CHARACTERISTIC CLASSES OF FIBERWISE BRANCHED SURFACE BUNDLES VIA ARITHMETIC GROUPS

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Abstract. This paper is about the cohomology of certain finite-index subgroups of mapping class groups and its relation to the cohomology of arithmetic groups. For $G = \mathbb{Z}/m\mathbb{Z}$ and for a regular G-cover $S \to \bar{S}$ (possibly branched), a finite index subgroup $\Gamma < \mathrm{Mod}(\bar{S})$ acts on $H_1(S;\mathbb{Z})$ commuting with the deck group action, thus inducing a homomorphism $\Gamma \to \mathrm{Sp}_{2g}^G(\mathbb{Z})$ to an arithmetic group. The induced map $H^*(\mathrm{Sp}_{2g}^G(\mathbb{Z});\mathbb{Q}) \to H^*(\Gamma;\mathbb{Q})$ can be understood using index theory. To this end, we describe a families version of the G-index theorem for the signature operator and apply this to (i) compute $H^2(\mathrm{Sp}_{2g}^G(\mathbb{Z});\mathbb{Q}) \to H^2(\Gamma;\mathbb{Q})$, (ii) re-derive Hirzebruch's formula for signature of a branched cover, (iii) compute Toledo invariants of surface group representations to $\mathrm{SU}(p,q)$ arising from Atiyah–Kodaira constructions, and (iv) describe how classes in $H^*(\mathrm{Sp}_{2g}^G(\mathbb{Z});\mathbb{Q})$ give equivariant cobordism invariants for surface bundles with a fiberwise G action, following Church–Farb–Thibault.

1 Introduction

Let $(S, \mathbf{z})$ be a closed oriented surface of genus $g \geq 2$ with $\mathbf{z} \subset S$ a finite set of marked points, and let $\mathrm{Mod}(S, \mathbf{z})$ be its (pure) mapping class group (see §2 for the definition). This note focuses on the cohomology of $\mathrm{Mod}(S, \mathbf{z})$ and its finite index subgroups. One source of cohomology classes is the representation $\mathrm{Mod}(S, \mathbf{z}) \to \mathrm{Sp}_{2g}(\mathbb{Z})$ arising from the action of $\mathrm{Mod}(S, \mathbf{z})$ on $H_1(S;\mathbb{Z}) \simeq \mathbb{Z}^{2g}$. The image lies in $\mathrm{Sp}_{2g}(\mathbb{Z})$ because $\mathrm{Mod}(S, \mathbf{z})$ preserves the algebraic intersection pairing on $H_1(S;\mathbb{Z})$. A theorem of Borel [Bor74] calculates $H^k(\mathrm{Sp}_{2g}(\mathbb{Z});\mathbb{Q})$ in the stable range, i.e. when $g \gg k$. One can then ask about the image of the stable classes under $H^*(\mathrm{Sp}_{2g}(\mathbb{Z});\mathbb{Q}) \to H^*(\mathrm{Mod}(S, \mathbf{z});\mathbb{Q})$. A well-known computation [Ati69] shows that their span is the algebra generated by the odd Miller–Morita–Mumford (MMM) classes $\{\kappa_{2i+1} : i \geq 0\}$.

In this paper, we extend the above example. Let G be a finite group of diffeomorphisms of S. Let $\mathbf{z} = \mathrm{Fix}(G)$, and define $\mathrm{Mod}^G(S, \mathbf{z}) < \mathrm{Mod}(S, \mathbf{z})$ as the subgroup of mapping classes that can be realized by diffeomorphisms that commute with G. The group G injects into $\mathrm{Sp}_{2g}(\mathbb{Z})$; let $\mathrm{Sp}_{2g}^G(\mathbb{Z})$ be its centralizer. The image of $\mathrm{Mod}^G(S, \mathbf{z})$ under α lands in $\mathrm{Sp}_{2g}^G(\mathbb{Z})$. Again, we can ask about the image of $\alpha^* : H^*(\mathrm{Sp}_{2g}^G(\mathbb{Z});\mathbb{Q}) \to H^*(\mathrm{Mod}^G(S, \mathbf{z});\mathbb{Q})$, and this is a reasonable question since $\mathrm{Sp}_{2g}^G(\mathbb{Z})$ is an arithmetic group, so it's cohomology can be computed in a range of degrees.

Our main result is a computation of the image of

$$\alpha^* : H^2(\mathrm{Sp}_{2g}^G(\mathbb{Z});\mathbb{Q}) \to H^2(\mathrm{Mod}^G(S, \mathbf{z});\mathbb{Q})$$

when $G \simeq \mathbb{Z}/m\mathbb{Z}$ is a finite cyclic group. To describe the image, we need some notation. Consider the classes in $H^2(\mathrm{Mod}^G(S, \mathbf{z});\mathbb{Q})$ that are pulled back from $H^2(\mathrm{Mod}(S, \mathbf{z});\mathbb{Q})$: there is the first MMM class κ_1, and for each $z \in \mathbf{z}$, there is an *Euler class* e_z. Fix a generator $G = \langle t \rangle$, and decompose the fixed set $\mathbf{z} = \sqcup_{j=1}^{m-1} \mathbf{z}_j$, where $\mathbf{z}_j$ is the set of fixed points z where t acts on T_zS by rotation by $\frac{2\pi j}{m}$. Set $\epsilon_j = \sum_{z \in \mathbf{z}_j} e_z$. By convention if $\mathbf{z}_j = \varnothing$, then $\epsilon_j = 0$.

Theorem 1. *Let S be a genus-g, closed, oriented surface with an orientation-preserving action of a cyclic group G of order m. Assume that the stabilizer of each $x \in S$ is either trivial or equal to G. Denote*

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$\mathbf{z} = \mathrm{Fix}(G)$, and define $\mathbf{z}_j \subset \mathbf{z}$ and $\epsilon_j \in H^2(\mathrm{Mod}^G(S,\mathbf{z});\mathbb{Q})$ as in the preceding paragraph. If the genus of S/G is at least 6, then the image of $\alpha^* : H^2(\mathrm{Sp}_{2g}^G(\mathbb{Z});\mathbb{Q}) \to H^2(\mathrm{Mod}^G(S,\mathbf{z});\mathbb{Q})$ is the subspace spanned by κ_1 and $\epsilon_j + \epsilon_{m-j}$ for $1 \le j < m/2$.

Remark. In §2, we show that the classes κ_1 and e_z are nonzero in $H^2(\mathrm{Mod}^G(S,\mathbf{z});\mathbb{Q})$ if $g \gg 0$. In particular, there is more information captured by α^* than by the map $H^*(\mathrm{Sp}_{2g}(\mathbb{Z});\mathbb{Q}) \to H^*(\mathrm{Mod}(S,\mathbf{z});\mathbb{Q})$.

Remark. The assumption that G is cyclic and the assumptions on point stabilizers in Theorem 1 are added to make the statement simpler and because it is all we need for our applications. The more general cases when G is not cyclic or some point stabilizer is a nontrivial proper subgroup of G can also be treated by the methods of this paper.

Theorem 1 has the following corollary for the cohomology of the Torelli subgroup of $\mathrm{Mod}^G(S,\mathbf{z})$.

Corollary 2. *Fix S and G as in Theorem 1. Define $\mathcal{I}^G < \mathrm{Mod}^G(S,\mathbf{z})$ as the kernel of $\mathrm{Mod}^G(S,\mathbf{z}) \to \mathrm{Sp}_{2g}^G(\mathbb{Z})$. For each $1 \le j < m/2$, if $\mathbf{z}_j \cup \mathbf{z}_{m-j} \neq \emptyset$, then $\epsilon_j + \epsilon_{m-j}$ is nontrivial and is in the kernel of $H^2(\mathrm{Mod}^G(S,\mathbf{z});\mathbb{Q}) \to H^2(\mathcal{I}^G;\mathbb{Q})$.*

At the heart of Theorem 1 is an index theorem, stated in Theorem 4 below. Before discussing this, we give some further applications of Theorem 4.

1.1 Applications.

In Section 5 we discuss the following applications of the index formula:

Geometric characteristic classes after Church–Farb–Thibault.

Following [CFT12], a characteristic class $c \in H^k\big(B\,\mathrm{Diff}(F)\big)$ is called *geometric with respect to cobordism* if two F bundles $M \to B^k$ and $M_1 \to B_1^k$ have the same characteristic numbers $c^\#(M \to B) = c^\#(M_1 \to B_1)$ whenever the manifolds M and M_1 are cobordant. In particular, such a characteristic class is insensitive to the fibering $M \to B$.

The cohomology $H^*(\mathrm{Mod}^G(S,\mathbf{z});\mathbb{Q})$ can be interpreted as the ring of characteristic classes for (S,G)-*bundles*, which we define as surface bundles $S \to M \to B$ with a fiberwise G action that can locally be identified with the given action $G < \mathrm{Diff}(S)$. One source of characteristic classes is from the Hodge bundle $\mathbf{E}$ (which is a complex vector bundle over the moduli space of Riemann surfaces – see §2): let $c_i(\mathbf{E}) \in H^{2i}(\mathrm{Mod}(S,\mathbf{z});\mathbb{Q})$ be its Chern classes. In the presence of the G action, the Hodge bundle decomposes into eigenbundles $\mathbf{E} = \bigoplus_{q^m=1} \mathbf{E}_q$, and this gives classes $c_i(\mathbf{E}_q) \in H^{2i}(\mathrm{Mod}^G(S,\mathbf{z});\mathbb{Q})$ for each m-th root of unity $q^m = 1$. Below we will abbreviate $c_{i,q} = c_i(\mathbf{E}_q)$.

Our first application is that $c_{1,q} \in H^2(\mathrm{Mod}^G(S,\mathbf{z});\mathbb{Q})$ is geometric with respect to G-cobordism.

Corollary 3. *Fix S and G as in Theorem 1. Let $S \to M \to \Sigma$ be an (S,G)-bundle over a surface. Then for each $q^m = 1$, the characteristic number $c_{1,q}^\#(M \to \Sigma)$ is a G-cobordism invariant, i.e. it depends only on the G-cobordism class of M.*

This corollary is an equivariant version of a theorem of Church–Farb–Thibault. For example, the standard Atiyah–Kodaira example is a surface bundle $S_6 \to M \to S_{129}$ with a fiberwise $G = \mathbb{Z}/2\mathbb{Z}$ action. The manifold M also fibers as $S_{321} \to M \to S_3$, and we have Hodge eigenbundles

$$\mathbb{C}^3 \to E_1 \to S_{129} \quad \text{and} \quad \mathbb{C}^3 \to E_{-1} \to S_{129} \quad , \quad \mathbb{C}^{104} \to E_1' \to S_3 \quad \text{and} \quad \mathbb{C}^{217} \to E_{-1}' \to S_3.$$

The corollary says that

$$\big\langle c_1(E_1), [S_{129}]\big\rangle = \big\langle c_1(E_1'), [S_3]\big\rangle \quad \text{and} \quad \big\langle c_1(E_{-1}), [S_{129}]\big\rangle = \big\langle c_1(E_{-1}'), [S_3]\big\rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the pairing between cohomology and homology.

Surface group representations. From a S-bundle $M \to \Sigma$ with monodromy in $\mathrm{Mod}^G(S,\mathbf{z})$ one obtains a collection of surface group representations $\pi_1(\Sigma) \to H$ where H is either $\mathrm{Sp}_{2k}(\mathbb{R})$ or $\mathrm{SU}(a,b)$.

The Toledo invariant of such a representation coincides with the Chern class $c_1(E_q)$ of one of the Hodge eigenbundles $E_q \to \Sigma$. The index formula (1) relates these Chern classes $c_{1,q} \in H^2(\mathrm{Mod}^G(S,\mathbf{z});\mathbb{Q})$ to the classes $\kappa_1, e_z \in H^2(\mathrm{Mod}^G(S,\mathbf{z});\mathbb{Q})$, which allows one to compute these Toledo invariants. For example, the Atiyah–Kodaira construction for $G = \mathbb{Z}/7\mathbb{Z}$ can be used to produce a representation

$$\alpha : \pi_1(\Sigma_{7^{17}+1}) \to \mathrm{SU}(8,13) \times \mathrm{SU}(9,12) \times \mathrm{SU}(10,11).$$

whose Toledo invariants (obtained by projecting to individual factors) are nonzero and distinct. See §5.

Hirzebruch's formula for signature of branched covers. The index formula can be used to express how the odd MMM classes behave under fiberwise branched covers. In the case of κ_1 this allows us to derive Hirzebruch's formula for the signature of a branched cover. One interesting thing about this derivation is that it emphasizes the connection to arithmetic groups.

1.2 Methods of proof.

The index theorem described below is the common ingredient in our main result and applications. As in §1.1, denote $\mathbf{E} = \bigoplus_{q^m=1} \mathbf{E}_q$ the decomposition of the Hodge bundle into eigenbundles. Denote the Chern character by $\mathrm{ch}(\mathbf{E}_q)$.

Theorem 4 (Index formula). *Fix S and G as in the statement of Theorem 1. Denote $\theta_j = \frac{2\pi j}{m}$. For $1 \le r \le m-1$,*

$$\sum_{q^m=1} q^r \left[\mathrm{ch}(\mathbf{E}_q) - \mathrm{ch}(\bar{\mathbf{E}}_{\bar{q}})\right] = \sum_{\substack{1\le j\le m-1\\ \mathbf{z}_j \neq \emptyset}} \coth\left(\frac{\epsilon_j + i\, r\theta_j}{2}\right). \tag{1}$$

Remark. The assumption on point stabilizers implies that $\mathbf{z}_j \neq \emptyset$ only if $\gcd(j,m) = 1$.

Theorem 4 is a families version of the G-index theorem for the signature operator. The left-hand side of (1) is a families version of the g-signature $\mathrm{Sig}(g,S)$ of [AS68c]. For $g = e^{2\pi i r/m} \neq 1$ this index is computed using the Atiyah–Segal localization theorem [AS68a] and this gives the right-hand side of (1). The case $g = 1$ is special, where one obtains instead the more familiar formula (see [AS68c, §6] and [AS71, Thm 5.1]) from the families index theorem:

$$\mathrm{ch}(\mathbf{E} - \bar{\mathbf{E}}) = \pi_!\left(\frac{x}{\tanh(x/2)}\right), \tag{2}$$

where $\pi_!(x^{2i}) = \kappa_{2i-1}$ (the power series on the right-hand side is a polynomial in x^2).

As a consequence formulas (1) and (2), we see that under $\alpha^* : H^*(\mathrm{Sp}_{2g}^G(\mathbb{Z});\mathbb{Q}) \to H^*(\mathrm{Mod}^G(S,\mathbf{z});\mathbb{Q})$, the stable cohomology is mapped into the subalgebra generated by the odd MMM classes $\{\kappa_{2i+1} : i \ge 0\}$ and the classes ϵ_j. Moreover, Theorem 1 describes the image precisely in degree 2. The precise image of $\alpha^* : H^k(\mathrm{Sp}_{2g}^G(\mathbb{Z});\mathbb{Q}) \to H^k(\mathrm{Mod}^G(S,\mathbf{z});\mathbb{Q})$ for $k > 2$ is more complicated.

Remark on proofs. Theorem 4 is obtained by combining the results of [AS68a, AS68c, AS71]. As far as the author knows this does not appear in the literature, although it is surely known to experts (see the last sentence of [AS71, §5]). Theorem 1 is proved by computing the stable cohomology of $\mathrm{Sp}_{2g}^G(\mathbb{Z})$ (following Borel), relating this cohomology to the Chern classes of the Hodge eigenbundles, and applying the index formula to reduce the problem to the linear algebra of circulant matrices.

1.3 Outline of paper.

In §2 we relate $\mathrm{Mod}^G(S,\mathbf{z})$ to a finite index subgroup of a mapping class group and interpret $H^*(\mathrm{Mod}^G(S,\mathbf{z}))$ as the ring of characteristic classes of (S,G)-bundles. The index formula of Theorem 4 is proved in §3. In §4 we prove Theorem 1, and in §5 we discuss the applications mentioned in §1.1.

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2 Mapping class groups, subgroups, and surface bundles

Throughout this section we assume S is a closed surface of genus at least 2 with a smooth action of a finite cyclic group G. Denote $\mathbf{z} \subset S$ the set points fixed by each $g \in G$.

It is well-known that a surface bundle $S \to M \to B$ is determined by its monodromy $\pi_1(B) \to \mathrm{Mod}(S)$. In this section we record an equivariant version of this fact that gives a monodromy characterization of (S, G)-*bundles* (defined below). Then we define the *liftable subgroup* $\mathrm{Mod}_\mu(\bar{S}, \bar{\mathbf{z}}) < \mathrm{Mod}(\bar{S}, \bar{\mathbf{z}})$ associated to a branched cover $\mu : S \to \bar{S}$ with branched set $\bar{\mathbf{z}} \subset \bar{S}$. We show that $\mathrm{Mod}_\mu(\bar{S}, \bar{\mathbf{z}})$ is finite index in $\mathrm{Mod}(\bar{S}, \bar{\mathbf{z}})$ and show that it contains a finite index subgroup that is a subgroup of $\mathrm{Mod}^G(S, \mathbf{z})$. Finally, we define the characteristic classes of (S, G) bundles that appear in this paper.

2.1 Defining (S, G) bundles. Let $\mathrm{Diff}(S, \mathbf{z})$ denote the group of orientation-preserving diffeomorphisms that fix each $z \in \mathbf{z}$, and define the mapping class group $\mathrm{Mod}(S, \mathbf{z}) = \pi_0\big(\mathrm{Diff}(S, \mathbf{z})\big)$. Our primary interest in $\mathrm{Mod}(S, \mathbf{z})$ is in its relation to surface bundles. In this paper by a surface bundle with fiber $(S, \mathbf{z})$ we mean is a locally trivial fibration $\pi : M \to B$ with structure group $\mathrm{Diff}(S, \mathbf{z})$. The bundle π is determined up to isomorphism by a homotopy class of map $B \to B\,\mathrm{Diff}(S, \mathbf{z})$ to the *classifying space*. The *monodromy* is the induced homomorphism

$$\pi_1(B) \to \pi_1\big(B\,\mathrm{Diff}(S, \mathbf{z})\big) \simeq \pi_0\big(\mathrm{Diff}(S, \mathbf{z})\big) \equiv \mathrm{Mod}(S, \mathbf{z})\,.$$

If the structure group of π reduces to the group $\mathrm{Diff}^G(S, \mathbf{z})$ of diffeomorphisms that commute with the $G < \mathrm{Diff}(S, \mathbf{z})$, then the total space M has a G action that preserves each fiber and this action can locally be identified with the G action on $(S, \mathbf{z})$. In this case we call $\pi : M \to B$ an (S, G)-*bundle*. The monodromy of an (S, G) bundle is contained in the centralizer $\mathrm{Mod}^G(S, \mathbf{z})$ of G in $\mathrm{Mod}(S, \mathbf{z})$.

Remark. The embedding $i : G \hookrightarrow \mathrm{Diff}(S, \mathbf{z})$ is part of the data of an (S, G) bundle. Since we fix i at the beginning, we omit it from the notation.

Remark. The primary known examples of (S, G) bundles are obtained by the fiberwise branched covering constructions of Atiyah–Kodaira [Ati69] and Morita [Mor01].

2.2 Classifying (S, G)-bundles. One of the miracles in the study of surface bundles is that an $(S, \mathbf{z})$ bundle $\pi : M \to B$ is determined by its monodromy. The following theorem gives the analogue for (S, G) bundles, and is a consequence of Earle–Schatz [ES70].

Theorem 5. *Fix a closed surface S with an action of a finite group G. For each manifold B, there is a bijection*

$$\left\{\begin{array}{c} (S,G)\text{-}\mathit{bundles} \\ M \to B \\ \mathit{up\ to\ isomorphism} \end{array}\right\} \leftrightarrow \left\{\begin{array}{c} \mathit{homomorphisms} \\ \pi_1(B) \to \mathrm{Mod}^G(S, \mathbf{z}) \\ \mathit{up\ to\ conjugacy} \end{array}\right\}$$

Proof. Let Diff_0 denote the path component of the identity in $\mathrm{Diff}(S, \mathbf{z})$. There is a fiber sequence

$$\mathrm{Diff}_0 \cap \mathrm{Diff}^G(S, \mathbf{z}) \to \mathrm{Diff}^G(S, \mathbf{z}) \to \mathrm{Mod}^G(S, \mathbf{z})$$

(which is also an exact sequence of groups). It follows from [ES70, §5(F)], that $\mathrm{Diff}_0 \cap \mathrm{Diff}^G_u(S, \mathbf{z})$ is contractible, which implies that $B\,\mathrm{Diff}^G(S, \mathbf{z}) \to B\,\mathrm{Mod}^G(S, \mathbf{z})$ is a homotopy equivalence, and the theorem follows. □

Remark.

(1) The theorem of Earle–Schatz says, in particular, that $\operatorname{Diff}^G(S,\mathbf{z})\cap\operatorname{Diff}_0$ is connected, so if $\phi\in\operatorname{Diff}^G(S,\mathbf{z})$ is isotopic to the identity, then it has an isotopy through diffeomorphisms that commute with G (compare with [BH72]). Consequently the surjection $\pi_0\big(\operatorname{Diff}^G(S,\mathbf{z})\big)\to\operatorname{Mod}^G(S,\mathbf{z})$ is an isomorphism.
(2) By Theorem 5, if the monodromy $\pi_1(B)\to\operatorname{Mod}(S,\mathbf{z})$ of an $(S,\mathbf{z})$-bundle $M\to B$ factors through $\operatorname{Mod}^G(S,\mathbf{z})$, then $M\to B$ has the structure of a (S,G)-bundle. Without the theorem of Earle–Schatz, it is not obvious why a bundle with monodromy $\operatorname{Mod}^G(S,\mathbf{z})$ admits a fiberwise G-action.
(3) As a consequence of (the proof of) Theorem 5, the cohomology $H^*\big(\operatorname{Mod}^G(S,\mathbf{z})\big)$ can be identified with the ring of characteristic classes of (S,G)-bundles.

2.3 Liftable subgroups. For a finite regular G-cover $\mu:S\to\bar S$ branched over $\bar{\mathbf{z}}$ with $\mathbf{z}=\mu^{-1}(\bar{\mathbf{z}})$, define the *liftable subgroup*

$$\operatorname{Mod}_\mu(\bar S,\bar{\mathbf{z}})=\{[f]\in\operatorname{Mod}(\bar S,\bar{\mathbf{z}})\mid f\text{ admits a lift }\widetilde f\in\operatorname{Diff}^G(S,\mathbf{z})\}.$$

By definition, there is an exact sequence

$$1\to G\to\operatorname{Mod}^G(S,\mathbf{z})\to\operatorname{Mod}_\mu(\bar S,\bar{\mathbf{z}})\to 1. \tag{3}$$

Our goal in this subsection is to explain the following proposition, which is a modification of an argument of Morita [Mor01, Lemma 4.13] to the case of a branched cover. Although our standing assumption is that G is cyclic, this proposition does not require this.

Proposition 6. *Let G be a finite group, and fix a regular G-cover $\mu:S\to\bar S$ branched over $\bar{\mathbf{z}}\subset\bar S$. Assume S is closed and $\chi(\bar S\setminus\bar{\mathbf{z}})<0$. Then the liftable subgroup $\operatorname{Mod}_\mu(\bar S,\bar{\mathbf{z}})<\operatorname{Mod}(\bar S,\bar{\mathbf{z}})$ is finite index and contains a finite-index subgroup over which the exact sequence (3) splits.*

Proof. Remove $\bar{\mathbf{z}}$ and $\mathbf{z}=\mu^{-1}(\bar{\mathbf{z}})$ to get an unbranched regular cover $\hat\Sigma\to\Sigma$. The group $\operatorname{Mod}(\bar S,\bar{\mathbf{z}})$ is isomorphic to a finite subgroup of $\operatorname{Mod}(\Sigma):=\pi_0\big(\operatorname{Diff}(\Sigma)\big)$ (namely, the subgroup where the punctures are not permuted). The same is true for $(S,\mathbf{z})$ and $\hat\Sigma$. Thus it suffices to prove the proposition for the liftable subgroup $\operatorname{Mod}_\mu(\Sigma)$ in $\operatorname{Mod}(\Sigma)$.

Set $\hat T=\pi_1(\hat\Sigma)$ and $T=\pi_1(\Sigma)$. The cover gives an exact sequence $1\to\hat T\to T\xrightarrow{q}G\to 1$. Give T the standard presentation $T=\langle a_1,b_1,\dots,a_g,b_g,p_1,\dots,p_n\mid\prod[a_i,b_i]\prod p_j=1\rangle$, where p_j is a loop around the j-th puncture (here $n=|\mathbf{z}|$). By assumption $\chi(\Sigma)<0$, so we may realize T as a Fuchsian group $T<\operatorname{PSL}_2(\mathbb{R})$ so that the a_i,b_i are hyperbolic and the p_j are parabolic. An automorphism $\phi\in\operatorname{Aut}(T)$ is called *type-preserving* if it preserves hyperbolic (resp. parabolic) elements. Denote $\mathcal{A}(T)<\operatorname{Aut}(T)$ the group of type-preserving automorphisms. By [MH75, Theorem 1] there are isomorphisms

$$\mathcal{A}(T)\simeq\operatorname{Mod}(\Sigma,*)\ \text{ and }\ \mathcal{A}(T)/\operatorname{Inn}(T)\simeq\operatorname{Mod}(\Sigma),$$

where $*\in\Sigma$ is a basepoint, and $\operatorname{Mod}(\Sigma,*)$ is the group of isotopy classes of diffeomorphisms of Σ that fix $*$. Similarly, we can define $\mathcal{A}(\hat T)$, and we have isomorphism $\mathcal{A}(\hat T)\simeq\operatorname{Mod}(\hat\Sigma,*)$ and $\mathcal{A}(\hat T)/\operatorname{Inn}(T)\simeq\operatorname{Mod}(\hat\Sigma)$.

Consider the group

$$\operatorname{Mod}_\mu(\Sigma,*)=\{\phi\in\mathcal{A}(T):\phi(\hat T)=\hat T\text{ and }q\circ\phi=q\}.$$

By definition, we have a homomorphism $\operatorname{Mod}_\mu(\Sigma,*)\to\mathcal{A}(\hat T)$. Transferring from group theory to topology, we have the following diagram.

$$\begin{tikzcd}
1 \arrow[r] & \hat T \arrow[r] & \operatorname{Mod}(\hat\Sigma,*) \arrow[r] & \operatorname{Mod}(\hat\Sigma) \arrow[r] & 1 \\
 & \operatorname{Mod}_\mu(\Sigma,*)\cap T \arrow[r, hook] & \operatorname{Mod}_\mu(\Sigma,*) \arrow[u] \arrow[ur, dashed, "r"'] \arrow[d, hook] & & \\
1 \arrow[r] & T \arrow[r] & \operatorname{Mod}(\Sigma,*) \arrow[r, "p"] & \operatorname{Mod}(\Sigma) \arrow[r] & 1
\end{tikzcd}$$

The top and bottom rows are instances of the Birman exact sequence [FM12, §4.2]. Note that the subgroup $\mathrm{Mod}_\mu(\Sigma,*) < \mathcal{A}(T) \simeq \mathrm{Mod}(\Sigma,*)$ is finite index because T has finitely many subgroups of index $|G|$ (permuted by $\mathcal{A}(T)$), the stabilizer of $\hat{T}$ acts on $G = T/\hat{T}$, and the group $\mathrm{Aut}(G)$ is finite.

By the argument in [Mor01, Lem 4.13], there is a finite-index subgroup $\Gamma < \mathrm{Mod}_\mu(\Sigma,*)$ such that $\Gamma \cap T = \{e\}$, so $\Gamma \hookrightarrow \mathrm{Mod}(\Sigma)$; furthermore, since $\mathrm{Mod}_\mu(\Sigma,*) < \mathrm{Mod}(\Sigma,*)$ is finite index and p is surjective, Γ is finite index in $\mathrm{Mod}(\Sigma)$. By construction we have finite-index subgroups $\Gamma < \mathrm{Mod}_\mu(\Sigma) < \mathrm{Mod}(\Sigma)$ with a homomorphism $r : \Gamma \to \mathrm{Mod}^G(\hat{\Sigma})$. This completes the proof. $\square$

2.4 Invariants of (S,G) bundles.

We describe the main invariants we will consider: the MMM classes κ_i, Euler classes e_z, and Chern classes of the Hodge (eigen)bundles $c_{i,q}$. These are viewed as elements in $H^*(\mathrm{Mod}^G(S,\mathbf{z});\mathbb{Q})$. The relation between these classes will be the focus of Sections 3 and 4.

Fix an $(S,\mathbf{z})$ bundle $\pi : M \to B$. Let $T_\pi M \to M$ be the vertical tangent bundle, and denote $e = e(T_\pi M) \in H^2(M)$ its Euler class. The *i-th MMM class* of the bundle π is defined as $\kappa_i(\pi) := \pi_!(e^{2i+1}) \in H^{2i}(B)$, where $\pi_! : H^*(M) \to H^*(B)$ is the Gysin (or push-forward) homomorphism. For more information see [Mor01, Ch. 4].

Since the structure group $\mathrm{Diff}(S,\mathbf{z})$ of $\pi : M \to B$ has fixed points, π admits a section $\sigma_z : B \to M$ for each $z \in \mathbf{z}$. The *Euler class* $e_z(\pi) \in H^2(B)$ associated to $z \in \mathbf{z}$ is defined as the image of e under $\sigma_z^* : H^*(M) \to H^*(B)$.

The cohomology classes $\kappa_i(\pi), e_z(\pi) \in H^*(B)$ are characteristic classes of $(S,\mathbf{z})$ bundles (i.e. they are natural with respect to bundle pullbacks). By the theory of classifying spaces, we can view these characteristic classes as elements $\kappa_i, e_z \in H^*(\mathrm{Mod}(S,\mathbf{z}))$. It's well-known (see [Mor87]) that κ_i is nonzero in $H^*(\mathrm{Mod}(S_g,\mathbf{z});\mathbb{Q})$ if $g \gg i$, and $\{e_z : z \in \mathbf{z}\}$ are linearly independent for all $g \geq 2$.

These characteristic classes are invariants of (S,G) bundles since $\mathrm{Mod}^G(S,\mathbf{z}) < \mathrm{Mod}(S,\mathbf{z})$.

Proposition 7. *Assume S/G has genus at least 2. Then the restriction of*

$$H^*(\mathrm{Mod}(S,\mathbf{z});\mathbb{Q}) \to H^*(\mathrm{Mod}^G(S,\mathbf{z});\mathbb{Q})$$

to the subalgebra generated by $\{e_z : z \in \mathbf{z}\}$ is injective.

Proof. The proposition would be obvious if $\mathrm{Mod}^G(S,\mathbf{z}) < \mathrm{Mod}(S,\mathbf{z})$ was finite index, since for a finite index subgroup $\Lambda' < \Lambda$ the induced map $H^*(\Lambda;\mathbb{Q}) \to H^*(\Lambda';\mathbb{Q})$ is injective (use the transfer map). Unfortunately, $\mathrm{Mod}^G(S,\mathbf{z}) < \mathrm{Mod}(S,\mathbf{z})$ is infinite index. Nevertheless, $\mathrm{Mod}^G(S,\mathbf{z})$ has the same rational cohomology as $\mathrm{Mod}_\mu(\bar{S},\bar{\mathbf{z}})$ because of the exact sequence (3); this follows easily from examining the associated spectral sequence, since $H^*(G;\mathbb{Q})$ is trivial. By Proposition 6, $\mathrm{Mod}_\mu(\bar{S},\bar{\mathbf{z}}) < \mathrm{Mod}(\bar{S},\bar{\mathbf{z}})$ is finite index. So the subalgebra of $H^*(\mathrm{Mod}(\bar{S},\bar{\mathbf{z}});\mathbb{Q})$ generated by $\{e_{\bar{z}} : \bar{z} \in \bar{\mathbf{z}}\}$ injects into $H^*(\mathrm{Mod}_\mu(\bar{S},\bar{\mathbf{z}});\mathbb{Q})$. The proposition now follows by observing that for $z \in \mathbf{z}$, the subspace $\mathbb{Q}\{e_z\}$ is the image of the subspace $\mathbb{Q}\{e_{\mu(z)}\} \subset H^*(\mathrm{Mod}_\mu(\bar{S},\bar{\mathbf{z}});\mathbb{Q})$ under $H^*(\mathrm{Mod}_\mu(\bar{S},\bar{\mathbf{z}})) \to H^*(\mathrm{Mod}^G(S,\mathbf{z}))$. To see this, fix an (S,G) bundle $M \to B$. Note that $M/G \to B$ is an $(\bar{S},\bar{\mathbf{z}})$ bundle. We can interpret e_z and $e_{\mu(z)}$ as the Euler classes of the normal bundles of $\sigma_z(B) \subset M$ and $\sigma_{\mu(z)}(B) \subset M/G$, respecitvely. Since $M \to M/G$ is a branched cover that sends $\sigma_z(B)$ diffeomorphically to $\sigma_{\mu(z)}(B)$, the two normal bundles have proportional Euler classes; see [Mor01, Prop. 4.12]. $\square$

Aside. Our original interest in studying $H^*(\mathrm{Sp}_{2g}^G(\mathbb{Z});\mathbb{Q}) \to H^*\big(\mathrm{Mod}^G(S,\mathbf{z});\mathbb{Q}\big)$ was to determine if the isomorphism $H^*\big(\mathrm{Mod}_\mu(\bar{S},\bar{\mathbf{z}});\mathbb{Q}\big) \simeq H^*\big(\mathrm{Mod}^G(S,\mathbf{z});\mathbb{Q}\big)$ produced any cohomology in the cokernel of the injection $H^*(\mathrm{Mod}(\bar{S},\bar{\mathbf{z}});\mathbb{Q}) \hookrightarrow H^*\big(\mathrm{Mod}_\mu(\bar{S},\bar{\mathbf{z}});\mathbb{Q}\big)$. Unfortunately this is not the case (at least stably) by Theorem 4 and the proof of Theorem 1 – as mentioned in the introduction, together they show that, in the stable range, the image of $\alpha^* : H^k(\mathrm{Sp}_{2g}^G(\mathbb{Z});\mathbb{Q}) \to H^k(\mathrm{Mod}^G(S,\mathbf{z});\mathbb{Q})$ is contained in the subalgebra generated by the odd MMM classes and the Euler classes.

Hodge bundle and eigenbundles. For an $(S,\mathbf{z})$ bundle $\pi : M \to B$, the *Hodge bundle* is the vector bundle $E \to B$ where the fiber over $b \in B$ is $H_1(S_b;\mathbb{R})$, where $S_b = \pi^{-1}(b)$. With this definition E is a real vector bundle, but if π is equipped with a fiberwise complex structure (possible if B is paracompact for example), then $E \to B$ is isomorphic to the realification of the complex vector bundle whose fiber over b is the space of holomorphic 1-forms on S_b. If π is an (S,G) bundle, then the G action on M induces a G action on E that covers the trivial action on B. Then we may decompose $E = \bigoplus_{q^m=1} E_q$. Fiberwise, this is the decomposition of the $\mathbb{C}G$ module $H_1(S_b;\mathbb{R})$ into isotypic components. The Chern classes $c_i(E_q) \in H^{2i}(B)$ are characteristic classes of (S,G) bundles, and so we can view them as elements $c_{i,q} \in H^{2i}(\mathrm{Mod}^G(S,\mathbf{z});\mathbb{Q})$.

Remark. The classes $c_{i,q}$ can be interpreted as the Chern classes of the "universal" Hodge bundle $\mathbf{E} = \bigoplus \mathbf{E}_q$ over an appropriate moduli space $\mathcal{M}^G(S,\mathbf{z})$ (which is an orbifold Eilenberg–Maclane space for $\mathrm{Mod}^G(S,\mathbf{z})$). We will not need the moduli space here, so we will not elaborate further on this point.

3 The index formula

In this section $G \simeq \mathbb{Z}/m\mathbb{Z}$ and for convenience we identify $G \simeq \{z \in \mathbb{C}^\times : q^m = 1\}$.

The goal of this section is to prove Theorem 4 by deriving the index formula (1). To the author's knowledge, this derivation (of a families version of the G-index theorem for the signature operator) is not detailed in the literature, although it can be obtained by combining the contents of [AS68b, AS68a, AS68c, AS71]. Since these references are quite accessible, we will be brief and refer the reader to these papers for more detail.

The index formula (1) is an equality between certain classes in $H^*(\mathrm{Mod}^G(S,\mathbf{z});\mathbb{Q})$. To prove it, it suffices to show that this equation holds for for every (S,G) bundle $\pi : M \to B$ where the base B is a finite complex. Fix such a bundle, and introduce a G-invariant fiberwise Riemannian metric. Denoting $S_b = \pi^{-1}(b)$ for $b \in B$, we have the de Rham complex $\Omega^*_{\mathbb{C}}(S_b)$, its exterior derivative d, the adjoint d^* of d (defined using the Hodge star operator $\star$), and a self-adjoint elliptic operator $D = d + d^*$. The operator $\tau : \Omega^p_{\mathbb{C}} \to \Omega^{2-p}_{\mathbb{C}}$ defined by $\tau = i^{p(p-1)+1}\star$ satisfies $\tau^2 = 1$ and $D\tau = -\tau D$, so D restricts to operators $D^\pm : \Omega^\pm \to \Omega^\mp$ on the ± 1 eigenspaces $\Omega^\pm$ of τ. The operators D^+ and D^- are mutually adjoint, and $\ker(D^\pm)$ is the ± 1 eigenspace $\mathcal{H}^*(S_b;\mathbb{C})^\pm$ of τ acting on harmonic forms $\mathcal{H}^*(S_b;\mathbb{C})$.

The collection $D^+_b : \Omega^+(S_b) \to \Omega^-(S_b)$ for $b \in B$ defines a family of G-invariant differential operators on S. The *(analytic) index* of the family $D^+ = \{D^+_b\}$ is defined as $\mathrm{ind}(D^+) = E^+ - E^- \in K_G(B)$, where $E^\pm$ is the (equivariant K-theory class of the) bundle whose fiber over $b \in B$ is $\mathcal{H}^*(S_b;\mathbb{C})^\pm$. It's not hard to see that the contribution of $\mathcal{H}^0(S_b;\mathbb{C})$ and $\mathcal{H}^2(S_b;\mathbb{C})$ to the index is zero [AS68c, §6]. Furthermore, the bundles $\bigcup_{b\in B} \mathcal{H}^1(S_b;\mathbb{C})^+ \to B$ and $\bigcup_{b\in B} \mathcal{H}^1(S_b;\mathbb{C})^- \to B$ are conjugate because $\mathcal{H}^1(S_b;\mathbb{C})^+$ and $\mathcal{H}^1(S_b;\mathbb{C})^-$ are conjugate vector spaces. Combining this with the fact that $\mathcal{H}^1(S_b;\mathbb{C})^+$ can be identified with the space of holomorphic 1-forms on S_b (with respect to the complex structure determined by the conformal class of the metric), the index is given by

$$\mathrm{ind}(D^+) = E - \bar{E} \in K_G(B),$$

where $E \to B$ is the Hodge bundle (§2.4).

The index theorem gives a topological description of the index: associated to D^+ is a *symbol class* $\sigma \in K_G(T_\pi M)$, where $T_\pi M \to M$ is the vertical (co)tangent bundle[1] of $\pi : M \to B$ (and $K_G(\cdot)$ denotes equivariant K-theory with compact supports). In [AS68c] (see also [Sha78, pg. 40] and [LM89, pgs. 236, 264]) it is shown that

$$\sigma = \Omega^+ - \Omega^- = (1 + \bar{L}) - (1 + L) = \bar{L} - L \in K_G(T_\pi M),$$

where L is the pullback of $T_\pi M \to M$ along $T_\pi M \to M$. The Thom isomorphism $K(M) \to K(T_\pi M)$ is given by multiplication by the Thom class $u \in K(T_\pi M)$, and in this case $u = 1 - L$. Note that

[1] The tangent and cotangent bundles are isomorphic.

$(1+\bar{L})(1-L) = \bar{L} - L$. Thus under inverse Thom isomorphism $K(T_\pi^* M) \to K(M)$ we have

$$\sigma = \bar{L} - L = (1+\bar{L})(1-L) \mapsto 1 + T_\pi^* M.$$

The topological index is defined as t-ind $= \pi_!(\sigma)$, where $\pi_! : K_G(M) \to K_G(B)$ is the push-forward in K-theory. By the index theorem, $\mathrm{ind}(D^+) = $ t-ind, so

$$\boxed{E - \bar{E} = \pi_!(1 + T_\pi^*).} \tag{4}$$

We want to understand (4) on the level of ordinary cohomology, i.e. under the map

$$\mathrm{ch}_g : K_G(B) \simeq K(B) \otimes R(G) \xrightarrow{1\otimes\chi_g} K(B) \otimes \mathbb{C} \xrightarrow{\mathrm{ch}} H^*(B) \otimes \mathbb{C}, \tag{5}$$

where $R(G)$ is the representation ring and $\chi_g : R(G) \to \mathbb{C}$ is the ring homomorphism that sends a representation V to its character $\chi_g(V)$ at $g \in G$.

The image of the left-hand side of (4) under (5) is easily expressed. In $K(B) \otimes R(G)$, we have

$$E = \sum_{q^m=1} E_q \otimes \rho_q \quad \text{and} \quad \bar{E} = \sum_{q^m=1} \bar{E}_q \otimes \rho_{\bar{q}},$$

where $\bar{q}$ denotes the complex conjugate of $q \in \mathbb{C}$, and ρ_q is the $\mathbb{C}G$ module $\mathbb{C}[x]/(x-q)$. It follows that

$$\mathrm{ch}_g\left(E - \bar{E}\right) = \sum_{q^m=1} \left(\mathrm{ch}(E_q) - \mathrm{ch}(\bar{E}_{\bar{q}})\right) \cdot \chi_g(\rho_q). \tag{6}$$

Observe that for $g = e^{2\pi i r/m}$, we have $\chi_g(\rho_q) = q^r$.

In the remainder of the section we compute $\mathrm{ch}_g\left(\pi_!(1+T_\pi^*)\right)$. The push-forward $\pi_!$ in K-theory is difficult to understand directly, so we want to commute ch and $\pi_!$, and compute $\pi_!$ in ordinary cohomology. This can be done after passing to the fixed point set M^g, using the Atiyah–Segal localization theorem [AS68a].

Atiyah–Segal localization theorem. The character homomorphism $\chi_g : R(G) \to \mathbb{C}$ factors through the localization $R(G)_g$ at the (prime) ideal $\ker(\chi_g)$. Denoting $K_G(M)_g$ the localization of the $R(G)$ module $K_G(M)$, there is a commutative diagram

$$\begin{tikzcd}
K_G(M)_g \arrow[r, "i^*/e"] \arrow[d, "\pi_!"'] & K_G(M^g)_g \arrow[d, "p_!"] \\
K_G(B)_g \arrow[r, equal] & K_G(B)_g
\end{tikzcd}$$

Here i^* is induced by $i : M^g \hookrightarrow M$, the map p is the restriction $\pi\big|_{M^g}$, and $e = 1 - T_\pi^* \in K(M^g)$ is the Thom class (in K-theory) of the normal bundle of $M^g \hookrightarrow M$. The top arrow is an isomorphism by [Seg68, Proposition 4.1]. Dividing by the Euler class makes the diagram commute (compare [LM89, pg. 261]).

In the diagram above, $\sigma = 1 + T_\pi^* \in K_G(M)_{(g)}$ maps to $\frac{1+T_\pi^*}{1-T_\pi^*} \in K_G(M^g)_g$. Now we can compute $\mathrm{ch}_g \circ p_! \left(\frac{1+T_\pi^*}{1-T_\pi^*}\right)$ using the following diagram

$$\begin{tikzcd}
K_G(M^g)_g \arrow[r, "\chi_g"] \arrow[d, "p_!^K"] & K(M^g) \otimes \mathbb{C} \arrow[r, "\mathrm{ch}"] \arrow[d, "p_!^K"] & H^*(M^g) \otimes \mathbb{C} \arrow[d, "p_!^H"] \\
K_G(B)_g \arrow[r, "\chi_g"] & K(B) \otimes \mathbb{C} \arrow[r, "\mathrm{ch}"] & H^*(B) \otimes \mathbb{C}
\end{tikzcd}$$

If $g = 1$, then the right square doesn't commute, but the failure to commute is the *defect formula* $\mathrm{ch}\left[\pi_!\left(\sigma(D)\right)\right] = \pi_!^H\left[\mathrm{ch}\left(\sigma(D)\right) \cdot \mathrm{Td}(T_\pi)\right]$, where Td is the Todd class (this formula is also called the

Grothendieck–Riemann–Roch computation). Since $\sigma(D) = 1 + T_\pi^*$, we have $\mathrm{ch}\,\big(\sigma(D)\big) = 1 + e^{-x}$, where $x = e(T_\pi M)$. Combining this with $\mathrm{Td}(T_\pi) = \frac{x}{1-e^{-x}}$ (see [AS68c, pg. 555]), this recovers (2).

If $g = e^{2\pi i r/m} \neq 1$, then the right square commutes because $p : M^g \to B$ is a covering map. To express the class $\mathrm{ch} \circ \chi_g \left(\frac{1+T_\pi^*}{1-T_\pi^*}\right)$ in $H(M^g) \otimes \mathbb{C}$, note that the character of $\chi_g(T_\pi^*)$ will vary on different components of M^g. Decompose $M^g = \sqcup M_j$, so that $e^{2\pi i/m} \in G$ acts on $T_\pi\big|_{M_j}$ by rotation by $\theta_j = \frac{2\pi j}{m}$. Let x_j denote the restriction of $e(T_\pi M)$ to M_j. Then, on M_j, we have

$$\mathrm{ch} \circ \chi_g \left(\frac{1 + T_\pi^*\big|_{M_j}}{1 - T_\pi^*\big|_{M_j}} \right) = \frac{1 + e^{-i\,r\theta_j} e^{-x_j}}{1 - e^{-i\,r\theta_j} e^{-x_j}} = \frac{e^{(x_j + i\,r\theta_j)/2} + e^{-(x_j + i\,r\theta)/2}}{e^{(x_j + i\,r\theta_j)/2} - e^{-(x_j + i\,r\theta_j)/2}} = \coth\left(\frac{x_j + i\,r\theta_j}{2}\right)$$

Combining these terms for all j, denoting $\epsilon_j = p_!(x_j)$, and combining with (6) gives the desired index formula

$$\sum_{q^m=1} \big[\,\mathrm{ch}(E_q) - \mathrm{ch}(\bar{E}_{\bar{q}})\big] \cdot q^r = \sum_{\substack{1 \le j \le m-1 \\ M_j \neq \emptyset}} \coth\left(\frac{\epsilon_j + i\,r\theta_j}{2}\right).$$

Remark. We record here for later use the first two terms of the Taylor series of $\coth\left(\frac{x+i\varphi}{2}\right)$ at $x = 0$:

$$\coth\left(\frac{x + i\varphi}{2}\right) \approx -i\cot(\varphi/2) + \frac{1}{2}\csc^2(\varphi/2)\,x. \tag{7}$$

Remarks.

(1) The discussion above works generally when S is replaced by an orientable manifold of even dimension; for more details, see [AS68c] and [ERW15].
(2) In the case $B = \mathrm{pt}$ and $G = \{e\}$ (i.e. the non-families, non-equivariant version of the index theorem), $\mathrm{ind}(D^+) \in K(\mathrm{pt}) = \mathbb{Z}$ is equal to $\dim \mathcal{H}^+ - \dim \mathcal{H}^-$, which is zero because the ± 1-eigenspaces of τ acting on $\mathcal{H}^1(S;\mathbb{C})$ are conjugate (as complex vector spaces), so in particular they have the same dimension. However, in the families and/or equivariant case, $\mathrm{ind}(D^+)$ is nontrivial in general.

4 Computing $\alpha^* : H^2\big(\mathrm{Sp}_{2g}^G(\mathbb{Z});\mathbb{Q}\big) \to H^2\big(\mathrm{Mod}^G(S,\mathbf{z});\mathbb{Q}\big)$

In this section we prove Theorem 1. We proceed as follows.

- Step 1: We define classes $x_q \in H^2(\mathrm{Sp}_{2g}^G(\mathbb{Z});\mathbb{Q})$ such that
$$H^2(\mathrm{Sp}_{2g}^G(\mathbb{Z});\mathbb{Q}) \simeq \mathbb{Q}\{x_q : q^m = 1,\ \mathrm{Im}(q) \ge 0\},$$
using results of Borel [Bor74]. For our computation, in order to be in the stable range, we require S/G to have genus $h \ge 6$.
- Step 2: We show that $c_{1,q} = \alpha^*(x_q) = c_{1,\bar{q}}$ in $H^2(\mathrm{Mod}^G(S,\mathbf{z}))$, where $c_{1,q}$ is the class defined in §2.4. This involves comparing two complex structures on the Hodge bundle of an (S,G) bundle.
- Step 3: The index formulas (1) and (2) give a system of linear equations relating $\kappa_1, e_1, \ldots, e_{m-1} \in H^2(\mathrm{Mod}^G(S,\mathbf{z});\mathbb{Q})$ classes to the classes $\alpha^*(x_q)$. Upon investigating this linear system, the result will follow from some character theory and a result about circulant matrices.

4.1 The arithmetic group $\mathrm{Sp}_{2g}^G(\mathbb{Z})$. In this section we compute $H^2(\mathrm{Sp}_{2g}^G(\mathbb{Z});\mathbb{Q})$. This involves working out some of the general theory of arithmetic groups in the special case $\mathrm{Sp}_{2g}^G(\mathbb{Z})$. Specifically, we (i) use restriction of scalars to show $\mathrm{Sp}_{2g}^G(\mathbb{Z})$ is a lattice in a group $\mathbb{G} = \mathrm{Sp}_{2h}(\mathbb{R}) \times \mathrm{Sp}_{2h'}(\mathbb{R}) \times \prod_q \mathrm{SU}(a_q, b_q)$, (ii) use Borel–Matsushima to relate $H^j(\mathrm{Sp}_{2g}^G(\mathbb{Z});\mathbb{Q})$ to the cohomology of a product of Grassmannians in some range $0 \le j \le N$, and (iii) determine the range N by giving a lower bound the $\mathbb{Q}$-rank of the irreducible factors of $\mathrm{Sp}_{2g}^G(\mathbb{Z})$. To those familiar with arithmetic groups and their cohomology, (i) and (ii) are routine exercises. Our proof of (iii) uses the topology of the branched cover $S \to S/G$ to find isotropic subspaces in sub-representations of $H_1(S;\mathbb{Q})$.

Restriction of scalars. The group $\mathrm{Sp}_{2g}^G(\mathbb{Z})$ acts by G-module maps on $H = H_1(S;\mathbb{Q})$, so it preserves the decomposition

$$H = \bigoplus_{k|m} H_k \tag{8}$$

into isotypic components for the irreducible representations of G over $\mathbb{Q}$. Recalling that the simple $\mathbb{Q}G$-modules are isomorphic to $\mathbb{Q}(\zeta_k)$, where $\zeta_k = e^{2\pi i/k}$ and $k \mid m$, the group H_k is defined as $\mathbb{Q}(\zeta_k) \otimes_{\mathbb{Q}G} H$. As we explain below, (8) leads to a decomposition $\mathrm{Sp}_{2g}^G(\mathbb{Z}) \doteq \prod_{k|m} \Gamma_k$ into irreducible lattices (here $\doteq$ means commensurable). Furthermore, we identify Γ_k and determine the real semisimple Lie group $\mathbb{G}_k$ that contains Γ_k as a lattice.

Fix $k \mid m$. For simplicity, denote $\zeta = \zeta_k$ and $\Gamma = \Gamma_k$. The representation $V = H_k$ is naturally a vector space over $\mathbb{Q}(\zeta)$, and the intersection form ω on H determines a form $\beta : V \times V \to \mathbb{Q}(\zeta)$ given by

$$\beta(u,v) = -i \sum_{j=1}^{k} \omega(u, t^j v) \cdot \zeta^j. \tag{9}$$

Compare [GLLM15, §3.1]. If $k = 1,2$, then β is symplectic, the group $\mathbb{G} = \mathrm{Sp}(V)$ preserving β is an algebraic group defined over $\mathbb{Q}$, and $\Gamma \doteq \mathbb{G}(\mathbb{Z})$. For $k \geq 3$, β is Hermitian with respect to the involution $\tau(\zeta) = \zeta^{-1}$ on $\mathbb{Q}(\zeta)$, the group $\mathbb{G} = \mathrm{SU}(V,\beta)$ of $\mathbb{Q}(\zeta)$-linear automorphisms preserving β with determinant 1 is an algebraic group defined over $F = \mathbb{Q}(\zeta + \zeta^{-1})$ (the maximal real subfield of $\mathbb{Q}(\zeta)$), and $\Gamma \doteq \mathbb{G}(\mathcal{O})$, where $\mathcal{O} \subset F$ is the ring of integers. For a similar discussion, see [Loo97].

Restriction of scalars applied to $\mathbb{G} = \mathrm{SU}(V,\beta)$ gives an algebraic group $\mathbb{G}'$ defined over $\mathbb{Q}$ such that $\mathbb{G}'(\mathbb{Z})$ is commensurable with $\mathbb{G}(\mathcal{O})$. To define $\mathbb{G}'$, define an embedding $\sigma_q : F \to \mathbb{R}$ by $\zeta + \zeta^{-1} \mapsto q + q^{-1}$ for each primitive k-th root of unity q with $\mathrm{Im}(q) > 0$, and denote $\mathbb{G}^\sigma = \mathrm{SU}(V, \sigma_q \circ \beta)$. By the restriction of scalars construction, $\mathbb{G}' = \prod \mathbb{G}^\sigma$ is an algebraic group over $\mathbb{Q}$, the $\mathbb{Z}$-points $\mathbb{G}'_{\mathbb{Z}}$ is a lattice in $\mathbb{G}'$, and $\mathbb{G}_{\mathcal{O}} \doteq \mathbb{G}'_{\mathbb{Z}}$. Hence Γ is a lattice in $\mathbb{G}'$. Furthermore, for each σ_q the real points of $\mathbb{G}^{\sigma_q}$ is $\mathrm{SU}(a_q,b_q)$ for some $a_q, b_q \geq 0$ [Mor15, Prop. 18.5.7].

In addition, we remark that $\Gamma < \mathbb{G}'$ is irreducible: By [Mor15, §5.3, Exercise 4], Γ is irreducible if and only if $\mathbb{G}'$ is a $\mathbb{Q}$-simple group. It's a basic property of restriction of scalars that $\mathbb{G}'$ is $\mathbb{Q}$-simple if $\mathbb{G}$ is F-simple [Mar91, §1.7], and the latter is well-known, c.f. [PR94, §2.3.4].

Varying over all k, we find that

$$\mathrm{Sp}_{2g}^G(\mathbb{Z}) \doteq \mathrm{Sp}(H_1)_{\mathbb{Z}} \times \mathrm{Sp}(H_2)_{\mathbb{Z}} \times \prod_{\substack{k|m \\ 2<k\leq \frac{m}{2}}} \mathrm{SU}(H_k, \beta_k)_{\mathcal{O}_k} \tag{10}$$

is a lattice in

$$\mathrm{Sp}_{2g}^G(\mathbb{R}) = \mathrm{Sp}_{2h}(\mathbb{R}) \times \mathrm{Sp}_{2h'}(\mathbb{R}) \times \prod_{\substack{q^m=1 \\ \mathrm{Im}(q)>0}} \mathrm{SU}(a_q,b_q). \tag{11}$$

The second factor on the right-hand side of (10) and (11) appears only when m is even.

Remark. In §5 we describe how to determine the integers a_q, b_q using the Chevalley–Weil formula and the degree-0 term in the index formula (1).

Borel–Matsushima. In this section we recall the Borel–Matsushima description of $H^j(\Gamma;\mathbb{Q})$ when $\Gamma = \Gamma_k$ is an irreducible factor of $\mathrm{Sp}_{2g}^G(\mathbb{Z})$ as in (10). In what follows we will only use the case $j = 2$. For $\Gamma \simeq \mathrm{Sp}_{2n}(\mathbb{Z})$, it is well-known that $H^2(\mathrm{Sp}_{2n}(\mathbb{Z});\mathbb{Q}) \simeq \mathbb{Q}$ when $n \geq 3$ (see [ERW15, Theorem 3.4] or [Put12, Theorem 5.3]). Thus we focus on the Hermitian case $k > 2$.

Proposition 8. *Let $\mathbb{G}$ be an algebraic group defined over a field F whose associated real semisimple Lie group $\mathbb{G}(\mathbb{R})$ is a product of unitary groups $\mathrm{SU}(a_q, b_q)$ for q in some set Q. For a lattice $\Gamma \doteq \mathbb{G}(\mathcal{O}_F)$, the map*

$$\varphi : B\Gamma \to \prod B\,\mathrm{SU}(a_q, b_q) \sim \prod BS\big(\mathrm{U}(a_q) \times \mathrm{U}(b_q)\big) \to \prod B\,\mathrm{U}(a_q) \tag{12}$$

induces an isomorphism on $H^j(-;\mathbb{Q})$ for $0 \le j \le \lfloor(\mathrm{rk}_F(\Gamma) - 1)/2\rfloor$, where $\mathrm{rk}_F(\Gamma) \equiv \mathrm{rk}_F(\mathbb{G})$ is the F-rank.

Focusing on degree 2, since $H^2\big(B\,\mathrm{U}(p);\mathbb{Q}\big) = \mathbb{Q}\{c_1\}$ for $p \ge 1$, combining with the computation for $H^2(\mathrm{Sp}_{2h}(\mathbb{Z});\mathbb{Q})$ mentioned above, we have

Corollary 9. *Let $\mathrm{Sp}_{2g}^G(\mathbb{Z}) < \mathrm{Sp}_{2g}^G(\mathbb{R})$ be as in (10) and (11). Assume $h, h' \ge 3$ and $a_q, b_q \ge 1$. If $2 \le \min_{2<k\le m/2} \lfloor(\mathrm{rk}_{F_k}(\Gamma_k) - 1)/2\rfloor$, then*

$$H^2(\mathrm{Sp}_{2g}^G(\mathbb{Z});\mathbb{Q}) = \mathbb{Q}\{x_q : q^m = 1,\ \mathrm{Im}(q) \ge 0\}, \tag{13}$$

where x_1 and x_{-1} are pulled back from $\mathrm{Sp}_{2g}^G(\mathbb{Z}) \to \mathrm{Sp}_{2h}(\mathbb{R})$ and $\mathrm{Sp}_{2g}^G(\mathbb{Z}) \to \mathrm{Sp}_{2h'}(\mathbb{R})$, respectively, and x_q is pulled back from $\mathrm{Sp}_{2g}^G(\mathbb{Z}) \to \mathrm{SU}(a_q, b_q) \sim S\big(\mathrm{U}(a_q) \times \mathrm{U}(b_q)\big) \to \mathrm{U}(a_q)$.

Proof of Proposition 8. The map on cohomology induced by (12) can be realized more geometrically as follows (compare [Bor74, Proposition 7.5] and [Gia09, §3.2]). The cohomology $H^*(\Gamma;\mathbb{Q})$ can be identified with the cohomology of the complex $\Omega^*(X)^\Gamma$ of Γ-invariant differential forms on the symmetric space $X = \mathbb{G}(\mathbb{R})/K$, where $K < \mathbb{G}(\mathbb{R})$ is a maximal compact subgroup. A first approximation to the cohomology of $\Omega^*(X)^\Gamma$ is the cohomology of the subalgebra $\Omega^*(X)^{\mathbb{G}(\mathbb{R})}$ of $\mathbb{G}(\mathbb{R})$-invariant forms, which can be identified with the cohomology $H^*(X_U;\mathbb{Q})$ of the compact dual symmetric space

$$X_U = \prod \mathrm{SU}(a_q + b_q)/\mathrm{S}\big(\mathrm{U}(a_q) \times \mathrm{U}(b_q)\big) \simeq \prod \mathrm{Gr}_{a_q}(\mathbb{C}^{a_q+b_q}).$$

According to Borel [Bor81, Theorem 4.4(ii)], the inclusion $\Omega^*(X)^{\mathbb{G}(\mathbb{R})} \to \Omega^*(X)^\Gamma$ induces an isomorphism $H^j(X_U;\mathbb{R}) \to H^j(\Gamma\backslash X;\mathbb{R})$ for $0 \le j \le \min\big\{c(\mathbb{G}),\ m(\mathbb{G}(\mathbb{R}))\big\}$. In our case $c(\mathbb{G}) \ge \big\lfloor(\mathrm{rk}_F(\mathbb{G}) - 1)/2\big\rfloor$ by [Bor74, §9(3)], and $m(\mathbb{G}(\mathbb{R})) \ge \mathrm{rk}_{\mathbb{R}}(\mathbb{G}(\mathbb{R}))/2$ by [Mat62, Theorem 2] (see also [Bor74, §9.4]). Since F-rank is always less than or equal to $\mathbb{R}$-rank, we get an isomorphism for $0 \le j \le \big\lfloor(\mathrm{rk}_F(\mathbb{G}) - 1)/2\big\rfloor$.

Furthermore, the obvious map $\mathrm{Gr}_a(\mathbb{C}^{a+b}) \to \mathrm{Gr}_a(\mathbb{C}^\infty) \simeq B\,\mathrm{U}(a)$ induces a map $X_U \to \prod B\,\mathrm{U}(a_q)$ that is a cohomology isomorphism in degrees $0 \le j \le 2\min b_q$. See [Hat02, Example 4.53]. Note that no b_q can be smaller than $\min_k\{\mathrm{rk}_{F_k}(\Gamma_k)\}$ because the F-rank for a unitary group is equal to the maximal dimension of an isotropic subspace [Mor15, Ch. 9].

In summary the map $H^*\big(\prod B\,\mathrm{U}(a_q)\big) \to H^*(X_U) \to H^*(\Gamma)$ induces an isomorphism in degrees $0 \le j \le \big\lfloor(\mathrm{rk}_F(\mathbb{G}) - 1)/2\big\rfloor$, as desired. □

***F*-rank and covers.** To apply Proposition 8, we need to compute the F-rank of our lattice $\Gamma_k < \mathrm{SU}(H_k, \beta_k)$, or at least bound it from below.

Proposition 10. *Let S be a surface with a $G = \mathbb{Z}/m\mathbb{Z}$ action, and let h be the genus of S/G. Take $\Gamma_k < \mathrm{SU}(H_k, \beta_k)$ as above (for any $k \mid m$, $k \ge 3$). Then $\mathrm{rk}_{F_k}(\Gamma_k) \ge h - 1$.*

Proof. By [Mor15, Ch. 9], the F_k-rank of $\mathrm{SU}(H_k, \beta_k)$ is the maximal dimension of an β_k-isotropic subspace of H_k (as a vector space over F_k). By the definition of β_k, to prove the proposition, it suffices to exhibit an $(h-1)$-dimensional ω-isotropic subspace of $H_1(S;\mathbb{Q})$ (as a vector space over $\mathbb{Q}$).

Denote $\bar{S} = S/G$ and let $\mu : S \to \bar{S}$ be the quotient map. After replacing the fixed points $\mathbf{z} = \mathrm{Fixed}(G) \subset S$ and $\mu(\mathbf{z})$ with boundary components, the map π induces a covering map $\Sigma \to \bar{\Sigma}$ between surfaces with boundary. Associated to this cover is a surjective homomorphism $\rho : \pi_1(\bar{\Sigma}) \to H_1(\bar{\Sigma};\mathbb{Z}) \to \mathbb{Z}/m\mathbb{Z}$. By Poincaré duality, there exists primitive $c \in H_1(\bar{\Sigma}, \partial\bar{\Sigma};\mathbb{Z})$ so that $\rho(\gamma) = [\gamma] \cdot c \mod m$.

Case 1. If $\partial\bar{\Sigma} = \varnothing$ (i.e. $\bar{S} = \bar{\Sigma}$), then the homology class c is represented by a simple closed curve $\alpha \subset \bar{\Sigma}$. The complement $\bar{\Sigma} \setminus \alpha$ contains a subsurface N of genus $(h-1)$ that lifts to $\Sigma = S$ (because it is disjoint from our representative for c). Furthermore, $H_1(N;\mathbb{Q})$ contains an $(h-1)$-dimensional isotropic subspace that lifts to an $(h-1)$-dimensional isotropic subspace of $H_1(S;\mathbb{Q})$, as desired.

Case 2. Assume $\bar{\Sigma}$ has $r \geq 1$ boundary components. We can express $\bar{\Sigma} = \Sigma_{h,1} \cup \Sigma_{0,r+1}$ as the union of a genus-h surface with one boundary component with a genus-0 surface with $(r+1)$ boundary components $b_0, \ldots, b_r$, where the boundary component of $\Sigma_{h,1}$ is glued to b_0. See Figure 1.

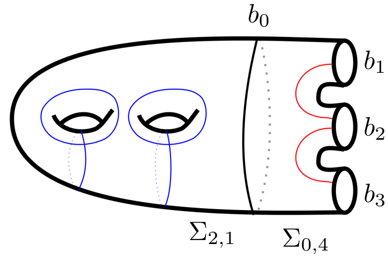

FIGURE 1. Decomposing $\bar{\Sigma} = \Sigma_{h,1} \cup \Sigma_{0,r+1}$. The blue and red curves generate A and A', respectively.

This leads by (relative) Mayer–Vietoris to a direct sum decomposition $H_1(\bar{\Sigma}, \partial\bar{\Sigma}) = A \oplus A'$, where $A \simeq H_1(\Sigma_{h,1})$ is spanned by curves on $\Sigma_{h,1}$, and $A' \simeq \mathbb{Z}^{r-1}$ is spanned by arcs on $\Sigma_{0,r+1}$ between distinct pairs of the boundary components $b_1, \ldots, b_r$.

Now the class $c \in H_1(\bar{\Sigma}, \partial\bar{\Sigma})$ can be expressed as $c = ka + a'$, where $a \in A \simeq H_1(\Sigma_{h,1})$ is primitive, $k \in \mathbb{Z}$, and $a' \in A'$ is represented by arcs supported in $\Sigma_{0,r+1} \subset \bar{\Sigma}$. Since $a \in H_1(\Sigma_{h,1})$ is primitive, we can represent it by a simple closed curve α [MP78, Thm. 1].

Similar to Case 1, the complement of α in $\Sigma_{h,1} \subset \bar{\Sigma}$ contains a subsurface N of genus $h-1$ that lifts to $\Sigma \subset S$ (because it is disjoint from our representative for c) and contributes an $(h-1)$-dimensional isotropic subspace to $H_1(S;\mathbb{Q})$. This completes the proof. $\square$

Since $\lfloor (\mathrm{rk}_{F_k}(\Gamma_k) - 1)/2 \rfloor \geq \lfloor (h-2)/2 \rfloor \geq 2$ for $h \geq 6$, this is the bound that appears in Theorem 1.

4.2 Relating $H^*(\mathrm{Sp}_{2g}^G(\mathbb{Z});\mathbb{Q})$ with Chern classes of the Hodge bundle. In order to study the image of $\alpha^* : H^2(\mathrm{Sp}_{2g}^G(\mathbb{Z})) \to H^2(\mathrm{Mod}^G(S,\mathbf{z}))$, we want to relate the classes $\alpha^*(x_q) \in H^2(\mathrm{Mod}^G(S,\mathbf{z}))$ to the Chern classes $c_{1,q} \in H^2(\mathrm{Mod}^G(S,\mathbf{z}))$ defined in §2.4. We will see that

$$c_{1,q} = \alpha^*(x_q) = c_{1,\bar{q}} \tag{14}$$

for $q^m = 1$ with $\mathrm{Im}(q) > 0$. This relation can be obtained by comparing two maps from $B\,\mathrm{Mod}^G(S,\mathbf{z})$ to the product of unitary groups. For the first map, consider the composition

$$B\,\mathrm{Mod}^G(S,\mathbf{z}) \to B\,\mathrm{Sp}_{2g}^G(\mathbb{Z}) \to B\,\mathrm{Sp}_{2g}(\mathbb{R}) \xrightarrow{\sim} B\,\mathrm{U}(g), \tag{15}$$

and note that it factors through $B\,\mathrm{U}(g)^G \to B\,\mathrm{U}(g)$. The group $\mathrm{U}(g)^G$ is a product of unitary groups, one for each $q^m = 1$. The map $B\,\mathrm{Mod}^G(S,\mathbf{z}) \to B\,\mathrm{Sp}_{2g}^G(\mathbb{Z}) \to B\,\mathrm{U}(g)^G$ classifies the Hodge eigenbundles (for the universal bundle).

The second map

$$B\,\mathrm{Mod}^G(S,\mathbf{z}) \to B\,\mathrm{Sp}_{2g}^G(\mathbb{Z}) \to B\,\mathrm{Sp}_{2g}^G(\mathbb{R}) \sim B\,\mathrm{U}(h) \times B\,\mathrm{U}(h') \times \prod_{\substack{q^m=1 \\ \mathrm{Im}(q)>0}} BS\big(\mathrm{U}(a_q) \times \mathrm{U}(b_q)\big) \tag{16}$$

is obtained using (11). On the bundle level this map is obtained by starting with an (S,G) bundle $M \to B$, taking the associated real vector bundle $H_1(S;\mathbb{R}) \to E \to B$, decomposing E according to the decomposition of $H_1(S;\mathbb{R})$ as a G-representation over $\mathbb{R}$, and giving this bundle a complex structure induced from the action of G (the Hermitian forms (9) on sub-representations of $H_1(S;\mathbb{Q})$ give $H_1(S;\mathbb{R})$ a natural complex structure).

The maps (15) and (16) classify the same bundle, but with respect to different complex structures. From this it follows that the terms in (14) differ by at most -1. The following proposition settles the difference. Although it suffices to work with the universal (S, G) bundle, we find it more convenient to work on the level of individual bundles.

Proposition 11. *Let $\pi : M \to B$ be an (S, G)-bundle with Hodge bundle $E = \bigoplus_{q^m=1} E_q \to B$. Then $c_1\big(E_q\big) = c_1\big(E_{\bar q}\big) = x_q(\pi)$ in $H^2(B)$, where $x_q(\pi)$ is the pullback of the class x_q defined in Corollary 9.*

Proof of Proposition 11. There are two natural complex structures on the bundle $H_1(S;\mathbb{R}) \to E \to B$ induced from different complex structures on $H_1(S;\mathbb{R})$. The first J is the Hodge star operator $\star^2 = -1$ on $H^1(S;\mathbb{R})$, and the second J' is induced by the G action on $H^1(S;\mathbb{R})$ (this depends on a choice of generator of G). The proposition is proved by comparing J and J' and recalling how the definition of the Chern classes is sensitive to a choice of complex structure (see Borel–Hirzebruch [BH58, §9.1]).

Decompose $H^1(S;\mathbb{R}) = H(1) \oplus H(-1) \oplus \bigoplus_{\substack{q^m=1 \\ \mathrm{Im}(q)>0}} H(q, \bar q)$ into isotypic components for the irreducible representations of G over $\mathbb{R}$. (Recall that the simple $\mathbb{R}G$ modules are the trivial representation $V(1)$, the sign representation $V(-1)$ (if m is even), and $V(q, \bar q) = \mathbb{R}[t]/\big(t^2 - (q + \bar q)t + 1\big)$ for $q^m = 1$ such that $\mathrm{Im}(q) > 0$.)

The complex structure J on $H^1(S;\mathbb{R})$ induces an isomorphism $H(q, \bar q) \simeq H^1(S;\mathbb{C})_q = H^{1,0}_q \oplus H^{0,1}_q$. This decomposition coincides with the decomposition of $H(q, \bar q)$ into positive-definite and negative-definite subspaces for the Hermitian form β in (9). Since $H^{1,0}$ and $H^{0,1}$ are $+i$ and $-i$ eigenspaces for J, the same holds for $H^{1,0}_q$ and $H^{0,1}_q$. This identifies the complex structure J on these two factors. On the other hand, if we view $V = H^{1,0}_q \oplus H^{0,1}_q$ as a real vector space $V(q, \bar q)^N$, the G action defines another complex structure J' such that for any $v \in V$, the orientation $(v, J'v)$ on $\mathbb{R}\{v, J'v\}$ agrees with the orientation $(v, \tau v)$, where $\tau = e^{2\pi i/m}$ generates G. From this description, it follows that if $\mathrm{Im}(q) > 0$, then J and J' agree on $H^{1,0}_q \simeq H^{0,1}_{\bar q}$, but differ by -1 on $H^{1,0}_{\bar q} \simeq H^{0,1}_q$.

Let $c_1(E_q)$ and $c_1'(E_q)$ denote the Chern class defined using J and J', respectively [BH58, §9.1]. The proposition follows by noting the following equalities: If $\mathrm{Im}(q) > 0$, then

$$c_1(E_q) = c_1'(E_q) = x_q(\pi), \tag{17}$$

and

$$c_1(E_{\bar q}) = -c_1'(E_{\bar q}) = c_1'(E_q) = x_q(\pi). \tag{18}$$

The first equality in (17) and (18) follows because $J = J'$ on $H^{1,0}_q$ and $J = -J'$ on $H^{1,0}_{\bar q}$. The middle equality in (18) holds because the bundle $E_q \oplus E_{\bar q} \to B$ is classified by a map $B \to B\,\mathrm{SU}(a_q, b_q) \sim B\mathrm{S}\big(\mathrm{U}(a_q) \times \mathrm{U}(b_q)\big)$ as in (16). Finally, $c_1'(E_q) = x_q(\pi)$ by (13). □

4.3 Applying the index formula. The degree-1 terms of the index formulas (2) and (1) give a system of linear equations:

$$c_{1,1} + c_{1,-1} + 2 \sum_{\substack{q^m=1 \\ \mathrm{Im}(q)>0}} c_{1,q} = \kappa_1/12, \tag{19}$$

and for $1 \le r \le m-1$,

$$c_{1,1} + (-1)^r c_{1,-1} + \sum_{\substack{q^m=1 \\ \mathrm{Im}(q)>0}} (q^r + \bar q^r)\, c_{1,q} = \sum_{1 \le j < m/2} \csc^2(r\theta_j/2)\, (\epsilon_j + \epsilon_{m-j})/4 \tag{20}$$

The term $c_1(E_{-1})$ appears only when m is even. Note that $\epsilon_j + \epsilon_{m-j}$ is nonzero only when $\mathbf{z}_j \cup \mathbf{z}_{m-j} \neq \varnothing$; furthermore, our assumption on the point stabilizers for G acting on S implies that $\mathbf{z}_j \cup \mathbf{z}_{m-j} \neq \varnothing$ only if $(j, m) = 1$.

Let $d = \lfloor m/2 \rfloor$, and let $1 \le j_1, \ldots, j_n < m/2$ be the indices for which $\mathbf{z}_j \cup \mathbf{z}_{m-j} \neq \varnothing$ (note that $n \le \phi(m)/2$). Equations (19) and (20) define a matrix equation of the form

$$J \begin{pmatrix} c_1(E_{\zeta^0}) \\ c_1(E_{\zeta^1}) \\ \vdots \\ c_1(E_{\zeta^d}) \end{pmatrix} = K \begin{pmatrix} \kappa_1 \\ \epsilon_{j_1} + \epsilon_{m-j_1} \\ \vdots \\ \epsilon_{j_n} + \epsilon_{m-j_n} \end{pmatrix}.$$

Here J is a $(d+1) \times (d+1)$ matrix and K is a $(d+1) \times (n+1)$ matrix.

We wish to show that $\operatorname{Im}\left[H^2(\mathrm{Sp}_{2g}^G(\mathbb{Z})) \to H^2(\mathrm{Mod}^G(S, \mathbf{z}))\right] = \mathbb{Q}\{\kappa_1, \epsilon_{j_1} + \epsilon_{m-j_1}, \ldots, \epsilon_{j_n} + \epsilon_{m-j_n}\}$. First we show that J is invertible, which implies the containment $\subseteq$. Then we show $\operatorname{rk}(K) \ge n+1$, which implies the other containment.

Proposition 12. *J is invertible.*

Proof. A row of J has the form $\begin{pmatrix} \chi_g(V_0) & \chi_g(V_1) & \cdots & \chi_g(V_d) \end{pmatrix}$ for fixed $g \in G \subset \mathbb{C}^\times$ with $\operatorname{Im}(g) \ge 0$, and where $V_j = \rho_{\zeta^j} + \rho_{\zeta^{-j}}$ for $1 \le j < m/2$ and $V_j = \rho_{\zeta^j}$ for $j = 0, m/2$.

If the columns of J are dependent, then there are constants $a_0, \ldots, a_d$ so that

$$a_0\, \chi_g(V_0) + \cdots + a_d\, \chi_g(V_d) = 0$$

for $g \in G \subset \mathbb{C}^\times$ with $\operatorname{Im}(g) \ge 0$. But then this equation holds for all $g \in G$ because $\chi_g(V_j) = \chi_{g^{-1}}(V_j)$. But this is impossible because the characters of irreducible representations of G are linearly independent. □

Proposition 13. $\operatorname{rk}(K) \ge n+1$.

Using (19) and (20), note that $K = \begin{pmatrix} 1/12 & 0 \\ 0 & K' \end{pmatrix}$, where K' is an $d \times n$ matrix. From inspection of (20), to prove the proposition it suffices to show the following proposition.

Proposition 14. *Fix $m \ge 2$. Let $V \simeq \mathbb{R}^{\phi(m)/2}$ be a real vector space with basis $\{e_\ell\}$ for $1 \le \ell < m/2$ and $\gcd(\ell, m) = 1$. Then the vectors*

$$v_k = \sum_{\substack{1 \le \ell < m/2 \\ \gcd(\ell, m) = 1}} \csc^2\left(\frac{\pi k \ell}{m}\right) e_\ell$$

$1 \le k \le \phi(m)/2$ also form a basis for V.

Proof. We will denote $\mathbb{Z}/m\mathbb{Z}$ by C_m. For simplicity we start with the case $m = p$ is prime. The case $m = p^n$ is a prime power follows easily from this. Then we explain the general case.

Case 1: $m = p$ is prime. Let $q = \frac{p-1}{2}$. Consider functions $f_k : (C_p)^\times \to \mathbb{R}$ defined by $f_k(x) = \csc^2\left(\frac{k\pi}{p}\, x\right)$, and $A = (A_{k,\ell})$ be the $q \times q$ matrix $A_{k,\ell} = f_k(\ell)$ for $1 \le k, \ell \le q$. To prove the proposition, it is enough to show that A is invertible.

To this end, define another $q \times q$ matrix B as follows. Consider the surjective homomorphism $\phi : (C_p)^\times \simeq C_{p-1} \to C_q$. For $0 \le i, j \le q-1$, define B_{ij} by $\csc^2\left(\frac{\pi}{p} \cdot y\right)$, where $\phi(y) = i + j$. This is well defined because $\csc^2(x)$ is an even function.

Now observe

(1) A and B are the same matrix, up to permuting rows and columns. Thus it suffices to show that $\det(B) \neq 0$. We will show the eigenvalues of B are all nonzero.

(2) B is a *circulant matrix*, up to permuting rows and columns (see [Dav79] for the definition). This is easy to see because B is obtained by taking the multiplication table for $\mathbb{Z}/q$ and applying a fixed function to each entry. (The multiplication table of a cyclic group is circulant up to permuting the rows.)

Now the eigenvalues/eigenvectors of a circulant matrix are easily computed [Dav79]. The eigenvalues have the form $\lambda_j = c_0 + c_1\omega^j + c_2\omega^{2j} + \cdots + c_{q-1}\omega^{(q-1)j}$, where $\omega = e^{2\pi i/q}$ and $0 \le j \le q-1$ and the c_i are in bijection with $\csc^2\left(\frac{k\pi}{p}\right)$, $1 \le k \le q$. If $\lambda_j = 0$, then ω^j is a solution to the polynomial $P(x) = c_0 + c_1x + \cdots + c_{q-1}x^{q-1}$ for some j. This is possible if and only if $c_0 = c_1 = \cdots = c_{q-1}$, which is not the case.

Case 2: $m = p^n$ is a prime power. An important feature of the above argument is that when m is prime, $(C_m)^\times \simeq C_{\phi(m)}$ is cyclic, as is $(C_m)^\times/\{\pm 1\}$, so its multiplication table is given by a circulant matrix whose determinant is easy to compute (even after applying a fixed function to each coordinate).

When p is an odd prime, then $(C_{p^n})^\times \simeq C_{\phi(p^n)}$ is cyclic, so we may repeat the argument of Case 1.

When $p = 2$, the group $(C_{2^n})^\times \simeq C_2 \times C_{2^{n-2}}$ is *not* cyclic. However, the fact that f_k is even implies that it factors through $(C_{2^n})^\times/\{\pm 1\}$, and the subgroup $\{\pm 1\} < (C_{2^n})^\times$ corresponds to the subgroup $C_2 \times \{0\} < C_2 \times C_{2^{n-2}}$. This means $f_k : C_2 \times C_{2^{n-2}} \to \mathbb{R}$ factors though the cyclic group $C_{2^{n-2}}$, and we can again apply the argument from Case 1.

Case 3: m is arbitrary. In this case we cannot assume that $(C_m)^\times$ is cyclic, and in most cases the multiplication table for $(C_m)^\times/\{\pm 1\}$ will not be circulant. However, if we write $m = p_1^{n_1} \cdots p_r^{n_r}$, then using the isomorphism $(C_m)^\times \simeq (C_{p_1^{n_1}})^\times \times \cdots \times (C_{p_r^{n_r}})^\times$, the multiplication table for $(C_m)^\times/\{\pm 1\}$ may be expressed as a special kind of block circulant matrix. Having this block circulant form will allow us to apply the argument of Case 1 iteratively.

We begin by examining what the structure of the multiplication table of a product of cyclic groups. Fix a finite group F and a cyclic group $C_d = \langle t \rangle$. If the multiplication table for F is given by a matrix A, then the multiplication table for $F \times C_d$ has the form

$$\begin{pmatrix} A & tA & \cdots & t^{d-1}A \\ tA & t^2A & \cdots & A \\ \vdots & & \ddots & \vdots \\ t^{d-1}A & A & \cdots & t^{d-2}A \end{pmatrix}$$

This matrix becomes block circulant after permuting the rows. Thus the multiplication table of a product of cyclic groups is an iterated block circulant matrix.

Next we determine the eigenvalues of a block circulant matrix. Fix $d, n \ge 1$, fix $A_0, \ldots, A_{d-1} \in M_n(\mathbb{R})$, and consider the block circulant matrix

$$B = \begin{pmatrix} A_0 & A_1 & \cdots & A_{d-1} \\ A_{d-1} & A_0 & \cdots & A_{d-2} \\ \vdots & & \ddots & \vdots \\ A_1 & A_2 & \cdots & A_0 \end{pmatrix}$$

Suppose that the matrices A_i share common eigenvectors $x_0, \ldots, x_{n-1}$, so that $A_ix_j = \lambda_{ij}x_j$. Denoting $\zeta = e^{2\pi i/d}$, the eigenvectors of B are

$$x_{kj} = \begin{pmatrix} x_j \\ \zeta^k x_j \\ \vdots \\ \zeta^{k(d-1)}x_j \end{pmatrix}$$

for $0 \le k \le d-1$ and $0 \le j \le n-1$, and the eigenvalues are

$$\eta_{kj} = \lambda_{0k} + \lambda_{1k}\zeta^j + \cdots + \lambda_{m-1,k}\zeta^{j(d-1)}$$

These facts are easily checked.

Now the group $(C_m)^\times/\{\pm 1\}$ is a product of cyclic groups, so its multiplication table is an iterated block circulant matrix B_0. The matrix $A = \big(f_k(\ell)\big)$ is equivalent to the matrix B obtained by applying $\csc^2(\frac{\pi}{m}\cdot)$ to each entry of B_0. Since all $n \times n$ circulant matrices have the same eigenvectors, the above computation applies for computing the eigenvalues of B. Now, as in Case 1, the eigenvalues are given as degree $m-1$ polynomials $P(\exp^{2\pi i/m})$ with (nonconstant!) coefficients among the $f_k(\ell)$, so $\det(B) \neq 0$. $\square$

Since J is invertible and $\mathrm{rk}(K) \ge n+1$, we conclude that $H^2(\mathrm{Sp}_{2g}^G(\mathbb{Z}))$ surjects to the subspace of $H^2(\mathrm{Mod}^G(S,\mathbf{z}))$ generated by κ_1 and $\{\epsilon_{j_i} + \epsilon_{m-j_i} : 1 \le i \le n\}$, which finishes the proof of Theorem 1.

5 Further application of the index formula

5.1 The real points of $\mathrm{Sp}_{2g}^G(\mathbb{Z})$. We remark on how the degree-0 term of the index formula can be used to determine the real semisimple Lie group $\mathrm{Sp}_{2g}^G(\mathbb{R})$ that contains $\mathrm{Sp}_{2g}^G(\mathbb{Z})$ as a lattice. Part of the work was already done in §4.1; it remains to determine the numbers a_q, b_q in equation (11). This computation is an elaboration of a remark in [McM13, §3] and will be used later in this section.

Chevalley–Weil. First one can use the Chevalley–Weil algorithm to determine the character χ_H of $H = H_1(S;\mathbb{R})$. Obviously $\chi_H(e) = \dim H = 2g$, and by the Lefschetz formula, $\chi_H(g) = 2 - \#\,\mathrm{Fix}(g)$ for $g \neq e$. Since a representation is determined by its character, this gives the integers n_q in the decomposition

$$H_1(S;\mathbb{R}) = V(1)^{n_1} \oplus \bigoplus_{\substack{q^m=1 \\ \mathrm{Im}(q)>0}} V(q,\bar{q})^{n_q} \oplus V(-1)^{n_{-1}}.$$

Here $V(\pm 1)$ are the trivial/alternating representations, and $V(q,\bar{q}) = \mathbb{R}[t]/(t^2 - (q+\bar{q})t + 1)$.

Hodge star and index formula. The Hodge star gives a complex structure to $H_1(S;\mathbb{R})$, and hence an isomorphism for each q with $\mathrm{Im}(q) > 0$

$$V(q,\bar{q})^{n_q} \simeq V(q)^{a_q} \oplus V(\bar{q})^{b_q},$$

where $V(q) = \mathbb{C}[t]/(t-q)$. The numbers a_q, b_q can be computed using the degree-0 term of the index formula (1)

$$\sum_{q^m=1,\,\mathrm{Im}(q)>0} (a_q - b_q)(q^r - \bar{q}^r) = -i \sum_{1 \le j \le m-1} \cot(r\,\theta_j/2) \cdot |\mathbf{z}_j|, \tag{21}$$

where $\mathbf{z}_j$ is as in the statement of Theorem 4.

Example. Here we consider a closed surface S of genus $g = \frac{(m-1)(m-2)}{2} + mh$ with an action of $G = \mathbb{Z}/m\mathbb{Z}$ with m fixed points. These surfaces arise in Morita's m-construction [Mor01, §4.3].

An explicit model for S can be obtained as follows. Take m disks, stacked horizontally, and attach m strips between each pair of adjacent levels, as pictured in Figure 2 (in the case $m=5$). This gives a surface of genus $\frac{(m-1)(m-2)}{2}$ with m boundary components. The rotation by $2\pi/m$ on the disk extends to an action of $\mathbb{Z}/m\mathbb{Z}$ on this surface with one fixed point in each disk. Along each boundary component, we can attach a genus-h surface (with one boundary component) to obtain a closed surface of genus $\frac{(m-1)(m-2)}{2} + mh$ with an action of $\mathbb{Z}/m\mathbb{Z}$ with m fixed points.

Using Chevalley–Weil, one easily computes the decomposition of $H_1(S;\mathbb{Q})$ into isotypic components (as was discussed in §4.1:

$$H_1(S;\mathbb{Q}) = \mathbb{Q}^{2h} \oplus \bigoplus_{\substack{k|m \\ k \ge 2}} \mathbb{Q}(\zeta_k)^{2h+m-2}.$$

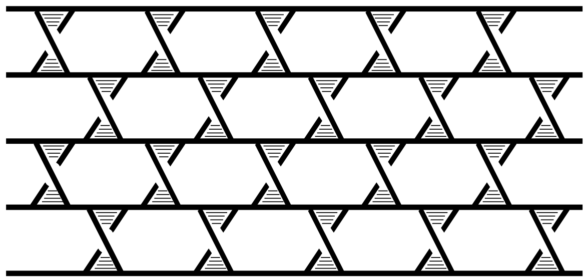

FIGURE 2. Schematic of a genus-6 surface with 5 boundary components and an action of $\mathbb{Z}/5\mathbb{Z}$.

In this case

$$\mathrm{Sp}_{2g}^G(\mathbb{Z}) \doteq \mathrm{Sp}_{2g}(\mathbb{Z}) \times \mathrm{Sp}_{2h+m-2}(\mathbb{Z}) \times \prod_{\substack{k|m \\ 2<k\leq m/2}} \mathrm{SU}(H_k, \beta_k)_{\mathcal{O}_k}.$$

The second factor appears only when m is even. Applying (21), we find that $\prod_{\substack{k|m \\ 2<k\leq m/2}} \mathrm{SU}(H_k, \beta_k)_{\mathcal{O}_k}$ is a lattice in $\mathbb{G}(\mathbb{R}) = \prod_{i=0}^N \mathrm{SU}(h+i, h+m-2-i)$, where $N = \lfloor \frac{m-1}{2} \rfloor$. Equivalently, the factors in $\mathbb{G}(\mathbb{R})$ are of the form $\mathrm{SU}(u+v_q)$ for $u, v_q \in \mathbb{Z}^2$, where $u = (h, h+m-2)$ and for each $q^m = 1$ with $\mathrm{Im}(q) > 0$, we define $v_q = (a, -a)$ where a the number of m-th roots of unity above the line from 1 to q in $\mathbb{C}$. See Figure 3 and also [McM13, Figure 1].

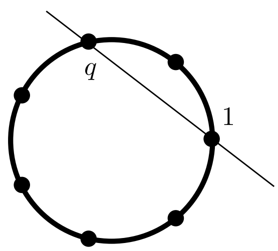

FIGURE 3. For $m = 7$ the group $\mathbb{G}(\mathbb{R}) = \mathrm{SU}(h, h+5) \times \mathrm{SU}(h+1, h+4) \times \mathrm{SU}(h+2, h+3)$.

5.2 Relation to Hirzebruch's signature formula. Hirzebruch [Hir69] explained how the signature changes in a branched cover. In this section we derive this result for surface bundles over surfaces from our viewpoint. For simplicity we restrict to 2-fold branched covers.

Let M be a closed oriented 4-manifold with a $G = \mathbb{Z}/2\mathbb{Z}$ action with fixed set $\mathrm{Fix}(G) = M_0$. In this case, Hirzebruch proved that

$$\mathrm{Sig}(M) = 2\ \mathrm{Sig}(M/G) - \mathrm{Sig}(M_0 \cdot M_0), \tag{22}$$

where $M_0 \cdot M_0$ is a closed oriented manifold and thus has a signature. This formula applies in the special case when M is the total space of an (S, G)-bundle over a surface. Our main observation here is that the terms $\mathrm{Sig}(M/G)$, $\mathrm{Sig}(M_0 \cdot M_0)$ can be understood in terms of cohomology of the arithmetic group $\mathrm{Sp}_{2g}^G(\mathbb{Z}) \doteq \mathrm{Sp}_{2h}(\mathbb{Z}) \times \mathrm{Sp}_{2h'}(\mathbb{Z})$.

To illustrate this, consider a G action on a genus-$2h$ surface with two fixed points $\mathbf{z} = \{z_1, z_2\}$. The quotient $\mu : S \to \bar{S}$ has genus h. Let $\bar{\mathbf{z}} = \mu(\mathbf{z})$. In this case $\mathrm{Sp}_{2g}^G(\mathbb{Z}) \doteq \mathrm{Sp}_{2h}(\mathbb{Z}) \times \mathrm{Sp}_{2h}(\mathbb{Z})$, and we have a commutative diagram

$$\begin{tikzcd} \mathrm{Mod}^G(S,\mathbf{z}) \arrow[r, "f"] \arrow[d, "\phi"'] & \mathrm{Sp}_{2g}^G(\mathbb{Z}) \arrow[d, "\psi"] \\ \mathrm{Mod}_\mu(\bar{S},\bar{\mathbf{z}}) \arrow[r, "g"] & \mathrm{Sp}_{2h}(\mathbb{Z}) \end{tikzcd}$$

The cohomology $H^2\big(\mathrm{Sp}_{2g}^G(\mathbb{Z});\mathbb{Q}\big)$ is generated by $\{x_1, x_{-1}\}$ as in Corollary 9. Also $H^2\big(\mathrm{Sp}_{2h}(\mathbb{Z});\mathbb{Q}\big) \simeq \mathbb{Q}\{y_1\}$ for $h \geq 3$. Let κ_1 and $\bar{\kappa}_1$ be the 1st MMM class in the cohomology of $\mathrm{Mod}^G(S,\mathbf{z})$ and $\mathrm{Mod}_\mu(\bar{S},\bar{\mathbf{z}})$, respectively, and let $e_i \in H^2\big(\mathrm{Mod}^G(S,\mathbf{z})\big)$ be the Euler class at the fixed point z_i for $i = 1,2$.

By (2), $g^*(y_1) = \bar{\kappa}_1/12$. Hirzebruch's formula will come from determining $\phi^*(\bar{\kappa}_1)$. Since the diagram commutes and $\psi^*(y_1) = x_1$, we want to compute $f^*(x_1)$. By the index formulas (19) and (20), we have $f^*(x_1 + x_{-1}) = \kappa_1/12$ and $f^*(x_1 - x_{-1}) = (e_1 + e_2)/4$, so

$$\phi^*(\bar{\kappa}_1) = f^*(12\,x_1) = \frac{1}{2}\,\kappa_1 + \frac{3}{2}\,(e_1 + e_2). \tag{23}$$

To conclude, let $M \to B$ be an S bundle with B a surface and with monodromy $\rho : \pi_1(B) \to \mathrm{Mod}^G(S,\mathbf{z})$. By Hirzebruch's signature formula $\mathrm{Sig}(M) = \frac{1}{3}\langle \rho^*(\kappa_1), [B]\rangle$. The fixed set $M_0 = \mathrm{Fix}(G)$ is a surface and $\langle \rho^*(e_1 + e_2), [B]\rangle = \#(M_0 \cdot M_0)$. Then (23) gives

$$\mathrm{Sig}(M/G) = \frac{\mathrm{Sig}(M)}{2} + \frac{\#(M_0 \cdot M_0)}{2}.$$

Since the signature of a 0-manifold is the number of points, this is the same as Hirzebruch's formula (22).

5.3 Toledo invariants of surface group representations. The Toledo invariant τ is an integer invariant of a representation $\alpha : \pi_1(\Sigma) \to H$, where Σ is a closed oriented surface (genus ≥ 1) and H is a *Hermitian* Lie group. In this section we will be interested in the case $H = \mathrm{SU}(p,q)$ with $1 \leq p \leq q$. To define $\tau(\alpha)$, first construct a smooth α-equivariant map $f : \widetilde{\Sigma} \to X$, where $\widetilde{\Sigma}$ is the universal cover of Σ and $X = \mathrm{SU}(p,q)/S\big(\mathrm{U}(p) \times \mathrm{U}(q)\big)$ is the symmetric space associated to H. The *Toledo invariant* is defined as

$$\tau(\alpha) = \frac{1}{2\pi}\int_F f^*\omega,$$

where ω is the Kähler form of X and $F \subset \widetilde{\Sigma}$ a fundamental domain for the action of $\pi_1(\Sigma)$.

Domic–Toledo [DT87] showed that $|\tau(\alpha)| \leq -p\,\chi(\Sigma)$, and Bradlow–Garcia-Prada–Gothen [BGPG03] have shown that components of the representation variety $\mathrm{Hom}\big(\pi_1(\Sigma), H\big)/H$ are in bijection with the values achieved by τ. Here we simply observe that the Atiyah–Kodaira construction gives examples of surface group representations whose Toledo invariant can be computed using the index formula.

We'll explain this in a special case (see [Mor01, §4.3] for a general discussion of the Atiyah–Kodaira construction). Let $G = \langle \tau \rangle \simeq \mathbb{Z}/7\mathbb{Z}$ and let $\bar{S} = \bar{S}_h$ be a closed surface with a free $\mathbb{Z}/7\mathbb{Z}$ action. The product bundle $\bar{S} \times \bar{S} \to \bar{S}$ admits 7 disjoint sections $\Gamma_{\mathbb{1}}, \Gamma_\tau, \ldots, \Gamma_{\tau^6}$, where Γ_f denotes the graph of $f : \bar{S} \to \bar{S}$. In order to branch over $\bigcup \Gamma_{\tau^i}$, we must first pass to a cover. Let $p : \Sigma \times \bar{S}$ be the $\mathbb{Z}/7\mathbb{Z}$ homology cover (Σ has genus $7^{2h}(h-1)+1$). The bundle $\Sigma \times \bar{S}$ has sections $\Gamma_p, \Gamma_{\tau p}, \ldots, \Gamma_{\tau^6 p}$, and admits a $\mathbb{Z}/7\mathbb{Z}$ branched cover $M \to \Sigma \times \bar{S}$ with branching locus $\bigcup \Gamma_{\tau^i p}$. Projecting $M \to \Sigma \times \bar{S} \to \Sigma$ defines a bundle with fiber S, which is a 7-fold branched cover $\mu : S \to \bar{S}$ branched along 7 points (S has genus $7h + 15$). The homology $H_1(S;\mathbb{Q})$ is isomorphic to $\mathbb{Q}^{2h} \oplus \mathbb{Q}(\zeta_7)^{2h+5}$ as a G-module. In this case, $\mathrm{Sp}_{2g}^G(\mathbb{Z}) \doteq \mathrm{Sp}_{2h}(\mathbb{Z}) \times \Gamma$, where Γ is an irreducible lattice in $\mathrm{SU}(h, h+5) \times \mathrm{SU}(h+1, h+4) \times \mathrm{SU}(h+2, h+3)$. Thus we have a homomorphism

$$\alpha : \pi_1(\Sigma) \to \Gamma \hookrightarrow \mathrm{SU}(h, h+5) \times \mathrm{SU}(h+1, h+4) \times \mathrm{SU}(h+2, h+3).$$

Let α_i be the representation obtained by projecting to the i-th factor, $i = 1, 2, 3$. By the index formula, one obtains the following equations. For $\zeta = e^{2\pi i/7}$,

$$c_{1,1} = a\,\kappa_1 + 4b\,\epsilon_1, \quad c_{1,\zeta} = a\,\kappa_1 + b\,\epsilon_1, \quad c_{1,\zeta^2} = a\,\kappa_1 - b\,\epsilon_1, \quad c_{1,\zeta^3} = a\,\kappa_1 - 2b\,\epsilon_1,$$

where $a = \frac{1}{84}$ and $b = \frac{1}{7}$. Since the signature of $M/G = \Sigma \times \bar{S}$ is zero, it follows that $c_{1,1} = 0$, which allows us to express $\tau(\alpha_i) = c_{1,\zeta^i}$ in terms of $\kappa_1/3$, which computes the signature. Thus the Toledo invariants are given by

$$\tau(\alpha_1) = \frac{3}{112}\sigma, \qquad \tau(\alpha_2) = \frac{5}{112}\sigma, \qquad \tau(\alpha_3) = \frac{6}{112}\sigma,$$

where $\sigma = \mathrm{Sig}(M)$.

Remark. The Toledo invariants of representations obtained in this way will never have maximal Toledo invariant. This is because the Gromov norm of the Toledo class decreases when pulled back the mapping class group [Kot98], so in fact, no representation $\pi_1(\Sigma) \to \mathrm{SU}(p, q)$ that factors through $\mathrm{Mod}(S)$ will be maximal. However, one could also ask whether these representations are *weakly maximal* in the sense of [BSBH+17].

5.4 Cobordism invariants. Church–Farb–Thibault [CFT12] show that the odd MMM classes κ_{2i-1} are *cobordism invariants*. This means that for an S bundle $M^{4i} \to B$, the characteristic number $\kappa^{\#}_{2i-1}(M \to B)$ depends only on the cobordism class of M. In particular, the class κ_{2i-1} cannot distinguish between different fiberings of a $4i$-manifold M.

If $M \to B$ admits a fiberwise G-action, we can ask about characteristic classes c that are *G-cobordism invariants*, i.e. the corresponding characteristic number $c^{\#}(M \to B)$ depends only on the *G-bordism class* of M (for more on the notion of G-bordsim, see e.g. [CF64, Chapter III]). Consider the case $\dim(M) = 4$. Of course $\kappa^{\#}_1(M \to B)$ is also a G-cobordism invariant; below we prove Corollary 3 thus exhibiting more classes that have this property.

Proof of Corollary 3. Let Σ be a closed surface and fix an (S, G) bundle $M^4 \to \Sigma$. Let $E \to \Sigma$ be the Hodge bundle with eigenbundles $E = \bigoplus_{q^m=1} E_q$. We aim to show that the numbers

$$c^{\#}_1(E_q \to \Sigma) = \big\langle c_1(E_q), [\Sigma] \big\rangle$$

depend only on the G-bordism class of M.

Suppose that there is a G-manifold W^5 such that $M = \partial W$ (as G-manifolds). To prove the corollary, we must show that $c^{\#}_1(E_q \to \Sigma) = 0$. First observe that, by Theorem 1, $c^{\#}_1(E_q \to \Sigma)$ is a linear combination of the signature $\mathrm{Sig}(M)$ and the intersection numbers $\#(M^\tau_j \cdot M^\tau_j)$, where τ generates $\mathbb{Z}/m\mathbb{Z}$, and we decompose the fixed set $M^\tau = \bigcup_{j=1}^{m-1} M^\tau_j$ according to the action of τ on the normal bundle (as in the statement of Theorem 4). Now $\mathrm{Sig}(M) = 0$ because $M = \partial W$, and we claim that $\#(M^\tau_j \cdot M^\tau_j) = 0$ as well. To see the latter, note that M^τ and W^τ are submanifolds (average a metric so that τ acts by isometries), and $M^\tau = \partial(W^\tau)$ because $M = \partial W$ as G-manifolds. It follows that $M^\tau \cdot M^\tau = \partial(W^\tau \cdot W^\tau)$. Since W^τ is a 3-manifold, $W^\tau \cdot W^\tau$ is a 1-manifold with boundary, and the boundary points occur in pairs, which implies that $\#(M^\tau \cdot M^\tau) = 0$, as desired. □

Remark. It would be interesting to determine precisely which elements of $H^*\big(\mathrm{Mod}^G(S, \mathbf{z}); \mathbb{Q}\big)$ are G-cobordism invariants following Church–Crossley–Giansiracusa [CCG13].

References

[AS68a] M. F. Atiyah and G. B. Segal. The index of elliptic operators. II. *Ann. of Math. (2)*, 87:531–545, 1968.

[AS68b] M. F. Atiyah and I. M. Singer. The index of elliptic operators. I. *Ann. of Math. (2)*, 87:484–530, 1968.

[AS68c] M. F. Atiyah and I. M. Singer. The index of elliptic operators. III. *Ann. of Math. (2)*, 87:546–604, 1968.

[AS71] M. F. Atiyah and I. M. Singer. The index of elliptic operators. IV. *Ann. of Math. (2)*, 93:119–138, 1971.

[Ati69] M. F. Atiyah. The signature of fibre-bundles. In *Global Analysis (Papers in Honor of K. Kodaira)*, pages 73–84. Univ. Tokyo Press, Tokyo, 1969.

[BGPG03] S. B. Bradlow, O. García-Prada, and P. B. Gothen. Surface group representations and $\mathrm{U}(p,q)$-Higgs bundles. *J. Differential Geom.*, 64(1):111–170, 2003.

[BH58] A. Borel and F. Hirzebruch. Characteristic classes and homogeneous spaces. I. *Amer. J. Math.*, 80:458–538, 1958.

[BH72] J. Birman and H. Hilden. Isotopies of homeomorphisms of Riemann surfaces and a theorem about Artin's braid group. *Bull. Amer. Math. Soc.*, 78:1002–1004, 1972.

[Bor74] A. Borel. Stable real cohomology of arithmetic groups. *Ann. Sci. École Norm. Sup. (4)*, 7:235–272 (1975), 1974.

[Bor81] A. Borel. Stable real cohomology of arithmetic groups. II. In *Manifolds and Lie groups (Notre Dame, Ind., 1980)*, volume 14 of *Progr. Math.*, pages 21–55. Birkhäuser, Boston, Mass., 1981.

[BSBH+17] G. Ben-Simon, M. Burger, T. Hartnick, A. Iozzi, and A. Wienhard. On weakly maximal representations of surface groups. *J. Differential Geom.*, 105(3):375–404, 2017.

[CCG13] T. Church, M. Crossley, and J. Giansiracusa. Invariance properties of Miller-Morita-Mumford characteristic numbers of fibre bundles. *Q. J. Math.*, 64(3):729–746, 2013.

[CF64] P. E. Conner and E. E. Floyd. *Differentiable periodic maps*. Ergebnisse der Mathematik und ihrer Grenzgebiete, N. F., Band 33. Academic Press Inc., Publishers, New York; Springer-Verlag, Berlin-Göttingen-Heidelberg, 1964.

[CFT12] T. Church, B. Farb, and M. Thibault. On the geometric nature of characteristic classes of surface bundles. *J. Topol.*, 5(3):575–592, 2012.

[Dav79] P. Davis. *Circulant matrices*. John Wiley & Sons, New York-Chichester-Brisbane, 1979. A Wiley-Interscience Publication, Pure and Applied Mathematics.

[DT87] A. Domic and D. Toledo. The Gromov norm of the Kaehler class of symmetric domains. *Math. Ann.*, 276(3):425–432, 1987.

[ERW15] J. Ebert and O. Randal-Williams. Torelli spaces of high-dimensional manifolds. *J. Topol.*, 8(1):38–64, 2015.

[ES70] C. J. Earle and A. Schatz. Teichmüller theory for surfaces with boundary. *J. Differential Geometry*, 4:169–185, 1970.

[FM12] B. Farb and D. Margalit. *A primer on mapping class groups*, volume 49 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 2012.

[Gia09] J. Giansiracusa. The diffeomorphism group of a $K3$ surface and Nielsen realization. *J. Lond. Math. Soc. (2)*, 79(3):701–718, 2009.

[GLLM15] F. Grunewald, M. Larsen, A. Lubotzky, and J. Malestein. Arithmetic quotients of the mapping class group. http://arxiv.org/abs/1307.2593, April 2015.

[Hat02] A. Hatcher. *Algebraic topology*. Cambridge University Press, Cambridge, 2002.

[Hir69] F. Hirzebruch. The signature of ramified coverings. In *Global Analysis (Papers in Honor of K. Kodaira)*, pages 253–265. Univ. Tokyo Press, Tokyo, 1969.

[Kot98] D. Kotschick. Signatures, monopoles and mapping class groups. *Math. Res. Lett.*, 5(1-2):227–234, 1998.

[LM89] H. B. Lawson, Jr. and M. Michelsohn. *Spin geometry*, volume 38 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1989.

[Loo97] E. Looijenga. Prym representations of mapping class groups. *Geom. Dedicata*, 64(1):69–83, 1997.

[Mar91] G. A. Margulis. *Discrete subgroups of semisimple Lie groups*, volume 17 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1991.

[Mat62] Y. Matsushima. On Betti numbers of compact, locally sysmmetric Riemannian manifolds. *Osaka Math. J.*, 14:1–20, 1962.

[McM13] C. T. McMullen. Braid groups and Hodge theory. *Math. Ann.*, 355(3):893–946, 2013.

[MH75] C. Maclachlan and W. J. Harvey. On mapping-class groups and Teichmüller spaces. *Proc. London Math. Soc. (3)*, 30(part 4):496–512, 1975.

[Mor87] S. Morita. Characteristic classes of surface bundles. *Invent. Math.*, 90(3):551–577, 1987.

[Mor01] S. Morita. *Geometry of characteristic classes*, volume 199 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 2001. Translated from the 1999 Japanese original, Iwanami Series in Modern Mathematics.

[Mor15] D. Witte Morris. *Introduction to arithmetic groups*. Deductive Press, Place of publication not identified, 2015.

[MP78] W. H. Meeks, III and J. Patrusky. Representing homology classes by embedded circles on a compact surface. *Illinois J. Math.*, 22(2):262–269, 1978.

[PR94] V. Platonov and A. Rapinchuk. *Algebraic groups and number theory*, volume 139 of *Pure and Applied Mathematics*. Academic Press, Inc., Boston, MA, 1994. Translated from the 1991 Russian original by Rachel Rowen.

[Put12] A. Putman. The Picard group of the moduli space of curves with level structures. *Duke Math. J.*, 161(4):623–674, 2012.

[Seg68] G. Segal. Equivariant K-theory. *Inst. Hautes Études Sci. Publ. Math.*, (34):129–151, 1968.

[Sha78] P. Shanahan. *The Atiyah-Singer index theorem*, volume 638 of *Lecture Notes in Mathematics*. Springer, Berlin, 1978.

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