_d\tau_c^{-1}$ also belongs to Γ. Since $\tau_d \in \Gamma$ this implies that $\tau_c \in \Gamma$, as desired. □

Remark 2.13. When applying Lemma 2.12(i) or (ii) we refer to it as the α^p- or α^2-symmetry breaking, respectively.

Lemma 2.14. *Dehn twists about the curves in Figure 2 are in Γ.*

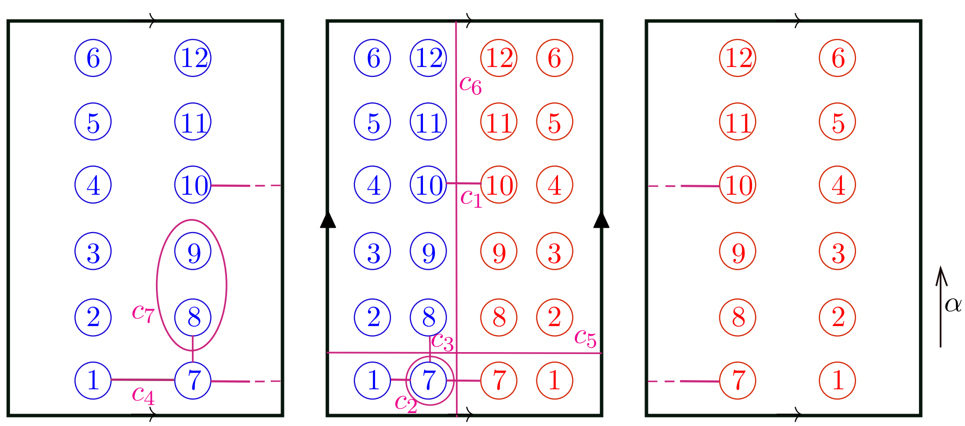

FIGURE 2. The curves used in the proof of Lemma 2.14. Here we are using the model for Σ_{23}, but the proof follows in the same way for similar types of curves on any Σ.

In Figure 2, we illustrate the case $k = 6, p = 3$. The corresponding curves in the general case belong to Γ by the exact same argument.

Proof of Lemma 2.14. First observe that τ_{c_1} and τ_{c_6} are in Γ because each is invariant under α^p. We deduce that $\tau_{c_2} \in \Gamma$ using α^2-symmetry breaking with $d = c_1$. Each of τ_{c_3} and τ_{c_4} are in Γ by α^p-symmetry breaking with $d = c_2$. Finally, both τ_{c_5} and τ_{c_7} are in Γ by α^p-symmetry breaking with $d = c_3$. □

Surface stabilization sequence. We stabilize with the sequence of curves represented in Figure 3. To get the initial subsurface S_0 we can take the subsurface spanned by the chain of curves $c_0, c_1, \ldots, c_4$. This subsurface has genus 2 and one boundary component, and these curves are Humphries generators for S_0 [FM12, Fig. 4.10]. Next we extend this chain with the curves $c_5, \ldots, c_{4k-1}$; at this point the left genus-0 subsurface has been filled. Next we stabilize with c_{4k} and the curves $(c_{4k+1}, c_{4k+2}, \cdots, c_{8k-2})$ that fill the right genus-0 subsurface; there is some choice in the order of curves we stabilize, but this is not important. Finally we stabilize with the curves c_{8k-1} and c_{8k} that generate $\pi_1(T^2)$.

All of the twists about the curves used are in Γ. In each case, this can be seen either directly from the statement of Lemma 2.14 or by an argument that is a small variation of its proof. Since this collection of curves fills Σ, we have shown that $\Gamma = \text{Mod}(\Sigma, *)$. This proves Proposition 2.10. □

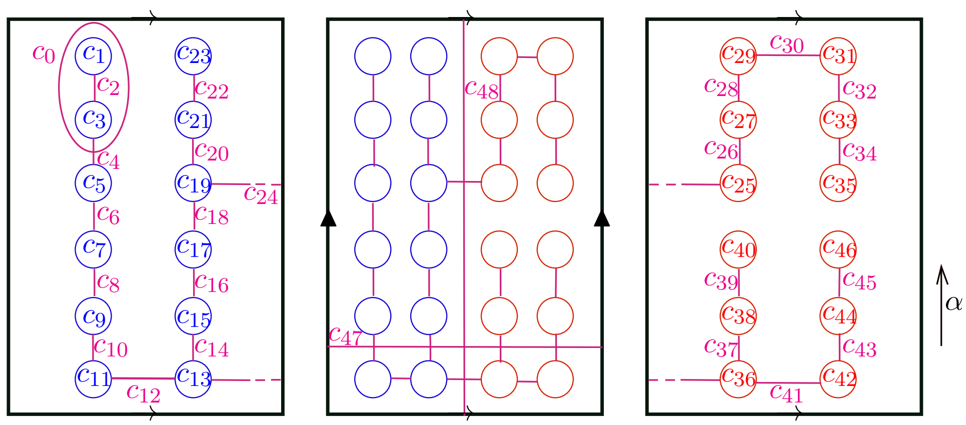

FIGURE 3. The stabilization sequence we use for the case $k = 6$ and $p = 3$.

2.4. **Step 4: conclusion.** By Proposition 2.3, $\widehat{\sigma}(\alpha)$, $\widehat{\sigma}(\alpha^2)$ and $\widehat{\sigma}(\alpha^p)$ all have the same fixed set, which is a circle $c \subset \widehat{X}$. The centralizers $C(\alpha^2)$ and $C(\alpha^p)$ preserve c. By Proposition 2.10, $\text{Mod}(\Sigma, *) = \langle C(\alpha^2), C(\alpha^p)\rangle$, so $\widehat{\sigma}\big(\text{Mod}(\Sigma, *)\big)$ preserves c. This contradicts the fact that $\widehat{\sigma}(\pi_1(\Sigma_g))$ acts as the deck group, which as a properly discontinuous action does not preserve any compact set.

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