ON THE NON-REALIZABILITY OF BRAID GROUPS BY DIFFEOMORPHISMS

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Abstract. For every compact surface S of finite type (possibly with boundary components but without punctures), we show that when n is sufficiently large there is no lift σ of the surface braid group $B_n(S)$ to $\mathrm{Diff}(S, n)$, the group of diffeomorphisms preserving n marked points and restricting to the identity on the boundary. Our methods are applied to give a new proof of Morita's non-lifting theorem in the best possible range. These techniques extend to the more general setting of spaces of codimension-2 embeddings, and we obtain corresponding results for spherical motion groups, including the string motion group.

1. Introduction

Let N^k and M^{k+2} be smooth manifolds. For any $n \geq 1$ the symmetric group S_n acts on the space $\mathrm{Emb}_n(N, M)$ of C^1 embeddings $\coprod_n N \to M$ by permuting the components of $\coprod_n N$. The quotient $\mathrm{Conf}_n(N, M) = \mathrm{Emb}_n(N, M)/S_n$ is the *configuration space*. The most familiar setting is for $k = 0$, so that $M = S$ is a surface and $N = \{*\}$ is a point. In this case $\mathrm{Conf}_n(\{*\}, S) = \mathrm{Conf}_n(S)$ is the configuration space of n-tuples of distinct, unordered points on S, and $\pi_1\big(\mathrm{Conf}_n(M)\big) =: B_n(S)$ is a *surface braid group*.

The group of C^1 diffeomorphisms[1] $\mathrm{Diff}(M)$ acts on $\mathrm{Conf}_n(N, M)$ with the stabilizer of $[\phi]$ denoted $\mathrm{Diff}(M, [\phi])$. Associated to this action is a homomorphism

$$\mathcal{P} : \pi_1\big(\mathrm{Conf}_n(N, M)\big) \to \pi_0\big(\mathrm{Diff}(M, [\phi])\big)$$

generalizing the *point-pushing map* $\mathcal{P} : B_n(S) \to \mathrm{Mod}(S, n)$ in the surface braid group setting. See Theorem 2.2 and Proposition 5.2 for detailed constructions. This note focuses on the *non-realizability* of $\mathcal{P}$ by C^1 diffeomorphisms. We say that $\mathcal{P}$ is *realized by (C^1) diffeomorphisms* if there exists a homomorphism $\sigma : \pi_1\big(\mathrm{Conf}_n(N, M)\big) \to \mathrm{Diff}(M, [\phi])$ such that the composition

$$\pi_1\big(\mathrm{Conf}_n(N, M)\big) \xrightarrow{\sigma} \mathrm{Diff}(M, [\phi]) \to \pi_0\big(\mathrm{Diff}(M, [\phi])\big)$$

is equal to $\mathcal{P}$. Such a σ, if it exists, is called a *lift* of $\mathcal{P}$.

Bestvina–Church–Souto [BCS13] show by a cohomological argument that $B_n(S)$ is not realized by diffeomorphisms when S is closed, $\mathrm{genus}(S) \geq 2$, and $n \geq 1$ $\big($note that $B_1(S) \cong \pi_1(S)\big)$. It

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[1]All diffeomorphisms considered in this paper will be orientation-preserving. Also, all diffeomorphisms are C^1 unless otherwise noted.

does not seem that their methods extend to surfaces with boundary or to surfaces of low genus. In particular, this leaves the case of the classical braid group $B_n = B_n(\mathbb{D}^2)$ unresolved.

Morita's non-lifting theorem [Mor87] shows that there is no lift of $\mathrm{Mod}(\Sigma_g) = \pi_0\big(\mathrm{Diff}(\Sigma_g)\big)$ to the group of C^2 diffeomorphisms $\mathrm{Diff}^2(\Sigma_g) \subset \mathrm{Diff}(\Sigma_g)$ by showing that $H^*(\mathrm{Mod}(\Sigma_g)) \to H^*\big(\mathrm{Diff}^2(\Sigma_g)\big)$ fails to be injective for g sufficiently large. It is tempting to try and follow this strategy for B_n, exploiting the fact that $B_n = \pi_0\big(\mathrm{Diff}(\mathbb{D}^2, n)\big)$. However, there is evidence that this approach will not work, as Nariman [Nar15] has shown that $H^*(B_n;\mathbb{Z})$ is a direct summand of $H^*(\mathrm{Diff}(\mathbb{D}^2 \setminus X_n);\mathbb{Z})$, where $X_n \subset \mathbb{D}^2$ is a set of n distinct points and $\mathrm{Diff}(\mathbb{D}^2 \setminus X_n)$ is the group of compactly supported diffeomorphisms of $\mathbb{D}^2 \setminus X_n$. We are able to sidestep these difficulties by using more geometric methods.

Theorem 1.1. *Let S be a compact surface. If $\partial S = \varnothing$ then $\mathcal{P} : B_n(S) \to \mathrm{Mod}(S, n)$ is not realized by C^1 diffeomorphisms for all $n \geq 6$. In the case $\partial S \neq \varnothing$, this can be improved to all $n \geq 5$.*

In Section 4, we use Theorem 1.1 to give a new proof of Morita's non-lifting theorem.

Theorem 1.2. *Let Σ_g be a closed surface of genus g. For $g \geq 2$, there is no homomorphism $\mathrm{Mod}(\Sigma_g) \to \mathrm{Diff}(\Sigma_g)$ which splits the natural projection $\mathrm{Diff}(\Sigma_g) \to \mathrm{Mod}(\Sigma_g)$.*

Morita's original argument [Mor87] showed there is no splitting $\mathrm{Mod}(\Sigma_g) \to \mathrm{Diff}^2(\Sigma_g)$ for $g \geq 18$. This was improved by Franks–Handel [FH09], who obtained the nonlifting theorem for C^1 diffeomorphisms and $g \geq 3$; see also Bestvina–Church–Souto [BCS13, Theorem 1.2]. Theorem 1.2 provides a further improvement, giving the best possible genus bound while avoiding the dynamical machinery lurking in the proof of Franks–Handel.

Remark 1.3. Much less is understood about realizing $B_n(S)$ by homeomorphisms. Thurston showed that B_3 *is* realized by homeomorphisms [Thu11]. In contrast, $B_6(S^2)$ is not realized by homeomorphisms (for otherwise, one could lift this realization to the branched cover $\Sigma_2 \to S^2$ to obtain a realization of $\mathrm{Mod}(\Sigma_2)$ by homeomorphisms, and this is impossible by work of Markovic–Šarić [MŠ08], building on the ideas of Markovic [Mar07]).

Along with surface braid groups, we will also be concerned with the space $\mathrm{Conf}_n(S^k, M)$ of configurations of unlinked, codimension-2 spheres in $M \in \{\mathbb{R}^{k+2}, S^{k+2}\}$ for $k \geq 1$. The fundamental group $B_n(S^k, M) = \pi_1\big(\mathrm{Conf}_n(S^k, M)\big)$ is called the *spherical motion group*. In the case $k = 1$, this group is closely related to the *ring group* studied by Brendle and Hatcher in [BH13] (see Section 7). The main result is as follows.

Theorem 1.4. *Let M be S^{k+2} or $\mathbb{R}^{k+2}$. Fix an unlinked embedding $\phi : \coprod_n S^k \hookrightarrow M$, and let $[\phi] \in \mathrm{Conf}_n(S^k, M)$ denote the corresponding configuration. Let $\mathcal{D}(M, [\phi]) \leq \mathrm{Diff}(M)$ be the group of compactly-supported C^1 diffeomorphisms isotopic the identity and such that $[f \circ \phi] = [\phi]$. If either*

(a) $M = \mathbb{R}^{k+2}$ and $n \geq 5$, or

(b) $M = S^{k+2}$ *and* $n \geq 6$,

then the "spherical push map" $\mathcal{P} : B_n(S^k, M) \to \pi_0\big(\mathcal{D}(M, \phi)\big)$ *is not realized by diffeomorphisms.*

Remark 1.5. The arguments of Theorems 1.1 and 1.4 can be extended to certain finite-index subgroups, but do not work, e.g. for the pure braid group $PB_n \leq B_n(\mathbb{D}^2)$. It is also not clear whether the bounds in Theorem 1.1 or Theorem 1.4 can be improved, although the methods of the current paper do not extend beyond the stated ranges. See Remark 1.3 for some related discussions.

In Theorem 1.4, the diffeomorphism groups under consideration are required to fix the image of ϕ pointwise up to permutation. In Section 7, we use work of Parwani [Par08] to give an extension of Theorem 1.4 that deals with the possibility of a lift of $\mathcal{P}$ that only fixes the image of ϕ *setwise*, in the case $k = 1$. We also treat a generalization of Theorem 1.1, where the marked points on S are replaced by boundary components.

The proof of Theorems 1.1 and 1.4 involves two main ingredients. The first is the Thurston stability theorem [Thu74], which can be used to impose restrictions on the homology of finitely-generated subgroups of diffeomorphisms. The second is the fact that B_n interacts poorly with these restrictions. The main theorems are proved by exhibiting suitable subgroups closely related to B_n in each of the braid or motion groups under consideration.

The paper is organized as follows. In Section 2, we briefly review Birman's theory of push maps for surface braid groups. In Sections 3 and 4 we prove Theorems 1.1 and 1.2, respectively. In Section 5 we develop a notion of push maps for spherical motion groups. In Section 6 we prove Theorem 1.4. Finally in Section 7, we prove some strengthenings of Theorems 1.1 and 1.4 in low dimensions.

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2. From configuration spaces to mapping class groups

In this section, we review how surface braid groups give rise to subgroups of mapping class groups via *push maps.* Let S be a surface. The *pure configuration space of* n *points in* S is defined as

$$\mathrm{PConf}_n(S) = \{(x_1, \ldots, x_n) \mid x_i \in \mathrm{int}(S) \text{ and } x_i \neq x_j \text{ if } i \neq j\}.$$

The *configuration space* is defined as the quotient $\mathrm{Conf}_n(S) = \mathrm{PConf}_n(S)/S_n$ by the (free) action of the symmetric group on n letters via permutation of coordinates.

Definition 2.1. The *braid group on n strands in S*, written $B_n(S)$, is defined to be $\pi_1\big(\operatorname{Conf}_n(S)\big)$. In the case $S = \mathbb{D}^2$, we write $B_n = B_n(\mathbb{D}^2)$.

The following is due to J. Birman. See [FM12, Section 9.1.4].

Theorem 2.2 (Birman). *Let S be a compact surface with possibly nonempty boundary. Let $X_n = \{x_1, \ldots, x_n\}$ be a set of n distinct points in S. There is a homomorphism*

$$\mathcal{P} : B_n(S) \to \pi_0\big(\operatorname{Diff}(S, \partial S, X_n)\big);$$

here $\operatorname{Diff}(S, \partial S, X_n)$ is the group of C^1 diffeomorphisms of S restricting to the identity on ∂S that preserve X_n setwise. The kernel of $\mathcal{P}$ is isomorphic to a quotient of $\pi_1\big(\operatorname{Diff}(S, \partial S)\big)$.

Remark 2.3. The condition $\pi_1\big(\operatorname{Diff}(S, \partial S)\big) = 1$ is satisfied whenever $\chi(S) < 0$, and also when $S = \mathbb{D}^2$ (see [EE69] and [ES70]). In the exceptional cases, $\pi_1\big(\operatorname{Diff}(S^2)\big) \cong \mathbb{Z}/2$, and $\pi_1\big(\operatorname{Diff}(T^2)\big) \cong \mathbb{Z}^2$. It follows that for all $n \geq 5$ (the cases under consideration in this paper), the map $\mathcal{P}$ is nonzero.

3. Proof of Theorem 1.1

The situation can be expressed diagrammatically as follows:

$$\begin{tikzcd} & \operatorname{Diff}(S, \partial S, X_n) \arrow[d, "\pi"] \\ B_n(S) \arrow[ur, dashed, "\sigma"] \arrow[r, "\mathcal{P}"'] & \pi_0\big(\operatorname{Diff}(S, \partial S, X_n)\big). \end{tikzcd}$$

We seek to obstruct the existence of a homomorphism σ lifting $\mathcal{P}$. Our method will be to reduce to the *Thurston stability theorem.*

Step 1: Local indicability and the Thurston stability theorem. The aim of this section is to show that certain diffeomorphism groups do not contain braid subgroups. We will be concerned with a property of groups known as *local indicability.*

Definition 3.1. A group G is said to be *locally indicable* if every nontrivial finitely-generated subgroup $\Gamma \leq G$ admits a surjection $\Gamma \to \mathbb{Z}$. Equivalently, G is locally indicable if every finitely-generated subgroup Γ has $H^1(\Gamma, \mathbb{R}) \neq 0$.

A group G is said to be *strongly non-indicable* if there exists a nontrivial finitely-generated subgroup Γ that is perfect, i.e. with $[\Gamma, \Gamma] = \Gamma$.

Remark 3.2. Suppose G is not locally indicable, and let $H \leq G$ be a subgroup witnessing this fact. If $N \lhd G$ is a normal subgroup with $H \cap N \neq H$, then HN/N witnesses the non-indicability of G/N. The same is true for strong non-indicability.

In [Thu74], Thurston showed that certain diffeomorphism groups are locally indicable.

Theorem 3.3 (Thurston stability theorem)**.** *Let M be a manifold, and let $x \in M$ be given. For a diffeomorphism g of M fixing x, we write $(Dg)_x \in GL(T_xM)$ for the derivative. Then the group*

$$\mathcal{G} = \{g \in \mathrm{Diff}(M) \mid g(x) = x,\ (Dg)_x = I\}$$

is locally indicable (and hence any subgroup of $\mathcal{G}$ is locally indicable as well).

The strategy for the remainder of the proof is to argue that a lift σ of $\mathcal{P}$ would force $\mathcal{G}$ to contain a non-locally-indicable subgroup. We will show that B_n is a suitable group.

Step 2: Braid groups are strongly non-indicable.

Proposition 3.4.

(i) For $n \geq 5$, the set

$$S = \{\sigma_i \sigma_{i+1}^{-1} \mid 1 \leq i \leq n-2\}$$

generates $[B_n, B_n]$. Moreover, the elements of S are all mutually conjugate within $[B_n, B_n]$.

(ii) (Gorin–Lin) *For $n \geq 5$, the commutator subgroup of the braid group B_n is* perfect*, i.e. $[B_n, B_n] = [[B_n, B_n], [B_n, B_n]]$.*

Consequently B_n is strongly non-indicable for $n \geq 5$.

Proof. We begin with the proof of (i). For $1 \leq i \leq n$, let $\sigma_i \in B_n$ denote the braid that passes the i^{th} strand over the $(i+1)^{st}$, with subscripts interpreted mod n. The elements $\sigma_1, \ldots, \sigma_n$ are all mutually conjugate, and the abelianization map $A : B_n \to \mathbb{Z}$ is given by the total exponent sum of all the generators. Consequently, the set

$$S = \{\sigma_i \sigma_{i+1}^{-1} \mid 1 \leq i \leq n-1\}$$

normally generates $[B_n, B_n]$ inside B_n.

To prove the claim, it therefore suffices to show that the subgroup $\langle S \rangle$ of B_n generated by S is normal, which in turn reduces to showing that $\sigma_j(\sigma_i\sigma_{i+1}^{-1})\sigma_j^{-1} \in \langle S \rangle$ for any $1 \leq j \leq n$. As $n \geq 5$, the generator σ_{i+3} commutes with σ_i and σ_{i+1}, from which

$$\sigma_j(\sigma_i\sigma_{i+1}^{-1})\sigma_j^{-1} = (\sigma_j\sigma_{i+3}^{-1})(\sigma_i\sigma_{i+1}^{-1})(\sigma_j\sigma_{i+3}^{-1})^{-1}.$$

The right-hand side exhibits $\sigma_j(\sigma_i\sigma_{i+1}^{-1})\sigma_j^{-1}$ as a product of elements of $\langle S \rangle$, and the result follows.

The next step is to show that the elements of S are all conjugate within $[B_n, B_n]$. Via the braid relations,

$$(\sigma_i\sigma_{i+1}\sigma_{i+2})\sigma_i\sigma_{i+1}^{-1}(\sigma_i\sigma_{i+1}\sigma_{i+2})^{-1} = \sigma_{i+1}\sigma_{i+2}^{-1}. \tag{1}$$

As above, the element

$$\sigma_i\sigma_{i+1}\sigma_{i+2}\sigma_{i+3}^{-3} \in [B_n, B_n]$$

also conjugates $\sigma_i\sigma_{i+1}^{-1}$ to $\sigma_{i+1}\sigma_{i+2}^{-1}$.

From what has been established above, to establish *(ii)*, it is sufficient to express each $\sigma_i\sigma_{i+1}^{-1}$ as a commutator in $[B_n, B_n]$. For $n \geq 5$, there is some j for which σ_j commutes with both σ_i and σ_{i+1}, and therefore the expression

$$\sigma_i\sigma_{i+1}^{-1} = [\sigma_{i+1}\sigma_i\sigma_j^{-2}, \sigma_{i+1}\sigma_j^{-1}]$$

(which holds as a result of the braid relations) proves the claim. □

Remark 3.5. In fact, $[B_n, B_n]$ is finitely generated for all $n \geq 2$. We content ourselves with the given proof because it is better suited to the applications in the present paper.

Step 3: Produce $B_m \leq B_n(S)$. The following is implied by a theorem of Paris-Rolfsen [PR00, Theorem 4.1(iii)].

Theorem 3.6 (Paris-Rolfsen)**.** *For $S \neq S^2$, the inclusion of subsurfaces $(\mathbb{D}, X_n) \hookrightarrow (S, X_n)$ induces an injective map $B_n \hookrightarrow B_n(S)$. In the case $S = S^2$, an inclusion $(\mathbb{D}, X_n) \hookrightarrow (S^2, X_{n+1})$ induces a homomorphism $B_n \to B_{n+1}(S^2)$. The kernel of this homomorphism is contained in the cyclic group $\langle\Delta\rangle$ generated by the Dehn twist of a boundary-parallel curve using the identification $B_n \cong \mathrm{Mod}(\mathbb{D}, n)$, and is contained in the center of B_n.*

Remark 3.7. By construction, the subgroup $B_{n-1} \leq B_n$ stabilizes $X_n \setminus X_{n-1}$. More precisely, if $\tau \in B_{n-1} \leq B_n(S)$ and $\phi \in \mathrm{Diff}(S, \partial S, X_n)$ is any representative of $\mathcal{P}(\tau) \in \pi_0\big(\mathrm{Diff}(S, \partial S, X_n)\big)$, then ϕ fixes the point $X_n \setminus X_{n-1}$. Similarly, the image of B_n inside $B_{n+1}(S^2)$ stabilizes $X_{n+1} \setminus X_n$.

Step 4: Reduction to Thurston stability. In order to apply the Thurston stability theorem, we must first study the derivative mapping at the global fixed point.

Lemma 3.8. *For $n \geq 5$, every homomorphism $f : B_n \to GL_2^+(\mathbb{R})$ has abelian image.*

This is a consequence of the following more general criterion (which we will employ again in Section 7).

Lemma 3.9. *Let G be a group generated by elements $\tau_1, \ldots \tau_n$ that satisfy the following properties:*

(1) The elements τ_i are all mutually conjugate.
(2) There exists $k \geq 2$ such that $[\tau_i, \tau_j] = 1$ for $|j - i| \geq k$ (here we mean distance in $\mathbb{R}/n\mathbb{Z}$).

Then for $n \geq 2k + 1$, every homomorphism $f : G \to \mathrm{GL}_2^+(\mathbb{R})$ has abelian image.

Proof. It suffices to show that the projection $\bar{f} : G \to \mathrm{GL}_2^+(\mathbb{R}) \to \mathrm{PSL}_2(\mathbb{R})$ has image contained in a one-parameter subgroup. This is because the preimage in $\mathrm{GL}_2^+(\mathbb{R})$ of any one-parameter subgroup in $\mathrm{PSL}_2(\mathbb{R})$ is abelian. For convenience, we will write $\bar{\tau}_i$ in place of $\bar{f}(\tau_i)$. By condition (1) above, if $\bar{f}$ is a nontrivial homomorphism, then each $\bar{\tau}_i \neq I$.

If the image of $\bar{f}$ is not contained in some one-parameter subgroup, then in particular, there must be some pair of elements $\bar{\tau}_i$ and $\bar{\tau}_j$ that do not commute. By relabeling if necessary, we may assume $i = 1$ and $2 \le j \le k$. Furthermore, we may assume j is the smallest integer between 2 and k for which $\bar{\tau}_1$ and $\bar{\tau}_j$ do not commute.

We wish to show $j = 2$. Suppose $j > 2$. If $\bar{\tau}_{j-1}$ and $\bar{\tau}_j$ do not commute, then by relabeling again, we may assume that $\bar{\tau}_1$ and $\bar{\tau}_2$ do not commute. If, on the other hand, $\bar{\tau}_{j-1}$ commutes with $\bar{\tau}_j$, then both $\bar{\tau}_1$ and $\bar{\tau}_j$ are contained in the centralizer $C_{\mathrm{PSL}_2(\mathbb{R})}(\bar{\tau}_{j-1})$. As the latter is a one-parameter subgroup, necessarily $\bar{\tau}_1$ and $\bar{\tau}_j$ commute, contrary to assumption. We conclude that up to a cyclic relabeling of the generators τ_i, we must have $\bar{\tau}_1$ and $\bar{\tau}_2$ noncommuting elements of $\mathrm{PSL}_2(\mathbb{R})$.

By condition (2) above and the assumption $n \ge 2k+1$, the element $\bar{\tau}_{k+2}$ commutes with both $\bar{\tau}_1$ and $\bar{\tau}_2$. Therefore, $\bar{\tau}_1$ and $\bar{\tau}_2$ are contained in the abelian subgroup $C_{\mathrm{PSL}_2(\mathbb{R})}(\bar{\tau}_{k+2})$, contrary to assumption. □

Proof. (of Lemma 3.8) We show that B_n satisfies the hypotheses of Lemma 3.9 for $k = 2$. Indeed, for $1 \le i \le n$, let $\tau_i = \sigma_i$, the i^{th} standard generator of B_n. We interpret σ_n to be the element crossing the n^{th} strand over the first, under a cyclic ordering of the strands. As the elements σ_i are mutually conjugate and $[\sigma_i, \sigma_j] = 1$ for $|j - i| \ge 2$, the result follows. □

Remark 3.10. The assumption $n \ge 5$ in Lemma 3.8 cannot be relaxed: it is well-known that there is a homomorphism $B_3 \to \mathrm{SL}_2\,\mathbb{Z}$ with nonabelian image. The case $n = 4$ follows from the existence of an exceptional surjective homomorphism $B_4 \to B_3$.

To complete the proof of Theorem 1.1, we begin with the case $\partial S = \varnothing$. Suppose, for a contradiction, that a lift $\sigma : B_n(S) \to \mathrm{Diff}(S, \partial\, S, X_n)$ (for $n \ge 6$) is given. By Theorem 3.6, there is a nontrivial homomorphism $B_{n-1} \to \mathrm{Mod}(S, n)$; it follows from Remark 3.2 that $\mathrm{Mod}(S, n)$ is strongly non-indicable. By Remark 3.7, the lift $\sigma(B_{n-1})$ fixes some point $x \in X_n \setminus X_{n-1}$. Let $D : B_{n-1} \to \mathrm{GL}_2^+(\mathbb{R})$ denote the derivative mapping at x. Via Lemma 3.8, $[B_{n-1}, B_{n-1}] \le \ker D$. Thurston stability (Theorem 3.3) then asserts that $[B_{n-1}, B_{n-1}]$ must be locally indicable, but this contradicts Theorem 3.4.

To obtain the improvement $n \ge 5$ in the case ∂S is non-empty, we simply apply the preceding arguments to any point $x \in \partial S$. Here, we do not need to pass to B_{n-1} in order to produce a fixed point a la Remark 3.7, and so the argument applies for all $n \ge 5$. □

4. The Morita non-lifting theorem

The purpose of this section is to show how the methods of Theorem 1.1 can be extended to give a new proof of Morita's non-lifting theorem. We are grateful to the referee for observing that our methods should be applicable to this situation, and for suggesting Steps 1 and 2 below.

Proof of Theorem 1.2. Suppose that there is a realization $\sigma : \mathrm{Mod}(\Sigma_g) \to \mathrm{Diff}(\Sigma_g)$. We will arrive at a contradiction. The argument is divided into four steps.

Step 1: A large subgroup with a finite orbit. In this step, we indicate a particular constraint on the dynamics of any realization of the mapping class group by diffeomorphisms. Let ι denote the hyperelliptic involution (as depicted in Figure 1). Let $C(\iota)$ denote the centralizer of ι inside $\mathrm{Mod}(\Sigma_g)$.

Lemma 4.1. *For any realization σ, the fixed set $\mathrm{Fix}(\sigma(\iota))$ consists of exactly $2g+2$ points.*

Proof. This is a standard argument that follows from the Lefschetz fixed-point theorem. See [FM12, Section 7.1.2] for details. □

A standard principle in the theory of group actions gives the following corollary.

Corollary 4.2. *The subgroup $\sigma(C(\iota)) \le \mathrm{Diff}(\Sigma_g)$ preserves the finite set $\mathrm{Fix}(\sigma(\iota))$; associated to this is a permutation representation $\rho : C(\sigma(\iota)) \to S_{2g+2}$, the symmetric group on $2g+2$ letters.*

Step 2: A non-indicable subgroup of $C(\iota)$.

Lemma 4.3. *For all $g \ge 2$, $C(\iota)$ contains a strongly non-indicable subgroup B isomorphic to a quotient of B_{2g+2}.*

Proof. Consider the family of $2g+1$ simple closed curves $c_1, \dots, c_{2g+1}$ indicated in Figure 1. As the geometric intersection $i(c_i, c_{i+1}) = 1$ for all i, and $i(c_i, c_j) = 0$ for $|i-j| \ge 2$, the subgroup $B \le \mathrm{Mod}(\Sigma_g)$ generated by the Dehn twists T_{c_i} satisfy the braid relations: B is a (nontrivial) quotient of B_{2g+2}. It follows from Theorem 3.4 and Remark 3.2 that B is strongly non-indicable.

As each c_i is invariant under the action of ι, it follows that each $T_{c_i} \in C(\iota)$; consequently $B \le C(\iota)$ as claimed. □

Remark 4.4. Let B' denote the image of B_{2g+1} in B. By the above arguments, B' is also strongly non-indicable for $g \ge 2$.

Step 3: The action of B_{2g+2} on $\mathrm{Fix}(\sigma(\iota))$. In this step, we explicitly identify the action of $\sigma(B)$ on $\mathrm{Fix}(\sigma(\iota))$.

Lemma 4.5. *There is a commutative diagram*

$$\begin{tikzcd} B_{2g+2} \arrow[r, "\mu"] \arrow[d] & S_{2g+2} \\ B \arrow[ur, "\rho\circ\sigma"'] & \end{tikzcd}$$

where the map $\mu : B_{2g+2} \to S_{2g+2}$ is the canonical permutation homomorphism. Letting $B' \le B$ be the subgroup defined in Remark 4.4, it follows that the action of B' on $\mathrm{Fix}(\sigma(\iota))$ has a global fixed point.

Proof. Let $\sigma_i \in B_{2g+2}$ denote the standard generator of B_{2g+2} interchanging strands i and $i+1$, so that $\mu(\sigma_i) = (i\ i+1)$. The homomorphism $B_{2g+2} \to B$ sends σ_i to the Dehn twist T_{c_i} indicated in Figure 1. Let $\widetilde{T}_{c_i}$ denote a realization of this Dehn twist supported on a neighborhood of c_i invariant under $\sigma(\iota)$. Then $\rho(\widetilde{T}_{c_i})$ is the involution $(i\ i+1) = \mu(\sigma_i) \in S_{2g+2}$.

We next claim that if $\alpha, \alpha' \in C(\sigma(\iota))$ are isotopic, then $\rho(\alpha) = \rho(\alpha')$. Modulo the claim the result follows easily, since by the above paragraph each element of B has some representative diffeomorphism (obtained by taking a suitable product of the $\widetilde{T}_{c_i}$) inducing the expected permutation.

The claim is most easily established by temporarily leaving concerns of smoothness behind. Let $A \subset \mathrm{Homeo}(\Sigma_g)$ denote the isotopy class of α, α' within $\mathrm{Homeo}(\Sigma_g)$. Letting $C_{\mathrm{Homeo}(\Sigma_g)}(\sigma(\iota))$ denote the centralizer of $\sigma(\iota)$ within $\mathrm{Homeo}(\Sigma_g)$, observe that ρ extends to a homomorphism $\rho : C_{\mathrm{Homeo}(\Sigma_g)}(\sigma(\iota)) \to S_{2g+2}$.

We claim that as a map of topological spaces (endowing $C_{\mathrm{Homeo}(\Sigma_g)}(\sigma(\iota))$ with the compact-open topology and S_{2g+2} with the discrete topology), ρ is continuous. Let $\phi \in C_{\mathrm{Homeo}(\Sigma_g)}(\sigma(\iota))$ and $x \in \mathrm{Fix}(\sigma(\iota))$ be given. Let $U \subset \Sigma_g$ be an open neighborhood such that $U \cap \mathrm{Fix}(\sigma(\iota)) = \{\phi(x)\}$. If $\psi \in C_{\mathrm{Homeo}(\Sigma_g)}(\sigma(\iota))$ is sufficiently close to ϕ, then $\psi(x) \in U$, but as $\psi(x) \in \mathrm{Fix}(\sigma(\iota))$, it follows that $\psi(x) = \phi(x)$.

To establish the claim, it therefore suffices to show that α and α' lie in the same connected component of $C_{\mathrm{Homeo}(\Sigma_g)}(\sigma(\iota))$. Proposition 9.4 of [FM12] asserts that if $\phi, \psi \in C_{\mathrm{Homeo}(\Sigma_g)}(\sigma(\iota))$ are isotopic, then there exists an isotopy through elements of $C_{\mathrm{Homeo}(\Sigma_g)}(\sigma(\iota))$. The claim, and hence the result, follows. $\square$

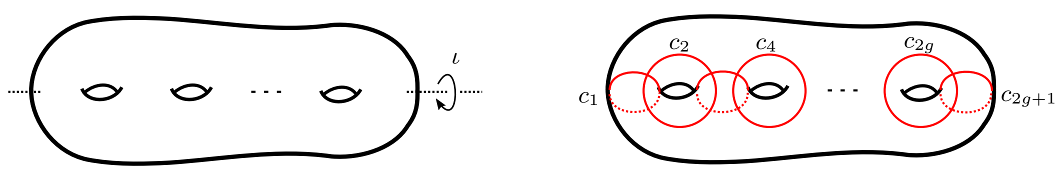

FIGURE 1. The hyperellitpic involution ι and curves c_i whose Dehn twists T_{c_i} generate a quotient of B_{2g+2}.

Step 4: Deriving the contradiction. From Steps 1 - 3 above, we have shown that if there is a realization $\sigma : \mathrm{Mod}(\Sigma_g) \to \mathrm{Diff}(\Sigma_g)$, then the strongly non-indicable subgroup $\sigma(B') \le \mathrm{Diff}(\Sigma_g)$ must act on Σ_g with a global fixed point $p \in \Sigma_g$. Consider the homomorphism

$$D_p \circ \sigma : B' \to \mathrm{GL}_2^+(\mathbb{R}).$$

According to Lemma 3.8, as B' is a quotient of B_{2g+1}, the image of $D_p \circ \sigma$ must be abelian. Letting $P \le B'$ denote any nontrivial finitely-generated perfect subgroup of B', it follows that $\sigma(P)$ acts trivially on the tangent space $T_p\Sigma_g$. Theorem 3.3 then asserts that P must be locally indicable, but this is impossible by assumption. $\square$

5. Push maps for spherical motion groups

We turn now to Theorem 1.4. It is first necessary to establish the existence of the push homomorphisms $\mathcal{P}$ that are the higher-dimensional analogues of the homomorphism in Theorem 2.2. Fix $k, n \geq 1$. For $M = \mathbb{R}^{k+2}$ or S^{k+2}, consider the space $\mathrm{Emb}_n(S^k, M)$ of C^1 embeddings $\coprod_n S^k \to M$. The symmetric group S_n acts on $\coprod_n S^k$ by permuting the components, and this induces an action on $\mathrm{Emb}_n(S^k, M)$ by precomposing an embedding by a permutation. Fix an embedding ϕ that is unlinked, and let $\mathrm{Emb}_n(S^k, M; \phi)$ denote the path component of ϕ. Define the *configuration space*

$$\mathrm{Conf}_n(S^k, M) = \mathrm{Emb}_n(S^k, M; \phi)/S_n.$$

An element of $\mathrm{Conf}_n(S^k, M)$ is a collection of disjoint, unordered, unlinked spheres, each of which comes with a parameterization.

Definition 5.1. Let $[\phi] \in \mathrm{Conf}_n(S^k, M)$ denote the equivalence class of the embedding ϕ. The group $B_n(S^k, M) := \pi_1\big(\mathrm{Conf}_n(S^k, M), [\phi]\big)$ is a *spherical motion group.*[2]

In order to state the analog of Theorem 2.2 for $B_n(S^k, M)$, let $\mathcal{D}(M) \leq \mathrm{Diff}(M)$ be the group of compactly-supported C^1 diffeomorphisms isotopic to the identity, and let $\mathcal{D}(M, [\phi]) \leq \mathcal{D}(M)$ be the subgroup of diffeomorphisms that satisfy $[f \circ \phi] = [\phi]$. Viewing ϕ as defining a parameterization on its image $\mathrm{Im}(\phi) \subset M$, diffeomorphisms of $\mathcal{D}(M, [\phi])$ preserve $\mathrm{Im}(\phi)$ together with the parameterization on each sphere, up to permutations. In particular, $f \in \mathcal{D}(M, [\phi])$ fixes *pointwise* any component of $\mathrm{Im}(\phi)$ taken to itself.

Proposition 5.2. *Fix $n \geq 1$. There is a homomorphism $\mathcal{P} : B_n(S^k, M) \to \pi_0\big(\mathcal{D}(M, [\phi])\big)$. The kernel of $\mathcal{P}$ is abelian.*

Proof. There is an evaluation map $\eta : \mathcal{D}(M) \to \mathrm{Conf}_n(S^k, M)$ defined by $f \mapsto [f \circ \phi]$. By Palais [Pal60] this map determines a fibration

$$\mathcal{D}(M, [\phi]) \to \mathcal{D}(M) \xrightarrow{\eta} \mathrm{Conf}_n(S^k, M).$$

The long exact sequence of homotopy groups of this fibration gives an exact sequence

$$\pi_1\big(\mathcal{D}(M)\big) \to B_n(S^k, M) \xrightarrow{\mathcal{P}} \pi_0\big(\mathcal{D}(M, [\phi])\big).$$

This defines $\mathcal{P}$. Note that as $\mathcal{D}(M)$ is a topological group, $\pi_1\big(\mathcal{D}(M)\big)$ is abelian, from which it follows that $\ker \mathcal{P}$ is as well. $\square$

[2]These groups were first studied by Dahm [Dah62].

6. Proof of Theorem 1.4

Once again, the situation can be expressed diagrammatically as follows:

$$\begin{tikzcd} & \mathcal{D}(M,[\phi]) \arrow[d, "\pi"] \\ B_n(S^k,M) \arrow[ur, dashed, "\sigma"] \arrow[r, "\mathcal{P}"'] & \pi_0\big(\mathcal{D}(M,[\phi])\big). \end{tikzcd}$$

We seek to obstruct the existence of a lift σ of $\mathcal{P}$. The outline of the proof is essentially the same as for Theorem 1.1. We will not reproduce Step 1 of Theorem 1.1, as the Thurston stability theroem holds for any smooth manifold. Also, Step 2 of Theorem 1.1, which concerns the group theory of B_n, needs no modification. As such, the proof of Theorem 1.4 will begin with finding a nontrivial homomorphism $B_n \to B_n(S^k, M)$.

Step 1: Produce nontrivial $B_n \to B_n(S^k, M)$. In this section we prove the following proposition.

Proposition 6.1. *If $M = \mathbb{R}^{k+2}$, then there is an embedding $B_n \hookrightarrow B_n(S^k, M)$. If $M = S^{k+2}$, then there is a homomorphism $B_n \to B_n(S^k, M)$ whose kernel is contained in the center $Z(B_n)$.*

Proof. To produce the desired homomorphism, we first find a subspace $\mathcal{C} \subset \mathrm{Conf}_n(S^k, M)$ such that $\pi_1(\mathcal{C})$ contains B_n. This uses work of Brendle–Hatcher. Then we will study the composition $B_n \hookrightarrow \pi_1(\mathcal{C}) \to \pi_1\big(\mathrm{Conf}_n(S^k, M)\big)$ by looking at the induced action of B_n on $\pi_1\big(M \setminus \coprod_n S^k\big) \simeq F_n$. For $M = \mathbb{R}^{k+2}$ this action coincides with the Artin representation $B_n \to \mathrm{Aut}(F_n)$, which is well-known to be injective. For $M = S^{k+2}$, we obtain instead a quotient of the Artin representation $B_n \to \mathrm{Out}(F_n)$, and we explain why its kernel is $Z(B_n)$.

To define $\mathcal{C}$, give $\mathbb{R}^{k+2}$ coordinates $(x, y, z, w_1, \ldots, w_{k-1})$ and fix an embedding $f : S^k \hookrightarrow \mathbb{R}^{k+2}$ whose image is the sphere of radius 1 centered at the origin in $\mathbb{R}^{k+1} \simeq \{(x, y, z, w_1, \ldots, w_{k-1}) : x = 0\}$. Consider the space $\mathcal{E}$ of embeddings $\phi : \coprod_n S^k \to \mathbb{R}^{k+2}$ where the embedding on each component is the composition of f with a dilation of $\mathbb{R}^{k+2}$ followed by a translation in the xy-plane. The quotient $\mathcal{C} = \mathcal{E}/S_n$ is a subspace of $\mathrm{Conf}_n(S^k, \mathbb{R}^{k+2})$. We also obtain $\mathcal{C} \subset \mathrm{Conf}_n(S^k, S^{k+2})$ by choosing an embedding $\mathbb{R}^{k+2} \hookrightarrow S^{k+2}$.

There is a map $a : \mathcal{C} \to \mathcal{UW}_n$ to the *untwisted wicket space* of Brendle–Hatcher [BH13] obtained by restricting an embedding $\phi : \coprod_n S^k \to \mathbb{R}^{k+2}$ to $\coprod_n V$, where $V \subset S^k$ is the subspace $f^{-1}\{(0, y, z, 0, \ldots, 0)\} \simeq S^1$. The map a is a homeomorphism by the construction of $\mathcal{C}$; furthermore, $\pi_1(\mathcal{UW}_n)$ contains a braid group by [BH13, Proposition 3.1]. In $\pi_1(\mathcal{C})$, this braid group is generated by motions $\rho_1, \ldots, \rho_{n-1} \in \pi_1(\mathcal{C})$ that exchange the i^{th} and $(i+1)^{st}$ spheres of a fixed embedding ϕ, passing the $(i+1)^{st}$ sphere through the i^{th} sphere. See Figure 2.

Next we determine how $B_n \leq \pi_1(\mathcal{C})$ acts on $\pi_1\big(M \setminus \coprod_n S^k\big)$. The homomorphism $\mathcal{P}$ of Proposition 5.2 gives a homomorphism $\pi_1(\mathcal{C}) \to \pi_0\big(\mathcal{D}(M, [\phi])\big)$. The latter group acts on $\pi_1(M \setminus \mathrm{Im}\,\phi)$. If $M = \mathbb{R}^{k+2}$, then we can define $\pi_0\big(\mathcal{D}(M, [\phi])\big) \to \mathrm{Aut}\big(\pi_1(M \setminus \mathrm{Im}\,\phi, *)\big)$ act by identifying $\mathbb{R}^{k+2} \simeq \mathrm{int}(\mathbb{D}^{k+2})$, identifying $\mathcal{D}(M, [\phi])$ with the corresponding subgroup of

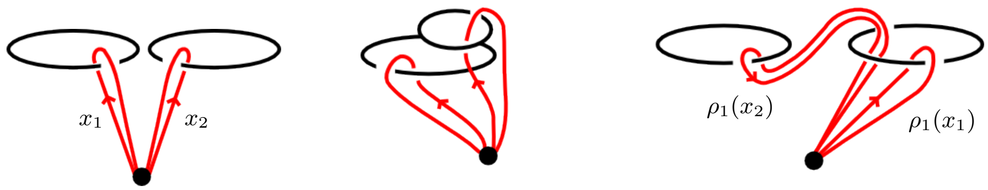

FIGURE 2. A 3-frame movie of the motion ρ_1 and the result on $\pi_1(M \setminus \coprod_n S^k)$.

$\mathrm{Diff}(\mathbb{D}^{k+2})$, and choosing $* \in \partial\mathbb{D}$. If $M = S^{k+2}$, then we cannot choose a global fixed point for the action of $\mathcal{D}(M,\phi)$ on $M \setminus \mathrm{Im}\,\phi$, and so we only obtain $\pi_0\big(\mathcal{D}(M,[\phi])\big) \to \mathrm{Out}\big(\pi_1(M \setminus \mathrm{Im}\,\phi)\big)$.

The group $\pi_1(M \setminus \mathrm{Im}\,\phi)$ is free by the following lemma.

Lemma 6.2. *Fix $k \geq 2$. Let $\coprod_n S^k \hookrightarrow \mathbb{R}^{k+2}$ be an unlinked embedding. Then $\pi_1\big(\mathbb{R}^{k+2} \setminus \coprod_n S^k\big)$ is isomorphic to the free group F_n.*

Proof. For definiteness, we will specify a particular embedding $\phi : \coprod_n S^k \to \mathbb{R}^{k+2}$ where the i^{th} sphere is mapped to the equator of the sphere of radius $1/4$ centered at $(i, 0, \ldots, 0) \in \mathbb{R}^{k+2}$. We proceed by induction on n. For the base case $n = 1$, first note that

$$S^{k+2} \cong \partial(\mathbb{D}^{k+1} \times \mathbb{D}^2) = S^k \times \mathbb{D}^2 \bigcup_{S^k \times S^1} \mathbb{D}^{k+1} \times S^1.$$

It follows that $\pi_1\big(S^{k+2} \setminus S^k\big) \cong \pi_1(\mathbb{D}^{k+1} \times S^1) \cong \mathbb{Z}$. Then also $\pi_1\big(\mathbb{R}^{k+2} \setminus S^k\big) \cong \mathbb{Z}$, since removing a single point from a $(m \geq 3)$-manifold does not change the fundamental group.

For the inductive step, take ϕ as above and decompose $\mathbb{R}^{k+2}$ into open sets

$$U = \{(x_1, \ldots, x_{k+2}) : x_1 < n - \frac{1}{2} + \varepsilon\} \quad \text{and} \quad V = \{(x_1, \ldots, x_{k+2}) : x_1 > n - \frac{1}{2} - \varepsilon\}$$

for any small positive ε. By construction U contains the first $n-1$ spheres and V contains the n^{th} sphere. Since $U \cap V$ is contractible, by Seifert–van Kampen, we have

$$\pi_1\big(\mathbb{R}^{k+2} \setminus \coprod_n S^k\big) \cong \pi_1\big(\mathbb{R}^{k+2} \setminus \coprod_{n-1} S^k\big) * \pi_1(\mathbb{R}^{k+2} \setminus S^k) \cong F_{n-1} * \mathbb{Z} \cong F_n.$$

The second isomorphism uses the inductive hypothesis and the base case. $\square$

Remark 6.3. The lemma obviously implies that $\pi_1\big(S^{k+2} \setminus \coprod_n S^k\big) \simeq F_n$.

We now have homomorphisms

$$\beta : B_n \to \pi_1(\mathcal{C}) \to \pi_1\big(\mathrm{Conf}_n(S^k, \mathbb{R}^{k+2})\big) \to \mathrm{Aut}(F_n)$$

and

$$\gamma : B_n \to \pi_1(\mathcal{C}) \to \pi_1\big(\mathrm{Conf}_n(S^k, S^{k+2})\big) \to \mathrm{Out}(F_n).$$

To prove Proposition 6.1 we show that β is injective and that $\ker\gamma = Z(B_n)$.

Lemma 6.4. *The homomorphism β is injective.*

Proof. There is another homomorphism $\alpha : B_n \to \mathrm{Aut}(F_n)$ (sometimes called the *Artin representation*) induced by the action of the mapping class group $\mathrm{Mod}(\mathbb{D}, n) \cong B_n$ on $\pi_1(\mathbb{D} \setminus \{n \text{ points}\}) \cong F_n$. It is a well-known theorem of Artin that α is injective (see [Art25] or [Bir74, Corollary 1.8.3]). We prove the lemma by showing that β and α coincide after making the right identifications.

Choose a configuration $Y = \{y_1, \ldots, y_n\} \subset \mathbb{D}$ as in Figure 3. Let $\{\sigma_1, \ldots, \sigma_{n-1}\}$ be the standard generating set for B_n (c.f. Lemma 3.8). The isomorphism $B_n \xrightarrow{\sim} \mathrm{Mod}(\mathbb{D}, n)$ is defined by sending σ_i to the mapping class that exchanges y_i and y_{i+1} by moving them counterclockwise around their midpoint. We choose generators η_i for $\pi_1(\mathbb{D} \setminus Y, *) \cong F_n$ as in Figure 3. It is easy to compute (c.f. Figure 3)

$$\alpha(\sigma_i) : \begin{cases} \eta_j \mapsto \eta_j & \text{if } j \neq i, i+1 \\ \eta_i \mapsto \eta_{i+1} & \\ \eta_{i+1} \mapsto \eta_{i+1}\eta_i\eta_{i+1}^{-1} & \end{cases}$$

On the other hand, the inclusion $B_n \hookrightarrow \pi_1\big(\mathrm{RConf}_n(S^k, \mathbb{D}^{k+2})\big)$ sends σ_i to the motion ρ_i defined above (Figure 2). We identify $\pi_1(\mathbb{D}^{k+2} \setminus \coprod_n S^k) \cong F_n$ as follows. Fix a basepoint $* \in \partial \mathbb{D}^{k+2}$, and choose an embedding $\coprod \mathbb{D}^{k+1} \to \mathbb{D}^{k+2}$ such that, in the image, the boundary of the i^{th} disk D_i is the i^{th} sphere. Then $\pi_1(\mathbb{D}^{k+2} \setminus \coprod_n S^k, *)$ is generated by loops $\gamma_1, \ldots, \gamma_n : [0,1] \to \mathbb{D}^{k+2} \setminus \coprod_n S^k$ such that $\gamma_i \cap D_j = \emptyset$ for $i \neq j$ and γ_i has a single, positive transverse intersection with D_i. Then for any $\gamma \in \pi_1(\mathbb{D}^{k+2} \setminus \coprod_n S^k, *)$, expressing $\rho_i(\gamma) \in F_n$ as a word in $\gamma_1, \ldots, \gamma_n$ reduces to computing the intersection of $\rho_i(\gamma)$ with the disks $D_1, \ldots, D_n$. From this it is easy to see ρ_i sends γ_i to γ_{i+1}, sends γ_{i+1} to $\gamma_{i+1}\,\gamma_i\,\gamma_{i+1}^{-1}$, and fixes γ_j for $j \neq i, i+1$; see Figure 4. Since $\rho_i = \beta(\sigma_i)$, this shows that β and α agree, as desired. □

Lemma 6.5. *The kernel of γ is equal to the center $Z(B_n)$.*

Proof. By the proof of Lemma 6.4, γ is the composition of the Artin representation $\alpha : B_n \to \mathrm{Aut}(F_n)$ with the projection $\mathrm{Aut}(F_n) \to \mathrm{Out}(F_n)$. Thus it suffices to understand this composition.

To describe $B_n \xrightarrow{\alpha} \mathrm{Aut}(F_n) \to \mathrm{Out}(F_n)$, we use a stronger version of the theorem of Artin mentioned in the proof of Lemma 6.4 that describes the image of α explicitly. Let F_n be generated by $\eta_1, \ldots, \eta_n$ as in Figure 3. Then $\phi \in \mathrm{im}(\alpha)$ if and only if there exists $A_1, \ldots, A_n \in F_n$ and $\tau \in S_n$ such that

(i) $\phi(\eta_i) = A_i\, \eta_{\tau(i)}\, A_i^{-1}$ for $1 \leq i \leq n$, and
(ii) $\phi(\eta_1 \cdots \eta_n) = \eta_1 \cdots \eta_n$.

See [Art25] or [Bir74, Theorem 1.9]. From this it quickly follows that $\phi \in \mathrm{Inn}(F_n) \cap \mathrm{im}(\alpha)$ if and only if ϕ is conjugation by $(\eta_1 \cdots \eta_n)^r$ for some $r \in \mathbb{Z}$ (we must have $A_1 = A_2 = \cdots = A_n$ and A_1 must commute with $\eta_1 \cdots \eta_n$). Now the lemma follows by checking that $\alpha(\Delta) = \mathrm{conj}(\eta_1 \cdots \eta_n)$, where $\Delta \in B_n$ is the full twist (which generates $Z(B_n)$). □

This completes the proof of Proposition 6.1. □

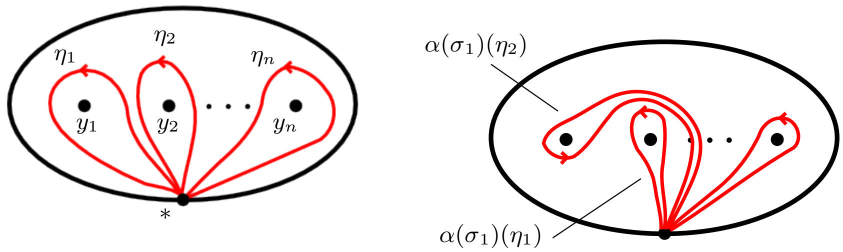

FIGURE 3. The braid group $B_n \cong \mathrm{Mod}(\mathbb{D}, n)$ acting on $\pi_1(\mathbb{D} \setminus Y_n)$.

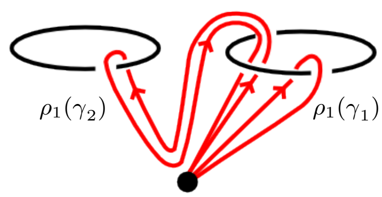

FIGURE 4. An illustration showing that $\rho_1(\gamma_1) = \gamma_2$ and $\rho_1(\gamma_2) = \gamma_2\,\gamma_1\,\gamma_2^{-1}$.

Step 2: Reduction to Thurston stability. For $M = \mathbb{R}^{k+2}$ we will use the following easy corollary to Thurston stability (Theorem 3.3).

Corollary 6.6. *Let M be a noncompact manifold. Then the group $\mathrm{Diff}_c(M)$ of compactly supported C^1 diffeomorphisms is locally indicable (and hence any subgroup is also locally indicable).*

Proof. Let $\Gamma \leq \mathrm{Diff}_c(M)$ be a finitely generated subgroup. The intersection of the supports of the generators is compact, so Γ acts trivially on a neighborhood of some $x \in M$. Thus $\Gamma \leq \mathcal{G}$, and there exists a surjection $\Gamma \to \mathbb{Z}$ by Thurston stability. □

For the spherical motion groups $B_n(S^k, S^{k+2})$, there is one additional step that is required in the reduction process. Below, $\mathrm{Diff}(S^{k+2}, S^k)$ denotes the group of diffeomorphisms of S^{k+2} that restrict to the identity on the image of a fixed embedding $S^k \to S^{k+2}$.

Proposition 6.7. *Let $\Gamma \leq \mathrm{Diff}(S^{k+2}, S^k)$ be finitely generated. If Γ is strongly non-indicable, then there is a homomorphism $f : \Gamma \to GL_2^+(\mathbb{R})$ with nonabelian image.*

Proof. Choose $x \in S^k$. Then there are coordinates in which any $g \in \mathrm{Diff}(S^{k+2}, S^k)$ has derivative given by

$$(Dg)_x = \left(\begin{array}{c|c} I_{k-2} & V_g \\ \hline 0 & A_g \end{array}\right).$$

In this setting, $V_g \in M_{k-2,2}(\mathbb{R})$ is a $(k-2) \times 2$ matrix, and $A_g \in GL_2^+(\mathbb{R})$. Denote by $p : \mathrm{Diff}(S^{k+2}, S^k) \to GL_2^+(\mathbb{R})$ the homomorphism given by $p(g) = A_g$.

Let $\Gamma \leq \mathrm{Diff}(S^{k+2}, S^k)$ be strongly non-indicable, and let $\Gamma' \leq \Gamma$ be a finitely-generated perfect subgroup. We claim that $p : \Gamma \to GL_2^+(\mathbb{R})$ has nonabelian image. If not, then $\Gamma' \leq \ker p$. In this case, $g \mapsto V_g$ defines a homomorphism $V : \Gamma' \to M_{k-2,2}(\mathbb{R})$, where $M_{k-2,2}(\mathbb{R})$ is viewed as a group under addition. As $M_{k-2,2}(\mathbb{R})$ is abelian and Γ' is perfect, V must be trivial. But then Thurston stability implies that Γ' is locally indicable, a contradiction. □

To complete the proof of Theorem 1.4, suppose $\sigma : B_n(S^k, M) \to \mathcal{D}(M, [\phi])$ is a lift of $\mathcal{P}$. If $M = \mathbb{R}^{k+2}$, then $B_n \leq B_n(S^k, M)$ by Proposition 6.1. By Proposition 5.2, $\sigma\big([B_n, B_n]\big) \leq \mathcal{D}(M, [\phi])$ is a nontrivial subgroup. Since it is finitely-generated and perfect, $\mathcal{D}(M, [\phi])$ is strongly non-indicable. However, $\mathcal{D}(M, [\phi]) \leq \mathrm{Diff}_c(\mathbb{R}^k)$, so this contradicts Corollary 6.6.

In the case $M = S^{k+2}$, consider the homomorphism $j : B_n \to B_n(S^k, M)$ provided by Proposition 6.1. Take a further subgroup $B_{n-1} \leq B_n$ so that $\sigma\big(j(B_{n-1})\big)$ fixes some component of $\mathrm{Im}(\phi)$ pointwise. By Propositions 5.2 and 6.1, the image of B_{n-1} in $\mathcal{D}(S^{k+2}, [\phi])$ is nontrivial, and $\sigma([B_{n-1}, B_{n-1}])$ is a nontrivial finitely-generated perfect subgroup. Consequently $\sigma(B_{n-1})$ is strongly non-indicable. By Proposition 6.7, there is a homomorphism $f : \sigma(B_{n-1}) \to \mathrm{GL}_2^+(\mathbb{R})$ with nonabelian image, but this contradicts Lemma 3.8. □

7. Extensions of the main theorems

In this section we give a strengthening of Theorems 1.1 and 1.4 using a result of Parwani [Par08, Theorem 1.4] building off of work of Deroin-Kleptsyn-Navas [DKN07].

Theorem 7.1 (Parwani)**.** *Let G and H be two finitely generated groups such that $H_1(G;\mathbb{Z}) = 0 = H_1(H;\mathbb{Z})$. Then for any C^1 action of $G \times H$ on S^1, either $G \times 1$ or $1 \times H$ acts trivially.*

7.1. **Surfaces.** Let S be a closed surface and let $X \subset S$ be finite. Let S' be the compact surface obtained by replacing each marked point $x \in X$ with a boundary component. In what follows, $\mathrm{Diff}(S')$ denotes the group of diffeomorphisms of S' where the boundary components of S' are *not* required to be fixed pointwise. It is well-known that $\pi_0 \mathrm{Diff}(S, X) \cong \pi_0 \mathrm{Diff}(S')$. Therefore, one can ask whether the homomorphism

$$\mathcal{P} : B_n(S) \to \pi_0 \mathrm{Diff}(S, X) \cong \pi_0 \mathrm{Diff}(S') \tag{2}$$

lifts to a homomorphism $B_n(S) \to \mathrm{Diff}(S')$.

Theorem 7.2. *Fix $n \geq 11$. Then $\mathcal{P} : B_n(S) \to \pi_0 \mathrm{Diff}(S')$ is not realized by diffeomorphisms. That is, there does not exist a homomorphism $B_n(S) \to \mathrm{Diff}(S')$ such that the composition $B_n(S) \to \mathrm{Diff}(S') \to \pi_0 \mathrm{Diff}(S')$ is equal to $\mathcal{P}$.*

Proof. Suppose for a contradiction that $\sigma : B_n(S) \to \mathrm{Diff}(S')$ is a lift of (2). By passing to a finite-index subgroup of $B_n(S)$ we may assume one component $C \subset \partial S'$ is fixed. By the assumption $n \geq 11$, this finite-index subgroup contains $B_5 \times B_5$. We may therefore take $G = [B_5, B_5] \times 1$ and $H = 1 \times [B_5, B_5]$ in Theorem 7.1 to conclude that, without loss of generality, G acts trivially on C. As G is non-locally-indicable (Theorem 3.4), the last stage of the argument of Theorem 1.4 for the case $M = S^{k+2}$ can be applied to derive a contradiction. $\square$

7.2. **Spheres.** Let $\mathrm{Emb}_n(S^k, S^{k+2}; \phi)$ be the embedding space defined in Section 5. Define the *(unparameterized) configuration space* $\overline{\mathrm{Conf}}_n(S^k, S^{k+2})$ as

$$\overline{\mathrm{Conf}}_n(S^k, S^{k+2}) = \mathrm{Emb}_n(S^k, S^{k+2}; \phi)/\operatorname{Diff}(\coprod_n S^k).$$

Note that $\overline{\mathrm{Conf}}_n(S^k, S^{k+2})$ is a quotient of $\mathrm{Conf}_n(S^k, S^{k+2})$, since $\operatorname{Diff}(\coprod_n S^k)$ is isomorphic to the wreath product $\operatorname{Diff}(S^k) \wr S_n$. An element of $\overline{\mathrm{Conf}}_n(S^k, S^{k+2})$ is a collection of disjoint, unordered, unlinked spheres (with no additional information about the parameterization).

Fix $X \in \overline{\mathrm{Conf}}_n(S^k, S^{k+2})$, and let $\bar{B}_n(S^k, S^{k+2}) = \pi_1\big(\overline{\mathrm{Conf}}_n(S^k, S^{k+2}), X\big)$. In the case $k = 1$, this group coincides with the *ring group* studied by Brendle and Hatcher in [BH13]. By the argument in Proposition 5.2, there is a homomorphism

$$\mathcal{P} : \bar{B}_n(S^k, S^{k+2}) \to \pi_0\big(\mathcal{D}(S^{k+2}, X)\big),$$

where $\mathcal{D}(S^{k+2}, X) \leq \mathcal{D}(S^{k+2})$ is the subgroup of diffeomorphisms that preserve X as a set.

We have the following strengthening of Theorem 1.4 in the case $k = 1$.

Theorem 7.3. *Fix $n \geq 15$. Then the homomorphism $\mathcal{P} : \bar{B}_n(S^1, S^3) \to \pi_0\big(\mathcal{D}(S^3, X)\big)$ is not realized by diffeomorphisms.*

Proof. Suppose for a contradiction that $\sigma : \bar{B}_n(S^1, S^3) \to \mathcal{D}(S^3, X)$ is a lift of $\mathcal{P}$. By the same argument as Proposition 6.1, there is a homomorphism $B_n \to \bar{B}_n(S^1, S^3)$ with kernel contained in $Z(B_n)$. By passing to a finite-index subgroup of $\bar{B}_n(S^1, S^3)$, we may assume that one component $C \cong S^1 \subset X$ is fixed. By the assumption $n \geq 15$, this finite-index subgroup contains $B_7 \times B_7$ and *a fortiori* contains $[B_7, B_7] \times [B_7, B_7]$. Taking $G = [B_7, B_7] \times 1$ and $H = 1 \times [B_7, B_7]$ in Theorem 7.1, it follows that (without loss of generality) G fixes C pointwise.

For the remainder of the argument, we follow the strategy in Step 2 of Theorem 1.4. In order to be able to derive a contradiction from Proposition 6.7, we must have that every homomorphism $f : [B_7, B_7] \to \mathrm{GL}_2^+(\mathbb{R})$ has abelian image.

The generating set S of Proposition 3.4.(i) satisfies the hypotheses of Lemma 3.9 for $k = 3$. It follows that every homomorphism $f : [B_7, B_7] \to \mathrm{GL}_2^+(\mathbb{R})$ has abelian image as desired. The argument in Step 2 of Theorem 1.4 can now be carried out showing that $[B_7, B_7] \times 1 \leq \bar{B}_n(S^1, S^3)$ lies in the kernel of any homomorphism $\sigma : \bar{B}_n(S^1, S^3) \to \mathcal{D}(S^3, X)$. Therefore $\bar{B}_n(S^1, S^3)$ cannot be realized by diffeomorphisms. $\square$

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