# HOMOTOPY-ANOSOV $\mathbb{Z}^2$ ACTIONS ON EXOTIC TORI

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ABSTRACT. We give examples of Anosov actions of $\mathbb{Z}^2$ on the $d$-torus $T^d$ that cannot be homotoped to a smooth action on $T^d\#\Sigma$, for certain exotic $d$-spheres $\Sigma$. This is deduced using work of Krannich, Kupers, and the authors that, in particular, computes the mapping class group of $T^d\#\Sigma$.

## 1. Introduction

An exotic $d$-torus $\mathfrak{T}$ is a closed smooth manifold that is homeomorphic but not diffeomorphic to the standard torus $T^d = \mathbb{R}^d/\mathbb{Z}^d$. For example, the connected sum $T^d\#\Sigma$ of $T^d$ with an exotic $d$-sphere $\Sigma$ is an exotic torus.

In this note we are interested in smooth group actions on exotic tori.

**Question 1.** Given an exotic torus $\mathfrak{T}$ and an action $G \curvearrowright T^d$ on the standard torus, is there an action of $G$ on $\mathfrak{T}$ that induces the same action on the fundamental group $\pi_1(\mathfrak{T}) \cong \mathbb{Z}^d \cong \pi_1(T^d)$? If so, we say the two actions are $\pi_1$-*equivalent*.

For example, if $\mathfrak{T} = T^d\#\Sigma$ and $G = \mathbb{Z}$, then for every action of $\mathbb{Z}$ on $T^d$, there exists a $\pi_1$-equivalent action of $\mathbb{Z}$ on $\mathfrak{T}$ (c.f. Remark 5). In contrast, for $G = \mathrm{SL}_d(\mathbb{Z})$ there exist $\mathfrak{T} = T^d\#\Sigma$ for which there is no action of $\mathrm{SL}_d(\mathbb{Z})$ on $T^d\#\Sigma$ that is $\pi_1$-equivalent to the linear action $\mathrm{SL}_d(\mathbb{Z}) \curvearrowright T^d$; this is shown by Krannich, Kupers, and the authors [BKKT23, Cor. C].

Below, for $G = \mathbb{Z}^2$, we show that not every action $\mathbb{Z}^2 \curvearrowright T^d$ is $\pi_1$-equivalent to an action on $T^d\#\Sigma$. For our examples, we can take the action $\mathbb{Z}^2 \curvearrowright T^d$ to be *Anosov*, i.e. some $g \in \mathbb{Z}^2$ acts as an Anosov diffeomorphism.

**Theorem 2.** *There exist exotic tori $\mathfrak{T} = T^d\#\Sigma$ and Anosov actions $\mathbb{Z}^2 \curvearrowright T^d$ for which there is no smooth $\mathbb{Z}^2$ action on $\mathfrak{T}$ that is $\pi_1$-equivalent to the given action $\mathbb{Z}^2 \curvearrowright T^d$.*

Theorem 2 is a direct consequence of Theorem 3 below. To state it, let $\Theta_d$ denote the Milnor–Kervaire group of homotopy $d$-spheres, let $\eta \in \pi_1^s \cong \mathbb{Z}/2$ denote the generator of the first stable homotopy group of spheres, and write $\eta \cdot \Sigma$ for the Milnor–Munkres–Novikov pairing $\pi_1^s \times \Theta_d \to \Theta_{d+1}$; see [Bre67] and also [BKKT23, §1.3.2].

Fixing a isomorphism $\pi_1(\mathfrak{T}) \cong \mathbb{Z}^d$, we write $\ell : \mathrm{Diff}^+(\mathfrak{T}) \to \mathrm{SL}_d(\mathbb{Z})$ for the homomorphism induced by the action on $\pi_1$. Recall that $A \in \mathrm{SL}_d(\mathbb{Z})$ is called hyperbolic if it has no eigenvalues on the unit circle.

**Theorem 3.** *Fix $d \geq 7$. Assume $\Sigma \in \Theta_d$ is a homotopy sphere such that $\eta \cdot \Sigma$ is not divisible by 2 in $\Theta_{d+1}$. Then there exist infinitely many conjugacy classes of subgroups*

*$G \cong \mathbb{Z}^2 < \mathrm{SL}_d(\mathbb{Z})$ such that (i) $G$ is generated by hyperbolic matrices, and (ii) the homomorphism $\ell : \mathrm{Diff}^+(T^d \# \Sigma) \to \mathrm{SL}_d(\mathbb{Z})$ does not split over $G$.*

**Remark 4.** The condition that $\eta \cdot \Sigma$ is not divisible by 2 in $\Theta_{d+1}$ holds for exotic spheres $\Sigma$ in infinitely many dimensions $d$; see [BKKT23, Rmk. 1.10].

We prove Theorem 3 in §2. To deduce Theorem 2 from Theorem 3, assume $\Sigma$ and $G < \mathrm{SL}_d(\mathbb{Z})$ satisfy the conditions in Theorem 3. The linear action of $G < \mathrm{SL}_d(\mathbb{Z})$ on $T^d$ is Anosov because $G < \mathrm{SL}_d(\mathbb{Z})$ contains a hyperbolic matrix. If this action $G \curvearrowright T^d$ is $\pi_1$-equivalent to an action on $T^d \# \Sigma$, then $\mathrm{Diff}^+(T^d \# \Sigma) \to \mathrm{SL}_d(\mathbb{Z})$ splits over $G$, contradicting the assumption on $G$.

**Remark 5.** In contrast to $T^d \# \Sigma$, if one considers exotic tori of the form $\mathfrak{T} \cong (T^{d-1} \# \Sigma^{d-1}) \times S^1$, then it is possible to give examples of (Anosov) $G \cong \mathbb{Z}$ acting on $T^d$ that are not $\pi_1$-equivalent to any smooth action on $\mathfrak{T}$. This is because for these $\mathfrak{T}$ the homomorphism $\mathrm{Diff}^+(\mathfrak{T}) \to \mathrm{SL}_d(\mathbb{Z})$ is not surjective [BKKT23, Lem. 3.1] (and one can choose $G$ generated by a hyperbolic matrix not in the image).

**Remark 6.** The $G$ constructed in the proof of Theorem 3 are *without rank-one factors*, c.f. [RHW14, Defn. 2.8]. Rodriguez-Hertz–Wang [RHW14, Cor. 1.2] show that if $G < \mathrm{SL}_d(\mathbb{Z})$ contains a hyperbolic element and is without rank-one factors, then no exotic $d$-torus $\mathfrak{T}$ has an *Anosov* action that is $\pi_1$-equivalent to the linear action of $G < \mathrm{SL}_d(\mathbb{Z})$ on $T^d$. Theorem 2 gives a stronger conclusion, with "Anosov" replaced by "smooth", albeit with additional assumptions on $\Sigma$ and $G$. Related to [RHW14], we remark that there are examples of Anosov actions of $\mathbb{Z}$ on exotic tori $T^d \# \Sigma$, due to Farrell–Jones and Farrell–Gogolev [FJ78, FG12].

**Remark 7.** With the same assumption on $\Sigma$ as in Theorem 3, Krannich, Kupers, and the authors show that the surjection $\mathrm{Diff}^+(T^d \# \Sigma) \twoheadrightarrow \mathrm{SL}_d(\mathbb{Z})$ does not split; in fact, there is no splitting of $\mathrm{Mod}(T^d \# \Sigma) \twoheadrightarrow \mathrm{SL}_d(\mathbb{Z})$, where $\mathrm{Mod}(-) = \pi_0 \mathrm{Diff}^+(-)$ is the mapping class group [BKKT23, Thm. A]. Theorem 3 is proved by finding $G \cong \mathbb{Z}^2 < \mathrm{SL}_d(\mathbb{Z})$ that are generated by hyperbolic matrices and such that the map $\mathrm{Mod}(T^d \# \Sigma) \to \mathrm{SL}_d(\mathbb{Z})$ does not split over $G$.

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## 2. Proof of Theorem 3

Fix $\Sigma \in \Theta_d$ as in the statement of the Theorem 3, and set $\mathfrak{T} := T^d \# \Sigma$. To show $\mathrm{Diff}^+(\mathfrak{T}) \to \mathrm{SL}_d(\mathbb{Z})$ does not split over $G < \mathrm{SL}_d(\mathbb{Z})$, it suffices to show that $\mathrm{Mod}(\mathfrak{T}) \to \mathrm{SL}_d(\mathbb{Z})$ does not split over $G$, where $\mathrm{Mod}(\mathfrak{T}) := \pi_0 \mathrm{Diff}^+(\mathfrak{T})$ is the mapping class group. We proceed in three steps.

**Step 1: Lie group reduction.** Fix $d \geq 7$. To show that $\mathrm{Mod}(\mathfrak{T}) \twoheadrightarrow \mathrm{SL}_d(\mathbb{Z})$ does not split over $G < \mathrm{SL}_d(\mathbb{Z})$ it suffices to show that the universal cover short exact sequence

$$1 \to \mathbb{Z}/2\mathbb{Z} \to \widetilde{\mathrm{SL}_d(\mathbb{R})} \to \mathrm{SL}_d(\mathbb{R}) \to 1 \tag{1}$$

does not split over $G < \mathrm{SL}_d(\mathbb{Z}) \hookrightarrow \mathrm{SL}_d(\mathbb{R})$. To explain this reduction, let

$$1 \to \mathbb{Z}/2\mathbb{Z} \to \widetilde{\mathrm{SL}}_d(\mathbb{Z}) \to \mathrm{SL}_d(\mathbb{Z}) \to 1$$

be the short exact sequence obtained by pullback of (1) along the inclusion $\mathrm{SL}_d(\mathbb{Z}) \hookrightarrow \mathrm{SL}_d(\mathbb{R})$. By [BKKT23, Thm. D], when $\eta \cdot \Sigma$ is not divisible by 2, there is an isomorphism $\mathrm{Mod}(\mathfrak{T}) \cong K \rtimes \widetilde{\mathrm{SL}}_d(\mathbb{Z})$ (where $K$ is a group whose precise form is not important here), and there is a commutative diagram

$$\begin{array}{ccc} & K \rtimes \widetilde{\mathrm{SL}}_d(\mathbb{Z}) \cong \mathrm{Mod}(\mathfrak{T}) & \\ \swarrow & & \searrow \\ \widetilde{\mathrm{SL}}_d(\mathbb{Z}) & \longrightarrow & \mathrm{SL}_d(\mathbb{Z}) \end{array}$$

This implies that if $\mathrm{Mod}(\mathfrak{T}) \to \mathrm{SL}_d(\mathbb{Z})$ splits over $G$, then $\widetilde{\mathrm{SL}}_d(\mathbb{Z}) \to \mathrm{SL}_d(\mathbb{Z})$ and hence also $\widetilde{\mathrm{SL}_d(\mathbb{R})} \to \mathrm{SL}_d(\mathbb{R})$ split over $G$.

**Step 2: a particular $\mathbb{Z}^2$ subgroup of $\mathrm{SL}_d(\mathbb{Z})$.** For each $d \geq 3$, we give a particular recipe for a pair of commuting hyperbolic matrices $A_1, A_2 \in \mathrm{SL}_d(\mathbb{Z})$ that generate a subgroup isomorphic to $\mathbb{Z}^2$; in Step 3 we prove that $\widetilde{\mathrm{SL}}_d(\mathbb{Z}) \to \mathrm{SL}_d(\mathbb{Z})$ does not split over $G = \langle A_1, A_2 \rangle$. Briefly, given $d \geq 3$, we write $d = n + 3$, and we define $A_i$ to be a block diagonal matrix $\begin{pmatrix} B_i & \\ & C_i \end{pmatrix}$, where $B_i \in \mathrm{SL}_3(\mathbb{Z})$ and $C_i \in \mathrm{SL}_n(\mathbb{Z})$ are hyperbolic matrices as defined in the following paragraphs.

First we construct commuting hyperbolic matrices $B_1, B_2 \in \mathrm{SL}_3(\mathbb{Z})$ that are conjugate in $\mathrm{SL}_3(\mathbb{R})$ to diagonal matrices of the form

$$\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \frac{1}{\lambda_1 \lambda_2} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \frac{1}{\mu_1 \mu_2} & 0 & 0 \\ 0 & \mu_1 & 0 \\ 0 & 0 & \mu_2 \end{pmatrix} \tag{2}$$

respectively, where $\lambda_1, \lambda_2, \mu_1, \mu_2$ are all negative and different from $-1$. As an explicit example, consider the polynomial $\xi = x^3 + x^2 - 2x - 1$. The totally real cubic field $K = \mathbb{Q}[x]/(\xi)$ has discriminant 49 (the smallest possible). Fixing a root $\alpha$ of $\xi$ in $K$, the group of units $\mathcal{O}_K^\times$, modulo its torsion subgroup (which is isomorphic to $\mathbb{Z}/2\mathbb{Z}$, generated by $-1$), is freely generated by $\epsilon_1 := \alpha^2 + \alpha - 1$ and $\epsilon_2 := -\alpha^2 + 2$. The action of the units $-\epsilon_1$ and $\epsilon_1\epsilon_2$ on the ring of integers $\mathcal{O}_K$ with the basis $\mathcal{O}_K \cong \mathbb{Z}\{1, \alpha, \alpha^2\}$ gives matrices as in (2). These claims are about this number field are contained in [Coh93, §B.4].

Next we recall that for each $n \geq 3$, there exists a subgroup $\mathbb{Z}^2 < \mathrm{SL}_n(\mathbb{Z})$ generated by hyperbolic matrices $C_1, C_2$ such that all eigenvalues of $C_1$ and $C_2$ are real and positive. Indeed, let $K/\mathbb{Q}$ be a degree $n$ totally real number field. Choose linearly independent units $\alpha_1, \alpha_2 \in \mathcal{O}_K^\times$, and let $C_i$ be the matrix for multiplication by $\alpha_i$ on $\mathcal{O}_K \cong \mathbb{Z}^n$ (with respect to any basis). Since the Galois conjugates of the $\alpha_i$ are real and not equal to $\pm 1$, they do not lie on the unit circle, so the matrices $C_i$ are hyperbolic. Furthermore, after replacing $\alpha_i$ by $\alpha_i^2$, we can ensure that the eigenvalues of $C_i$ are positive.

**Step 3: computing the obstruction to splitting.** Let $G = \langle A_1, A_2 \rangle \cong \mathbb{Z}^2$ be the subgroup of $\mathrm{SL}_d(\mathbb{Z})$ defined in Step 2 above. To complete the proof of Theorem 3, it remains to show that the short exact sequence

$$1 \to \mathbb{Z}/2\mathbb{Z} \to \widetilde{\mathrm{SL}_d(\mathbb{R})} \to \mathrm{SL}_d(\mathbb{R}) \to 1 \tag{3}$$

does not split over $G$.

Recall the following algorithm for deciding if the sequence (3) splits over $G \cong \mathbb{Z}^2 \hookrightarrow \mathrm{SL}_d(\mathbb{R})$. Compare with [Han92].

(i) Choose lifts $\widetilde{A}_1, \widetilde{A}_2 \in \widetilde{\mathrm{SL}_d(\mathbb{R})}$ of the generators of $G$. Using the definition of the universal cover as a set of paths, choosing lifts amounts to choosing paths from $A_i$ to the identity in $\mathrm{SL}_d(\mathbb{R})$.
(ii) Compute the commutator $[\widetilde{A}_1, \widetilde{A}_2]$; this element belongs to the kernel group $\mathbb{Z}/2\mathbb{Z}$, which can be identified with $\pi_1\big(\mathrm{SL}_d(\mathbb{R})\big)$ (the commutator defines a loop in $\mathrm{SL}_d(\mathbb{R})$ based at the identity). The sequence (3) splits over $G$ if and only if the loop $[\widetilde{A}_1, \widetilde{A}_2]$ represents the trivial element of $\pi_1\big(\mathrm{SL}_d(\mathbb{R})\big)$.

To apply this algorithm, we first define particular paths $\widetilde{A}_i$ from $A_i$ to the identity for which the obstruction $[\widetilde{A}_1, \widetilde{A}_2]$ is easy to compute. First, by conjugating, we may assume $A_1, A_2$ are diagonal (note that commuting hyperbolic matrices are simultaneously diagonalizable). Next we choose paths $\gamma_1(t)$ and $\gamma_2(t)$, $0 \le t \le 1$, within the group of diagonal matrices between $A_1$ and $A_2$ and $D_1 = (-1, -1, 1, 1, \ldots, 1)$ and $D_2 = (1, -1, -1, 1, \ldots, 1)$, respectively (recall how $A_1, A_2$ were defined in Step 2). We orient the paths $\gamma_i$ so that $\gamma_i(0) = D_i$ and $\gamma_i(1) = A_i$. The matrices $D_i$ belong to $\mathrm{SO}(3) < \mathrm{SL}_3(\mathbb{R}) < \mathrm{SL}_d(\mathbb{R})$. Next consider the paths $\eta_i(t)$, $0 \le t \le 1$,

$$\eta_1(t) = \begin{pmatrix} \cos(\pi t) & -\sin(\pi t) & 0 \\ \sin(\pi t) & \cos(\pi t) & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \eta_2(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\pi t) & -\sin(\pi t) \\ 0 & \sin(\pi t) & \cos(\pi t) \end{pmatrix}.$$

The concatenation $\eta_i * \gamma_i$ is a path in $\mathrm{SL}_d(\mathbb{R})$ from the identity to $A_i$ and is our specified lift $\widetilde{A}_i \in \widetilde{\mathrm{SL}_d(\mathbb{R})}$.

Having chosen $\widetilde{A}_i$, we compute the commutator $[\widetilde{A}_1, \widetilde{A}_2]$. Recall that the multiplication in $\widetilde{\mathrm{SL}_d(\mathbb{R})}$ of two paths $\lambda(t), \mu(t)$ in $\mathrm{SL}_d(\mathbb{R})$ based at the identity is the pointwise product path $t \mapsto \lambda(t) \cdot \mu(t)$ (this holds in any Lie group). Since $\widetilde{A}_i = \eta_i * \gamma_i$ and the paths $\gamma_1, \gamma_2$ pointwise commute (being contained in the diagonal group), it suffices to compute the commutator $[\eta_1, \eta_2]$ for the paths $\eta_i$ from the identity to $D_i$. For this, it is helpful to recall that the pointwise product of paths $\lambda, \mu$ is homotopic to the concatenation $\lambda * (\lambda(1) \cdot \eta)$ of $\lambda$ with the path $t \mapsto \lambda(1) \cdot \eta(t)$ (again this holds in any Lie group). Consequently, the path $\eta_1 \eta_2 \eta_1^{-1} \eta_2^{-1}$ is homotopic to the concatenation of paths

$$\eta_1 * (D_1 \cdot \eta_2) * (D_1 D_2 \cdot \eta_1^{-1}) * (D_1 D_2 D_1^{-1} \cdot \eta_2^{-1}).$$

Note that $D_1 D_2 D_1^{-1} = D_2$. One can compute directly that this loop represents a generator of $\pi_1\big(\mathrm{SO}(3)\big) \cong \mathbb{Z}/2\mathbb{Z}$. A picture of this path is given in Figure 1.

This shows that $G = \langle A_1, A_2 \rangle \hookrightarrow \mathrm{SL}_d(\mathbb{R})$ does not lift to $\widetilde{\mathrm{SL}_d(\mathbb{R})}$, as desired. This concludes the proof of Theorem 3. □

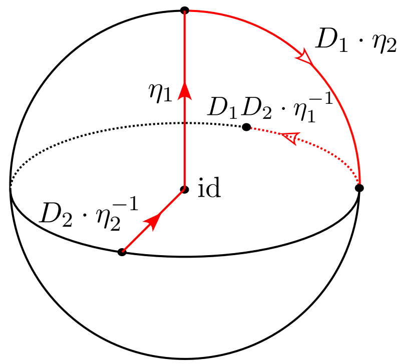


FIGURE 1. Loop homotopic to $[\eta_1, \eta_2]$ in $\mathrm{SO}(3) \cong \mathbb{R}P^3$, viewed as the quotient of the unit 3-ball by the antipodal map on its boundary. A point $v$ in the ball corresponds to the rotation with axis $v$ and angle $|v|\pi$ (counterclockwise according to the right-hand rule). The pictured loop is homotopically nontrivial.

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