MAPPING CLASS GROUPS OF CIRCLE BUNDLES OVER A SURFACE

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ABSTRACT. In this paper, we study the algebraic structure of mapping class group $\mathrm{Mod}(X)$ of 3-manifolds X that fiber as a circle bundle over a surface $S^1 \to X \to S_g$. There is an exact sequence $1 \to H^1(S_g) \to \mathrm{Mod}(X) \to \mathrm{Mod}(S_g) \to 1$. We relate this to the Birman exact sequence and determine when this sequence splits.

1. INTRODUCTION

For $g \geq 1$, let S_g denote the closed oriented surface of genus g, and for $k \in \mathbb{Z}$, let X_g^k denote the closed 3-manifold that fibers

$$S^1 \to X_g^k \to S_g$$

as an oriented circle-bundle with Euler number k. Assuming $(g,k) \neq (1,0)$, the mapping class group $\mathrm{Mod}(X_g^k) := \pi_0\big(\mathrm{Homeo}^+(X_g^k)\big)$ fits into a short exact sequence

$$1 \to H^1(S_g;\mathbb{Z}) \to \mathrm{Mod}(X_g^k) \to \mathrm{Mod}(S_g) \to 1. \tag{1}$$

This paper is motivated by the following question.

Question 1.1. For which values of g, k is the extension in (1) split?

Interestingly, the extension does split for $k = 2 - 2g$, in which case X_g^k is unit tangent bundle US_g. In fact, there is a natural action of $\mathrm{Mod}(S_g)$ on US_g by homeomorphisms, which gives a splitting of (1) upon taking isotopy classes. For $g \geq 2$, this action comes from the action of the punctured mapping class group $\mathrm{Mod}(S_{g,1})$ on triples of points on the boundary of hyperbolic space $\mathbb{H}^2$. This construction dates back to the work of Nielsen. See [FM12, §5.5.4, §8.2.6] and [Sou10, §1].

In general, Question 1.1 reduces to a question about group cohomology. The extension (1) splits if and only if its Euler class $eu_k \in H^2\big(\mathrm{Mod}(S_g); H^1(S_g;\mathbb{Z})\big)$ vanishes [Bro82, §IV.3]. Here the coefficients are twisted via the natural action of $\mathrm{Mod}(S_g)$ on $H^1(S_g;\mathbb{Z})$. However, a computation of $H^2\big(\mathrm{Mod}(S_g); H^1(S_g;\mathbb{Z})\big)$ does not appear to be in the literature.

The extension (1) is related to the Birman exact sequence

$$1 \to \pi_1(S_g) \to \mathrm{Mod}(S_{g,1}) \to \mathrm{Mod}(S_g) \to 1.$$

By taking quotients by the commutator subgroup $\pi' \equiv [\pi_1(S_g), \pi_1(S_g)]$, we obtain the following extension

$$1 \to H_1(S_g) \to \mathrm{Mod}(S_{g,1})/\pi' \to \mathrm{Mod}(S_g) \to 1. \tag{2}$$

Our main result relates the sequences (1) and (2).

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Theorem A. *Fix $g \geq 1$ and $k \in \mathbb{Z}$. Assume $(g, k) \neq (1, 0)$. There is a map between the short exact sequences* (1) *and* (2)

$$\begin{array}{ccccccccc} 1 & \longrightarrow & H_1(S_g) & \longrightarrow & \mathrm{Mod}(S_{g,1})/\pi' & \longrightarrow & \mathrm{Mod}(S_g) & \longrightarrow & 1 \\ & & \downarrow k\delta & & \downarrow & & \| & & \\ 1 & \longrightarrow & H^1(S_g;\mathbb{Z}) & \longrightarrow & \mathrm{Mod}(X_g^k) & \longrightarrow & \mathrm{Mod}(S_g) & \longrightarrow & 1 \end{array}$$

The homomorphism $k\delta$ is the Poincaré duality isomorphism δ composed with multiplication by k. In particular, when $k = 1$, the exact sequences (1) *and* (2) *are isomorphic.*

Theorem A implies the Euler classes of the extensions (1) satisfy $eu_k = k\, eu_1$ for fixed g. Next we determine the subgroup generated by eu_1 in $H^2\big(\mathrm{Mod}(S_g); H^1(S_g;\mathbb{Z})\big)$.

Theorem 1.2. *Fix $g \geq 1$, and let eu_1 be the Euler class of the extension* (1)*. Then eu_1 has order $2g - 2$ in $H^2\big(\mathrm{Mod}(S_g); H_1(S_g;\mathbb{Z})\big)$. Furthermore, if $g \geq 8$, then eu_1 generates this group, i.e.*

$$H^2\big(\mathrm{Mod}(S_g); H^1(S_g;\mathbb{Z})\big) \cong \mathbb{Z}/(2g-2)\mathbb{Z}.$$

Combining Theorem A and Theorem 1.2 we obtain the following answer to Question 1.1.

Corollary 1.3. *For $g \geq 2$ and $k \in \mathbb{Z}$, the extension* (1) *splits if and only if k is divisible $2g - 2$. For $g = 1$ the extension splits for each k.*

When a splitting exists, the different possible splittings (up to the action of $H^1(S_g;\mathbb{Z})$ on $\mathrm{Mod}(X_g^k)$ by conjugation) are parameterized by elements of $H^1\big(\mathrm{Mod}(S_g); H^1(S_g;\mathbb{Z})\big)$ [Bro82, Ch. IV, Prop. 2.3]. This group vanishes for $g \geq 1$ [Mor85, Prop. 4.1], so the splitting, when it exists, is unique.

Connection to Nielsen realization. Instead of Question 1.1, one can ask whether there is a splitting of the composite surjection

$$\mathrm{Homeo}(X_g^k) \to \mathrm{Mod}(X_g^k) \to \mathrm{Mod}(S_g).$$

This is an instance of a Nielsen realization problem. Of course, if $\mathrm{Mod}(X_g^k) \to \mathrm{Mod}(S_g)$ does not split, then neither does $\mathrm{Homeo}(X_g^k) \to \mathrm{Mod}(S_g)$, and Corollary 1.3 gives examples of this. Since $\mathrm{Mod}(S_g)$ has a natural action on US_g, the surjection $\mathrm{Homeo}(X_g^k) \to \mathrm{Mod}(S_g)$ does split for $k = \pm(2g - 2)$. This is somewhat surprising since mapping class groups are rarely realized as groups of surface homeomorphisms [Mar07, Che19, CS22]. We wonder whether this splitting is unique, or if a splitting exists for other values k divisible by $2g-2$ (for example, $k = 0$). We plan to study this in a future paper.

Previous work and proof techniques. Waldhausen [Wal68, §7] proved that the group $\pi_0\big(\mathrm{Homeo}(X_g^k)\big)$ is isomorphic to the outer automorphism group $\mathrm{Out}\,\big(\pi_1(X_g^k)\big)$. From this, the short exact sequence (1) can be derived from work of Conner–Raymond [CR77] and the Dehn–Nielsen–Baer theorem; alternatively, see McCullough [McC91, §3]. The Dehn–Nielsen–Baer theorem also plays a central role in Theorem A, since it allows us to translate back and forth between topology and group theory. There is a mix of both in the proof of Theorem A in §3.

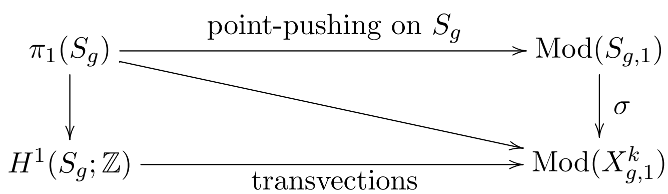
To prove Theorem A, we consider a version of Question 1.1 where X_g^k and S_g are punctured. For the punctured manifolds, similar to (1), there is a short exact sequence

$$1 \to H^1(S_g; \mathbb{Z}) \to \mathrm{Mod}(X_{g,1}^k) \to \mathrm{Mod}(S_{g,1}) \to 1,$$

and we construct a splitting

$$\sigma : \mathrm{Mod}(S_{g,1}) \to \mathrm{Mod}(X_{g,1}^k).$$

See Corollary 3.1. A key part of our proof of Theorem A is to determine the image of the point-pushing subgroup $\pi_1(S_g) < \mathrm{Mod}(S_{g,1})$ under σ. For this we relate three natural surface group representations $\pi_1(S_g) \to \mathrm{Mod}(X_{g,1}^k)$ that appear in the following diagram, where the diagonal map is point pushing on X_g^k (not a commutative diagram).



See Proposition 3.4 for a precise statement.

In order to deduce Corollary 1.3, we use a spectral sequence argument to prove that eu_1 generates a subgroup of $H^2\big(\mathrm{Mod}(S_g); H^1(S_g;\mathbb{Z})\big)$ isomorphic to $\mathbb{Z}/(2g-2)\mathbb{Z}$. A different spectral sequence computation proves that eu_1 generates $H^2\big(\mathrm{Mod}(S_g); H^1(S_g;\mathbb{Z})\big)$ when g is large. These computations use several known computations, including work of Morita [Mor85].

Section outline. In §2 we collect the results we need about the manifolds X_g^k and their mapping class groups, including Waldhausen's work. Theorem A is proved in §3; this section is the core of the paper. In §4, we do two spectral sequence computations to prove Theorem 1.2.

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2. Circle bundles over surfaces

Here we review some results about circle bundles over surfaces that we will need in future sections.

2.1. **Classification.** By an oriented circle bundle we mean a fiber bundle

$$S^1 \to E \to B$$

with structure group $\mathrm{Homeo}^+(S^1)$, the group of orientation preserving homeomorphisms of S^1. The inclusion of the rotation group $\mathrm{SO}(2)$ in $\mathrm{Homeo}^+(S^1)$ is a homotopy equivalence, so circle bundles are in bijection with rank-2 real vector bundles. The classifying space $B\,\mathrm{SO}(2)$ is homotopy equivalent to $\mathbb{C}P^\infty$, which is an Eilenberg–Maclane space $K(\mathbb{Z}, 2)$. Thus each circle bundle is uniquely determined up to isomorphism by its Euler class $eu(E) \in H^2(B;\mathbb{Z})$, which is the primary obstruction to a section of the bundle.

When $B = S_g$ is a closed, oriented surface, $H^2(S_g;\mathbb{Z}) \cong \mathbb{Z}$, we can speak of the Euler *number*. We use X_g^k to denote the total space of the circle bundle

$$S^1 \to X_g^k \to S_g$$

with Euler number k. For example, for the unit tangent bundle $eu(US_g) = 2 - 2g$ (the Euler characteristic), so $US_g \cong X_g^{2-2g}$. We also note that X_g^k and X_g^{-k} are homeomorphic 3-manifolds, since the sign of the Euler number of a circle bundle over S_g depends on the choice of orientation on S_g.

2.2. **Fundamental group $\pi_1(X_g^k)$ and its automorphisms.** From the long exact sequence of a fibration, we have an exact sequence

$$1 \to \mathbb{Z} \to \pi_1(X_g^k) \to \pi_1(S_g) \to 1.$$

The group $\pi_1(X_g^k)$ has a presentation

$$\pi_1(X_g^k) = \big\langle A_1, B_1, \ldots, A_g, B_g, z \mid z \text{ central}, [A_1, B_1] \cdots [A_g, B_g] = z^k \big\rangle. \tag{3}$$

Using this, one finds $\langle z\rangle \cong \mathbb{Z}$ is the center of $\pi_1(X_g^k)$ as long as $(g, k) \neq (1, 0)$. When $g \geq 2$ follows from the fact that the group $\pi_1(S_g)$ has trivial center; the case $g = 1$ can be treated directly.

Given this computation of the center, any automorphism of $\pi_1(X_g^k)$ induces an automorphism of $\langle z\rangle \cong \mathbb{Z}$ and descends to an automorphism of $\pi_1(S_g)$. The latter gives a homomorphism

$$\mathrm{Aut}\big(\pi_1(X_g^k)\big) \to \mathrm{Aut}\big(\pi_1(S_g)\big)$$

that restricts to an isomorphism between the inner automorphism groups

$$\mathrm{Inn}\big(\pi_1(X_g^k)\big) \cong \pi_1(S_g) \cong \mathrm{Inn}\big(\pi_1(S_g)\big) \tag{4}$$

and hence descends to a homomorphism

$$\mathrm{Out}\big(\pi_1(X_g^k)\big) \to \mathrm{Out}\big(\pi_1(S_g)\big). \tag{5}$$

Orientations. It will be convenient to define

$$\mathcal{AUT}\big(\pi_1(S_g)\big) < \mathrm{Aut}\big(\pi_1(S_g)\big)$$

as the subgroup that acts trivially on $H_2(\pi_1(S_g);\mathbb{Z}) \cong \mathbb{Z}$ (the "orientation-preserving" subgroup). We define

$$\mathcal{AUT}\big(\pi_1(X_g^k)\big) < \mathrm{Aut}\big(\pi_1(X_g^k)\big)$$

as the group of automorphisms that project into $\mathcal{AUT}\big(\pi_1(S_g)\big)$ and that act trivially on the center $\langle z\rangle \cong \mathbb{Z}$. In particular, $\mathcal{AUT}\big(\pi_1(X_g^k)\big)$ has index 4 in $\mathrm{Aut}\big(\pi_1(X_g^k)\big)$.

These orientation-preserving subgroups contain the (respective) inner automorphism groups, and we denote the quotients $\mathcal{OUT}\big(\pi_1(X_g^k)\big)$ and $\mathcal{OUT}\big(\pi_1(S_g)\big)$.

2.3. **Mapping class group $\mathrm{Mod}(X_g^k)$.** Fix $g \geq 1$ and $k \in \mathbb{Z}$, and assume $(g,k) \neq (1,0)$. Let $\mathrm{Homeo}^+(X_g^k)$ denote the group of homeomorphisms whose image in $\mathrm{Out}\left(\pi_1(X_g^k)\right)$ is contained in $\mathcal{OUT}\left(\pi_1(S_g)\right)$. Define

$$\mathrm{Mod}(X_g^k) := \pi_0\big(\,\mathrm{Homeo}^+(X_g^k)\big).$$

Waldhausen [Wal68, Cor. 7.5] proved that the natural homomorphism

$$\pi_0\big(\,\mathrm{Homeo}(X_g^k)\big) \to \mathrm{Out}\left(\pi_1(X_g^k)\right)$$

is an isomorphism. Then, by the definitions, this homomorphism restricts to an isomorphism $\mathrm{Mod}(X_g^k) \cong \mathcal{OUT}\left(\pi_1(X_g^k)\right)$. Waldhausen also proved that $\pi_0\,\mathrm{Homeo}(X_g^k)$ is isomorphic to the group of fiber-preserving homeomorphisms modulo homeomorphisms that are isotopic to the identity through fiber-preserving isotopies; see [Wal68, Rmk. following Cor. 7.5]. Consequently, there is a homomorphism

$$\mathrm{Mod}(X_g^k) \to \mathrm{Mod}(S_g). \tag{6}$$

Altogether, we have the following commutative diagram relating the maps (5) and (6).

$$\begin{array}{ccc} \mathrm{Mod}(X_g^k) & \longrightarrow & \mathrm{Mod}(S_g) \\ \cong\big\downarrow & & \big\downarrow\cong \\ \mathcal{OUT}\left(\pi_1(X_g^k)\right) & \longrightarrow & \mathcal{OUT}\left(\pi_1(S_g)\right) \end{array} \tag{7}$$

The right vertical map is an isomorphism by the Dehn–Nielsen–Baer theorem [FM12, Thm. 8.1]. Furthermore, by Conner–Raymond [CR77, Thm. 8] that there is a short exact sequence

$$1 \to \mathrm{Hom}(\pi_1(S_g),\mathbb{Z}) \to \mathcal{OUT}\left(\pi_1(X_g^k)\right) \to \mathcal{OUT}\left(\pi_1(S_g)\right) \to 1. \tag{8}$$

This establishes the short exact sequence (1) in the introduction. We will give a concrete derivation of this exact sequence in Corollary 3.1 below.

3. Relating $\mathrm{Mod}(X_g^k)$ to the Birman exact sequence

In this section, we prove Theorem A. To construct the map of short exact sequences in Theorem A, our main task is to first define a homomorphism $\mathrm{Mod}(S_{g,1}) \to \mathrm{Mod}(X_g^k)$ and then to compute that its kernel is the commutator subgroup of $\pi_1(S_g) < \mathrm{Mod}(S_{g,1})$ (the point-pushing subgroup). We do this is §3.1 and §3.2.

3.1. **A homomorphism $\Psi : \mathrm{Mod}(S_{g,1}) \to \mathrm{Mod}(X_g^k)$.** Fix a basepoint $* \in S_g$, and set $S_{g,1} = S_g \setminus \{*\}$. By the Dehn–Nielsen–Baer theorem, $\mathrm{Mod}(S_{g,1})$ is isomorphic to $\mathrm{Out}^*(F_{2g})$, where F_{2g} is the free group of rank $2g$ and $\mathrm{Out}^*(F_{2g}) < \mathrm{Out}(F_{2g})$ is the subgroup that preserves the conjugacy class corresponding to the free homotopy class of the curve around the puncture in $S_{g,1}$. We construct Ψ as a composition

$$\Psi : \mathrm{Mod}(S_{g,1}) \cong \mathrm{Out}^*(F_{2g}) \xrightarrow{\sigma} \mathcal{AUT}\left(\pi_1(X_g^k)\right) \to \mathcal{OUT}\left(\pi_1(X_g^k)\right) \cong \mathrm{Mod}(X_g^k). \tag{9}$$

To define σ, fix a generating set $\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g$ for F_{2g} such that $c = \prod_{i=1}^g [\alpha_i, \beta_i]$ represents the conjugacy class of the curve around the puncture. Let

$$\iota : F_{2g} \to \pi_1(X_g^k) \tag{10}$$

be the homomorphism defined by $\alpha_i \mapsto A_i$ and $\beta_i \mapsto B_i$. Given $f \in \mathrm{Out}^*(F_{2g})$, fix an automorphism $\tilde{f} : F_{2g} \to F_{2g}$ that represents f, and assume that $\tilde{f}(c) = c$ (this can always be achieved by composing any lift with an inner automorphism of F_{2g}). Next we define $\sigma(f)$ on generators of $\pi_1(X_g^k)$ by

$$\sigma(f)(A_i) = \iota\tilde{f}(\alpha_i), \quad \sigma(f)(B_i) = \iota\tilde{f}(\beta_i), \quad \sigma(f)(z) = z. \tag{11}$$

To show that $\sigma(f)$ extends to a homomorphism of $\pi_1(X_g^k)$, we check that the relation $[A_1, B_1]\cdots[A_g, B_g] = z^k$ is preserved under $\sigma(f)$:

$$\prod_i [\sigma(f)(A_i), \sigma(f)(B_i)] = \prod_i [\iota\tilde{f}(\alpha_i), \iota\tilde{f}(\beta_i)] = \iota(c) = z^k = \sigma(f)(z^k).$$

The second equality uses the the fact that $\tilde{f}(c) = c$. The map $\sigma(f)$ is independent of the choice of $\tilde{f}$ because different choices of $\tilde{f}$ differ by conjugation by powers of c (because the centralizer of c in F_{2g} is the cyclic subgroup $\langle c\rangle$)[1] and $\iota(c) = z^k$ is central in $\pi_1(X_g^k)$. The homomorphism $\sigma(f) : \pi_1(X_g^k) \to \pi_1(X_g^k)$ is an automorphism and belongs to $\mathcal{AUT}\left(\pi_1(X_g^k)\right)$ by definition. Furthermore, $f \mapsto \sigma(f)$ is a homomorphism, which is easy to check using the observation that if $w = \iota w'$, then $\sigma(f)(w) = \iota\tilde{f}(w')$.

Composing σ with $\mathcal{AUT} \to \mathcal{OUT}$ gives the desired homomorphism Ψ. As a corollary of this construction, we have proved the following.

Corollary 3.1. *Fix $g \geq 1$ and $k \in \mathbb{Z}$, and assume $(g, k) \neq (1, 0)$. The natural map $\Phi : \mathcal{AUT}\left(\pi_1(X_g^k)\right) \to \mathcal{AUT}\left(\pi_1(S_g)\right)$ (see §2.2) fits into an exact sequence*

$$1 \to \mathrm{Hom}(\pi_1(S_g), \mathbb{Z}) \to \mathcal{AUT}\left(\pi_1(X_g^k)\right) \xrightarrow{\Phi} \mathcal{AUT}\left(\pi_1(S_g)\right) \to 1, \tag{12}$$

and this exact sequence splits.

Proof. First we compute the kernel of Φ. Using the presentation for $\pi_1(X_g^k)$ in (3), if $f \in \ker(\Phi)$, then

$$f(A_i) = A_i z^{m_i} \quad \text{and} \quad f(B_i) = B_i z^{n_i}$$

for some $m_1, n_1, \ldots, m_g, n_g \in \mathbb{Z}$. The map $a_i \mapsto m_i$, $b_i \mapsto n_i$ extends to a homomorphism $\tau(f) : \pi_1(S_g) \to \mathbb{Z}$. It is elementary to check that the map $\ker(\Phi) \to \mathrm{Hom}(\pi_1(S_g), \mathbb{Z})$ defined by $f \mapsto \tau(f)$ is an isomorphism.

The homomorphism σ defined above shows that Φ is a split surjection. Note that the $\mathrm{Mod}(S_g, *) \cong \mathrm{Mod}(S_g \setminus \{*\})$ (basepoint vs. puncture), so by Dehn–Nielsen–Baer there is an isomorphism $\mathcal{AUT}\left(\pi_1(S_g)\right) \cong \mathrm{Out}^*(F_{2g})$, and we use this isomorphism to view σ as a splitting of Φ. □

Remark 3.2. We call elements of $\ker(\Phi) \cong H^1(S_g; \mathbb{Z})$ *transvections.*

Remark 3.3. The homomorphism Ψ can be constructed on the level of topology as follows. Fix a regular neighborhood D of the puncture on $S_{g,1}$ (so D is a once-punctured disk). Given a mapping class $f \in \mathrm{Mod}(S_{g,1})$, choose a representing homeomorphism $\mathfrak{f}$. Without loss of generality, we can assume that $\mathfrak{f}$ is the identity on D. The bundle $X_g^k \to S_g$ can be trivialized over $S \setminus D$ (because the classifying space $B\,\mathrm{SO}(2)$ is simply connected). Fixing a trivialization $(S \setminus D) \times S^1$ over $S \setminus D$, we lift $\mathfrak{f}$ to the product homeomorphism $\mathfrak{f} \times \mathrm{id}_{S^1}$. This homeomorphisms is the identity on the boundary $\partial(S \setminus D) \times S^1$, so we can extend by the identity to obtain a homeomorphism $\tilde{\mathfrak{f}}$ of X_g^k. The map sending $f \in \mathrm{Mod}(S_{g,1})$ to

[1]Note that the centralizer is isomorphic to $\mathbb{Z}$ and contains $\langle c\rangle$. It is only bigger if $c = u^i$ for some $u \in F_{2g}$ and $i \geq 2$. By contradiction, if $c = u^i$ for $i \geq 2$, then u is cyclically reduced because c is. This implies that u is a subword of $c = \prod[\alpha_i, \beta_i]$, which is absurd.

the isotopy class $\left[\tilde{\mathfrak{f}}\right] \in \mathrm{Mod}(X_g^k)$ is the topological version of the homomorphism Ψ. Note that the isotopy class $[\mathfrak{f}]$ is only well-defined up to Dehn twists about ∂D which is a loop around the puncture. This is analogous to the ambiguity encountered in the definition of σ, which ultimately does not affect the definition of Ψ.

Corollary 3.1 and equation (4) combine to give the short exact sequence of outer automorphism groups (8).

Warning. The splitting of the short exact sequence (12) does *not* give a splitting of the short exact sequence (8). Indeed we will show the latter sequence does *not* always split (Corollary 1.3). The subtlety comes from the fact that the inner automorphism group $\mathrm{Inn}\left(\pi_1(X_g^k)\right) \cong \pi_1(S_g)$ does not coincide with the image of $\pi_1(S_g) \cong \mathrm{Inn}\left(\pi_1(S_g)\right) < \mathrm{Aut}\left(\pi_1(S_g)\right)$ under the section σ. Proposition 3.4 below describes the precise relationship.

3.2. **Kernel of $\boldsymbol{\Psi : \mathrm{Mod}(S_{g,1}) \to \mathrm{Mod}(X_g^k)}$.** Observe that the kernel of Ψ is contained in the point-pushing subgroup $\pi_1(S_g) < \mathrm{Mod}(S_{g,1})$. This is because Ψ composed with the natural map $\mathrm{Mod}(X_g^k) \to \mathrm{Mod}(S_g)$ is the natural map $\mathrm{Mod}(S_{g,1}) \to \mathrm{Mod}(S_g)$, whose kernel is the point-pushing subgroup. Thus we want to understand the image of the point-pushing subgroup under the section σ used to define Ψ. What we find is a simple relationship between three surface group representations:

$$\begin{array}{ccccc} & & \pi_1(S_g) & & \\ & & \big\downarrow {\scriptstyle \sigma} \; \textit{inner auts of } \pi_1(S_g), \textit{ lifted} & & \\ \pi_1(S_g) & \xrightarrow[\textit{inner auts of } \pi_1(X_g^k)]{} & \mathcal{AUT}\left(\pi_1(X_g^k)\right) & \xleftarrow[\textit{transvections}]{} & \pi_1(S_g) \end{array}$$

The main results are Proposition 3.4 and Corollary 3.5 below. In order to state Proposition 3.4, we need the following notation. Let

$$\delta : H_1(S_g;\mathbb{Z}) \to H^1(S_g;\mathbb{Z})$$

be the Poincaré duality map, given explicitly by $\gamma \mapsto \langle -, \gamma \rangle$, where

$$\langle -, - \rangle : H_1(S_g;\mathbb{Z}) \times H_1(S_g;\mathbb{Z}) \to \mathbb{Z}$$

is the algebraic intersection form. We use $\hat{\delta}$ denote the composition

$$\hat{\delta} : H_1(S_g;\mathbb{Z}) \xrightarrow{\delta} H^1(S_g;\mathbb{Z}) \hookrightarrow \mathcal{AUT}\left(\pi_1(X_g^k)\right).$$

This map is given explicitly by $\hat{\delta}(\gamma)(w) = w \cdot z^{\langle [\bar{w}], \gamma \rangle}$, where $\bar{w}$ is the image of w under $\pi_1(X_g^k) \to \pi_1(S_g)$ and $[\bar{w}] \in H_1(S_g;\mathbb{Z})$ is the corresponding homology class.

Fix a basepoint $\star \in S_{g,1}$. Recall that we have fixed a standard generating set $\{\alpha_i, \beta_i\}$ of $\pi_1(S_{g,1}, \star) \cong F_{2g}$ so that $c := \prod_i [\alpha_i, \beta_i]$ is a loop around the puncture $*$ of $S_{g,1} = S_g \setminus \{*\}$. Define

$$\Pi : \pi_1(S_{g,1}, \star) \to \pi_1(S_g, *) \tag{13}$$

by $\gamma \mapsto \epsilon.\gamma.\bar{\epsilon}$, where ϵ is a fixed arc from $*$ to $\star$.

Proposition 3.4. *Fix $t \in \pi_1(S_g, *)$, and let $\operatorname{Push}(t) \in \operatorname{Mod}(S_{g,1}) \cong \operatorname{Out}^*(F_{2g})$ be the point-pushing mapping class. If $\tilde{t} \in \pi_1(S_{g,1}, \star)$ is any lift of t (i.e. $\Pi(\tilde{t}) = t$), then*

$$\sigma\big(\operatorname{Push}(t)\big) = C_{\iota\tilde{t}} \circ \hat{\delta}([kt]), \tag{14}$$

Here C_x denotes conjugation by x, and the maps $\iota : F_{2g} \to \pi_1(X_g^k)$ and $\sigma : \operatorname{Out}^*(F_{2g}) \to \mathcal{AUT}\left(\pi_1(X_g^k)\right)$ are defined in (10) and (11).

As a sanity check, observe that $C_{\iota\tilde{t}}$ does not depend on the choice of lift $\tilde{t}$ because any two lifts differ by an element of the normal closure of c in $\pi_1(S_{g,1}, \star) = F_{2g}$, and conjugation by any such element is trivial on $\pi_1(X_g^k)$.

Proof of Proposition 3.4. It suffices to prove the lemma for $t \in \pi_1(S_g, *)$ that are represented by a non-separating simple closed curve. To see this, first note that $\pi_1(S_g, *)$ is generated by these curves. Furthermore, the groups $\operatorname{Inn}\left(\pi_1(X_g^k)\right)$ and $H^1(S_g; \mathbb{Z})$ commute in $\operatorname{Aut}\left(\pi_1(X_g^k)\right)$, so

$$\left[C_{\iota\tilde{t}_1} \circ \hat{\delta}([t_1])\right] \circ \left[C_{\iota\tilde{t}_2} \circ \hat{\delta}([t_2])\right] = C_{\iota(\tilde{t}_1 * \tilde{t}_2)} \circ \hat{\delta}([t_1 * t_2]).$$

Assume now that $t \in \pi_1(S_g, *)$ is represented by a non-separating simple closed curve. After an isotopy, we can assume that t contains ϵ as a sub-arc. Choose $\tilde{t}$ as pictured in Figure 1.

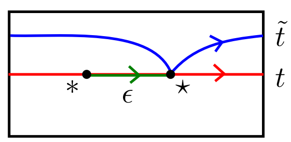

Figure 1. A small regular neighborhood of a loop representing $t \in \pi_1(S_g, *)$ and a lift $\tilde{t} \in \pi_1(S_{g,1}, \star)$.

We want to show that

$$\sigma\big(\operatorname{Push}(t)\big)(w) = \left[C_{\iota\tilde{t}} \circ \hat{\delta}([t])\right](w)$$

for each $w \in \pi_1(X_g^k)$. Since this is obviously true for $w = z$, it suffices to show this equality for $w = \iota(s)$ for $s \in \pi_1(S_{g,1}, \star)$; furthermore, it suffices to show the equality on any generating set of $\pi_1(S_{g,1}, \star)$. We use the (infinite) generating set consisting of curves of one of the forms pictured in Figure 2 (the intersection of these curves with the annulus around t has one component).

Note that $\operatorname{Push}(t)$ fixes the basepoint $\star$, so we can compute the action of $\operatorname{Push}(t)$ on $s \in \pi_1(S_{g,1}, \star)$. We compute the action of $\operatorname{Push}(t)$ on the elements in Figure 2 as follows. See Figure 3 for an illustration.

$$s_1 \mapsto (\tilde{t})^{-1} s_1 \tilde{t} c^{-1} \quad \text{and} \quad s_2 \mapsto (\tilde{t})^{-1} s_2 \tilde{t} \quad \text{and} \quad s_3 \mapsto c(\tilde{t})^{-1} s_3 \tilde{t} c^{-1} \quad \text{and} \quad c \mapsto c.$$

This proves that, for example, that

$$\sigma\big(\operatorname{Push}(t)\big)(\iota s_1) = (\iota\tilde{t})^{-1}(\iota s_1)(\iota\tilde{t}) z^{-k} = \left[C_{\iota\tilde{t}} \circ \hat{\delta}([kt])\right](\iota s_1).$$

We conclude similarly for the generators s_2, s_3. This proves the desired formula for $\sigma\big(\operatorname{Push}(t)\big)$. ☐

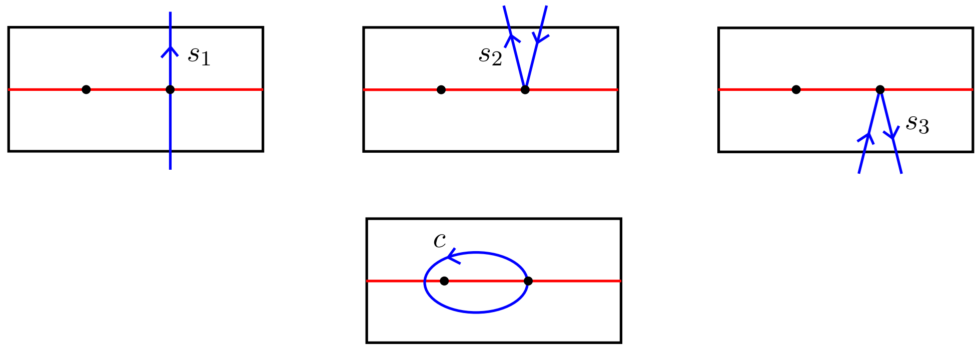

FIGURE 2. The group $\pi_1(S_{g,1},\star)$ is generated by $\tilde{t}$ and loops of the form pictured above.

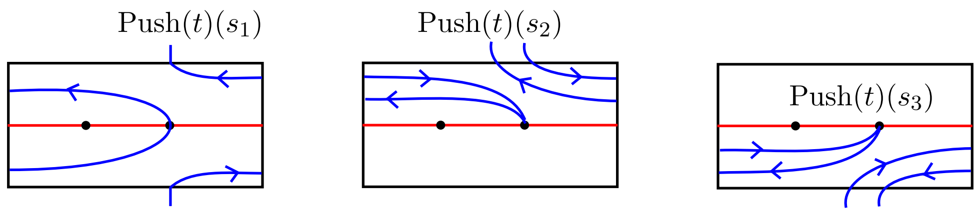

FIGURE 3. Action of point-pushing about t on the loops in Figure 2. The curve c is fixed up to isotopy (up to isotopy Push(t) is the identity on a neighborhood of t that contains c).

The following corollary is an immediate consequence of Proposition 3.4.

Corollary 3.5. *Consider the composition*

$$\Psi : \mathcal{AUT}\left(\pi_1(S_g)\right) \xrightarrow{\sigma} \mathcal{AUT}\left(\pi_1(X_g^k)\right) \to \mathcal{OUT}\left(\pi_1(X_g^k)\right). \tag{15}$$

The restriction of Ψ *to* $\pi_1(S_g) \cong \operatorname{Inn}\left(\pi_1(S_g)\right)$ *factors as follows.*

$$\begin{array}{ccc} \pi_1(S_g) & \xrightarrow{\text{conjugation}} & \mathcal{AUT}\left(\pi_1(S_g)\right) \\ {\scriptstyle k\delta\circ \mathrm{ab}}\downarrow & & \downarrow{\scriptstyle \Psi} \\ H^1(S_g;\mathbb{Z}) & \xrightarrow[\text{transvections}]{} & \mathcal{OUT}\left(\pi_1(X_g^k)\right) \end{array}$$

Here ab *denotes the abelianization map* $\pi_1(S_g,*) \to H_1(S_g;\mathbb{Z})$.

3.3. **Proof of Theorem A.** Using the isomorphisms between mapping class groups and automorphism groups, the desired diagram is equivalent to the following one.

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi_1(S_g)^{\mathrm{ab}} & \longrightarrow & \mathcal{AUT}\left(\pi_1(S_g)\right)/\pi' & \longrightarrow & \mathcal{OUT}\left(\pi_1(S_g)\right) & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle k\delta} & & \downarrow & & \Vert & & \\ 1 & \longrightarrow & \operatorname{Hom}(\pi_1(S_g),\mathbb{Z}) & \longrightarrow & \mathcal{OUT}\left(\pi_1(X_g^k)\right) & \longrightarrow & \mathcal{OUT}\left(\pi_1(S_g)\right) & \longrightarrow & 1 \end{array}$$

The map Ψ in (15) descends to the middle vertical map and restricts to the left vertical map by Corollary 3.5. The fact that σ is a section (Corollary 3.1) implies that the middle vertical map descends to the identity map on $\mathcal{OUT}\left(\pi_1(S_g)\right)$. When $k = 1$, the middle vertical map is an isomorphism by the five lemma. This concludes the proof of Theorem A.

4. Spectral sequence computation

In this section we prove Theorem 1.2. This is achieved by two different computations using the Lyndon–Hochschild–Serre (LHS) spectral sequence. Recall that this spectral sequence takes input a short exact sequence of groups $1 \to N \to G \to Q \to 1$ and a G-module A, has E_2 page

$$E_2^{p,q} = H^p\big(Q; H^q(N; A)\big),$$

and converges to $H^{p+q}(G; A)$. For both computations we use the Birman exact sequence, but with different choices of the module A.

Notational note. To simplify the notation, we use the convention that cohomology groups have $\mathbb{Z}$ coefficients unless otherwise specified.

4.1. Euler class computation.

Our goal in this section is to prove Proposition 4.1 below, which implies Corollary 1.3.

Proposition 4.1. *Fix $g \geq 1$. Let eu_k be the Euler class of the extension* (1)*. Then $eu_k = k\, eu_1$, and eu_1 has order $2g-2$ in $H^2\big(\mathrm{Mod}(S_g); H^1(S_g)\big)$.*

Proof. The relation $eu_k = k\, eu_1$ already follows from Theorem A. Indeed, choosing a set-theoretic section for the sequence in the top row of the diagram in Theorem A gives a cocycle representative for eu_k that is k times the cocycle representative for e_1.

Now we prove that eu_1 generates a cyclic subgroup isomorphic to $\mathbb{Z}/(2g-2)\mathbb{Z}$ in $H^2\big(\mathrm{Mod}(S_g); H^1(S_g)\big)$. Our method is to apply the LHS spectral sequence to the Birman exact sequence with the module $A = H^1(S_g)$. Here

$$E_2^{p,q} \cong H^p\big(\mathrm{Mod}(S_g); H^q(S_g; A)\big).$$

A portion of the associated 5-term exact sequence is as follows.

$$0 \to H^1\big(\mathrm{Mod}(S_g); H^1(S_g)\big) \to H^1\big(\mathrm{Mod}(S_{g,1}); H^1(S_g)\big) \xrightarrow{A} \mathrm{Hom}\left(H_1(S_g), H^1(S_g)\right)^{\mathrm{Mod}(S_g)} \xrightarrow{d_2^{0,1}} H^2\big(\mathrm{Mod}(S_g); H^1(S_g)\big)$$

This sequence has been studied by Morita. Morita [Mor85, Prop. 4.1] computes that the first term vanishes, so the map A is injective. The group $\mathrm{Hom}\left(H_1(S_g), H^1(S_g)\right)^{\mathrm{Mod}(S_g)}$ is isomorphic to $\mathbb{Z}$ and generated the Poincaré duality isomorphism δ. Morita [Mor85, proof of Prop. 6.4] shows that the image of A is $(2g-2)\mathbb{Z}$. Consequently, the differential $d_2^{0,1}$ descends to an injection $\mathbb{Z}/(2g-2)\mathbb{Z} \to H^2\big(\mathrm{Mod}(S_g); H^1(S_g)\big)$.

It remains to show that $d_2^{0,1}$ sends a generator to eu_1. The differential $d_2^{0,1}$ is the transgression; see e.g. [NSW08, Prop. 1.6.6, Thm. 2.4.3]. By standard knowledge of the transgression applied to our situation, we find that $d_2^{0,1}$ sends a generator to $\delta_*(eu)$, where eu is the Euler class of the extension (2), and

$$\delta_* : H^2\big(\mathrm{Mod}(S_g); H_1(S_g)\big) \to H^2\big(\mathrm{Mod}(S_g); H^1(S_g)\big)$$

is the isomorphism induced by the Poincaré duality isomorphism δ. (For this property of the transgression, see [NSW08, §I.6, Exercise 1-2]. While that reference is mainly concerned with finite or profinite groups, the analysis of the transgression contained given there applies more generally.) Finally, we observe that $\delta_*(eu) = eu_1$ by Theorem A. □

4.2. Computation of $H^2\big(\operatorname{Mod}(S_g); H^1(S_g)\big)$. Running the LHS spectral sequence with the trivial module $A = \mathbb{Z}$, we prove that if $g \geq 8$, then

$$H^2\big(\operatorname{Mod}(S_g); H^1(S_g)\big) \cong \mathbb{Z}/(2g-2)\mathbb{Z}. \tag{16}$$

Combining this with Proposition 4.1 proves Theorem 1.2. The relevant portion of the spectral sequence appears below.

	0	1	2	3	4
2	$H^0(\operatorname{Mod}(S_g); H^2(S_g))$	$d_2^{0,2}$ ↘			
1	0	0	$H^2(\operatorname{Mod}(S_g); H^1(S_g))$	$d_2^{2,1}$ ↘	
0	$\mathbb{Z}$	0	$H^2(\operatorname{Mod}(S_g))$	$H^3(\operatorname{Mod}(S_g))$	$H^4(\operatorname{Mod}(S_g))$

The computations in the first column are easy. In the second column, Morita [Mor85, Prop. 4.1] computed $H^1\big(\operatorname{Mod}(S_g); H^1(S_g)\big) = 0$ for $g \geq 1$. The other computation $H^1\big(\operatorname{Mod}(S_g)\big) = 0$ holds for $g \geq 1$ because the abelianization of $\operatorname{Mod}(S_g)$ is finite [FM12, §5.1.2-3].

According to [BT01, Cor. 1.2],

$$H_*\big(\operatorname{Mod}(S_{g,1})\big) \cong H_*\big(\operatorname{Mod}(S_g)\big) \otimes \mathbb{Z}[x]$$

in degrees $g \geq 2*$. Here x has degree 2. Applying this and using the universal coefficients theorem, we conclude that

$$H^i\big(\operatorname{Mod}(S_g)\big) \to H^i\big(\operatorname{Mod}(S_{g,1})\big)$$

is an isomorphism if $i = 3$ and $g \geq 6$, and it is injective if $i = 4$ and if $g \geq 8$.

Since the map $H^4\big(\operatorname{Mod}(S_g)\big) \to H^4\big(\operatorname{Mod}(S_{g,1})\big)$ is injective, the differential $d_2^{2,1}$ is zero. Since the map $H^3\big(\operatorname{Mod}(S_g)\big) \to H^3\big(\operatorname{Mod}(S_{g,1})\big)$ is an isomorphism, the differential $d_2^{0,2}$ is surjective.

Thus, the filtration of $H^2\big(\operatorname{Mod}(S_{g,1})\big)$ coming from the E_∞ page gives an exact sequence

$$\begin{aligned} 0 \to H^2\big(\operatorname{Mod}(S_g)\big) \to H^2\big(\operatorname{Mod}(S_{g,1})\big) &\xrightarrow{F} H^0\big(\operatorname{Mod}(S_g); H^2(S_g)\big) \cong \mathbb{Z} \\ &\xrightarrow{d_2^{0,2}} H^2\big(\operatorname{Mod}(S_g); H^1(S_g)\big) \to 0. \end{aligned}$$

For $g \geq 4$,

$$H^2\big(\operatorname{Mod}(S_g)\big) \cong \mathbb{Z}[e_1] \quad \text{and} \quad H^2\big(\operatorname{Mod}(S_{g,1})\big) \cong \mathbb{Z}[e, e_1]$$

and the map $\mathbb{Z}[e_1] \to \mathbb{Z}[e, e_1]$ is the obvious one $e_1 \mapsto e_1$. We claim that $F(e) = 2 - 2g$. From this we deduce the desired isomorphism (16). The claim follows from the fact that the extension that defines e, when restricted to the point-pushing subgroup $\pi_1(S_g) < \operatorname{Mod}(S_{g,1})$, gives the extension

$$1 \to \mathbb{Z} \to \pi_1(US_g) \to \pi_1(S_g) \to 1$$

where US_g is the unit tangent bundle. See [FM12, §5.5.5]. This extension has Euler class $2 - 2g$, so the claim follows.

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