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Borel's stable range for the cohomology of arithmetic groups

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Abstract. In this note, we remark on the range in Borel's theorem on the stable cohomology of the arithmetic groups $\mathrm{Sp}_{2n}(\mathbb{Z})$ and $\mathrm{SO}_{n,n}(\mathbb{Z})$. The main result improves the range stated in Borel's original papers, an improvement that was known to Borel. The proof is a technical computation involving the Weyl group action on roots and weights.
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1. Introduction

Let G be a semi-simple algebraic group defined over $\mathbb{Q}$, and let Γ be a finite-index subgroup of $G(\mathbb{Z})$. For V an algebraic representation of G, Borel [Bor74, Bor81] computed the cohomology $H^i(\Gamma;V)$ in a *stable range*, i.e. for $i\le N$ for some constant $N=N(G,V)$ that depends only on G,V.

In some cases, the constant $N(G,V)$ that appears in [Bor74, §9] and [Bor81] can be improved. This is remarked by Borel in [Bor81, §3.8]. In this note, we supply the details of Borel's remark when G is one of the algebraic groups

$$\mathrm{SO}_{n,n}=\{g\in\mathrm{SL}_{2n}(\mathbb{C}):g^t\begin{pmatrix}0&I\\I&0\end{pmatrix}g=\begin{pmatrix}0&I\\I&0\end{pmatrix}\}$$

or

$$\mathrm{Sp}_{2n}=\{g\in\mathrm{SL}_{2n}(\mathbb{C}):g^t\begin{pmatrix}0&I\\-I&0\end{pmatrix}g=\begin{pmatrix}0&I\\-I&0\end{pmatrix}\}.$$

Theorem 1.1 (Borel stability for $\mathrm{SO}_{n,n}(\mathbb{Z})$)**.** *Fix $n\ge4$. Let V be an irreducible rational representation of $\mathrm{SO}_{n,n}$, and let $\Gamma<\mathrm{SO}_{n,n}(\mathbb{Z})$ be a finite-index subgroup. If $k\le n-2$, then $H^k(\Gamma;V)$ vanishes when V is nontrivial, and agrees with the stable cohomology of $\mathrm{SO}_{n,n}(\mathbb{Z})$ when V is the trivial representation.*

Theorem 1.2 (Borel stability for $\mathrm{Sp}_{2n}(\mathbb{Z})$)**.** *Fix $n\ge3$. Let V be an irreducible rational representation of Sp_{2n}, and let $\Gamma<\mathrm{Sp}_{2n}(\mathbb{Z})$ be a finite-index subgroup. If $k\le n-1$, then $H^k(\Gamma;V)$ vanishes when V is nontrivial, and agrees with the stable cohomology of $\mathrm{Sp}_{2n}(\mathbb{Z})$ when V is the trivial representation.*

The cases $\mathrm{SO}_{2,2}$ and $\mathrm{SO}_{3,3}$ are exceptional because $\mathrm{SO}_{n,n}$ is isogenous to $\mathrm{SL}_2 \times \mathrm{SL}_2$ when $n = 2$ and SL_4 when $n = 3$. For $\mathrm{SL}_2(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z})$, the stable cohomology is trivial and there is no vanishing theorem. We remark further on the case of $\mathrm{SO}_{3,3}(\mathbb{Z})$ in §2.

The bound in Theorem 1.1 is nearly sharp. For example, when n is odd, [Tsh19] proves that there is $\Gamma < \mathrm{SO}_{n,n}(\mathbb{Z})$ with $H^n(\Gamma;\mathbb{Q}) \neq 0$, whereas if $i \le n-2$ is odd, then $H^i(\Gamma;\mathbb{Q}) = 0$ by Theorem 1.1 and the determination of the stable cohomology of $\mathrm{SO}_{n,n}(\mathbb{Z})$ [Bor74, §11].

Theorems 1.1 and 1.2 are likely well-known to experts on the cohomology of arithmetic groups, but it seems the proofs have not been written down. This article has the modest goal of filling this gap in the literature, which is of interest in applications. In particular, Theorem 1.2 is used in Hain's important work [Hai97] on the Torelli group (where is it stated without proof; c.f. Thm 3.2), and both Theorems 1.1 and 1.2 have been used by Kupers–Randal-Williams [KRW19] in their study of diffeomorphisms groups of manifolds $\#_n(S^d \times S^d)$ when $d \ge 3$. In this direction, we also mention that Theorems 1.1 and 1.2 make the hypothesis on the degree of the representation V in [ERW15, Prop. 3.9] unnecessary. Finally, we remark that this note originally appeared in a draft of [Tsh19], where it was used toward producing new characteristic classes of manifold bundles; however, an alternate approach not using Borel's theorem was found, so we have moved the computation into this separate note.

About the proof. Theorems 1.1 and 1.2 are deduced from the contents of [Bor74] together with a representation-theoretic computation.

We start by briefly summarizing Borel's approach to computing $H^*(\Gamma;V)$ in a range; see also [Bor74, Bor81]. Fix a semi-simple algebraic group G such that $G(\mathbb{R})$ is of noncompact type, and let $X = G(\mathbb{R})/K$ be the associated symmetric space. For a lattice $\Gamma < G(\mathbb{R})$, computing $H^*(\Gamma;V)$ is equivalent to computing the homology of the complex $\Omega^*(X;V)^\Gamma$ of V-valued, Γ-invariant differential forms on X. The subcomplex $I^*_{G,V} \subset \Omega^*(X;V)^\Gamma$ of $G(\mathbb{R})$-invariant forms consists of closed forms, so there is a homomorphism

$$j : I^*_{G,V} \to H^*(\Gamma;\mathbb{R}),$$

whose image is known as the *stable cohomology*. The ring $I^*_{G,V}$ is easily computed: it is isomorphic to $H^*(X_u;V)$, where X_u is the compact symmetric space dual to X, and it is also identified with Lie algebra cohomology $H^*(\mathfrak{g},K;V)$. In particular, if V is irreducible and nontrivial, then $H^*(\mathfrak{g},K;V)$ is trivial [BW00, Ch. II, Cor. 3.2]. Borel showed that j^* is bijective in a range $i \le \min\{m(G(\mathbb{R})), c(G,V)\}$. See [Bor74, Thm. 7.5] and [Bor81, Thm. 4.4].

To apply Borel's theorem, one wants to understand the constants $m(G(\mathbb{R}))$ and $c(G,V)$. According to [Bor81, §4], $m(G(\mathbb{R})) \ge \mathrm{rk}_{\mathbb{R}}\, G(\mathbb{R}) - 1$ for every G that is almost simple over $\mathbb{R}$ (this includes $\mathrm{SO}_{n,n}$ and Sp_{2n}, both of which have rank n). The constant $c(G,V)$ can be computed with some representation theory.

The constant $c(G,V)$. Let $\mathfrak{g}$ be the Lie algebra of $G(\mathbb{C})$, and let $B \subset G(\mathbb{C})$ be a minimal parabolic (i.e. Borel) subgroup with Levi decomposition $B = U \rtimes A$. Let $\mathfrak{a}$ and $\mathfrak{u}$ be the corresponding Lie algebras. Here $\mathfrak{a} \subset \mathfrak{g}$ is a maximal abelian (i.e.

Cartan) subalgebra. The weights of $\mathfrak{a}$ acting on $\mathfrak{g}$ are called the *roots* of $\mathfrak{g}$, and the subset of weights of $\mathfrak{a}$ acting on $\mathfrak{u}$ are called *positive*. Let ρ be half the sum of the positive roots. A positive root is called *simple* if it cannot be expressed as a nontrivial sum of positive roots. The simple positive roots $\{\alpha_k\}$ form a basis for $\mathfrak{a}^*$. An element $\phi \in \mathfrak{a}^*$ is called *dominant* (resp. *dominant regular*), denoted $\phi \geq 0$ (resp. $\phi > 0$), if $\phi = \sum c_k\,\alpha_k$ with $c_k \geq 0$ (resp. $c_k > 0$) for each k.

Borel's constant $c(G,V)$ is the largest q so that $\rho + \mu > 0$ for every weight μ of $\Lambda^q\mathfrak{u}^* \otimes V$, c.f. [Bor74, §2 and Thm. 4.4].

A better constant $C(G,V)$. According to [Bor81, Rmk. 3.8] (see also [GH68, Thm. 3.1] and [Zuc83, (3.20) and (4.57)]), there is a better constant $C(G,V) \geq c(G,V)$ so j^* bijective in degrees $i \leq \{m(G(\mathbb{R})), C(G,V)\}$. To define this constant, let W be the Weyl group of $G(\mathbb{C})$. For each $q \geq 0$, let $W^q \subset W$ be the subset of elements that send exactly q positive roots to negative roots. Denoting the highest weight of V by λ, define

$$C(G,V) = \max\{q : \sigma(\rho+\lambda) > 0 \text{ for all } \sigma \in W^q\}.$$

As Borel remarks, $C(G,V)$ can be interpreted as the largest q for which $\rho+\mu > 0$ for every weight μ of $H^q(\mathfrak{u};V)$. Since the Lie algebra cohomology $H^*(\mathfrak{u};V)$ is the homology of the complex $\Lambda^*\mathfrak{u}^* \otimes V$, it follows that the weights of the former are a subset of the weights of the latter, so $c(G,V) \leq C(G,V)$.

In the remainder of this note, we compute the value of $C(G,V)$ when G is $\mathbb{Q}$-split of type C_n or D_n, i.e. $G(\mathbb{Z})$ is commensurable with $\mathrm{Sp}_{2n}(\mathbb{Z})$ or $\mathrm{SO}_{n,n}(\mathbb{Z})$.

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2. Computation for $\mathrm{SO}_{n,n}$

The main goal of this section is to prove the following proposition.

Proposition 2.1. *Fix $n \geq 4$, and let $G = \mathrm{SO}_{n,n}$. Then $C(G,V) \geq n-2$ for each irreducible finite dimensional rational representation V of G.*

Our proof is divided into two steps: we first show $C(G,\mathbb{C}) = n-2$ for the $\mathbb{C}$ the trivial representation (Proposition 2.2), and then we show $C(G,V) \geq C(G,\mathbb{C})$ for any other representation (Proposition 2.3).

To begin, we need the following information from [Bou68, pg. 256-258]. Below $\epsilon_1,\ldots,\epsilon_n$ are the standard coordinate functionals on $\mathfrak{a} \subset \mathfrak{g}$.

- The simple roots are $\alpha_1 = \epsilon_1 - \epsilon_2$, …, $\alpha_{n-1} = \epsilon_{n-1} - \epsilon_n$, and $\alpha_n = \epsilon_{n-1} + \epsilon_n$.
- The half-sum of positive roots is $\rho = \sum_{i=1}^n r_i\,\alpha_i$, where $r_i = \frac{(2n-i-1)i}{2}$ for $1 \leq i \leq n-2$ and $r_{n-1} = r_n = \frac{n(n-1)}{4}$.
- The Weyl group $W = (\mathbb{Z}/2\mathbb{Z})^{n-1} \rtimes S_n$ acts as the even signed permutation group of $\{\pm\epsilon_1,\ldots,\pm\epsilon_n\}$, i.e. the symmetric group S_n acts by permuting the indices of $\epsilon_1,\ldots,\epsilon_n$, and $(\mathbb{Z}/2\mathbb{Z})^{n-1}$ acts by an even number of sign changes.

Let $\tau_i \in W$ be the reflection fixing the orthogonal complement of α_i (with respect to the inner product where the ϵ_i are othonormal). The τ_i generate W, and we write

$$S = \{\tau_1, \ldots, \tau_n\}.$$

The action of τ_i on the roots sends α_i to $-\alpha_i$ and permutes the remaining positive roots. Thus $\tau_i \in W^1$, and it's not hard to show that $\sigma \in W^q$ if and only if the word length of σ with respect to S is q. See [Hum78, §10.3] for details.

In what follows we will work in the basis $(\epsilon_1, \ldots, \epsilon_n)$ instead of $(\alpha_1, \ldots, \alpha_n)$. We record how these two bases are related: if $\sum x_i\alpha_i = \sum y_i\epsilon_i$, then

$$\begin{aligned} x_k &= y_1 + \cdots + y_k & k \le n-2 \\ x_{n-1} &= \tfrac{1}{2}(y_1 + \cdots + y_{n-1} - y_n) \\ x_n &= \tfrac{1}{2}(y_1 + \cdots + y_{n-1} + y_n) \end{aligned} \tag{1}$$

For $i = 1, \ldots, n-1$, the reflection τ_i interchanges ϵ_i and ϵ_{i+1} (and acts trivially on the remaining ϵ_j), while τ_n interchanges ϵ_{n-1} and ϵ_n and changes their signs. In ϵ_i-coordinates,

$$\rho = (n-1, n-2, \ldots, 2, 1, 0).$$

Proposition 2.2. *Fix $n \ge 3$, and let $G = \mathrm{SO}_{n,n}$. Then $C(G, \mathbb{C}) = n-2$.*

Proof. First observe that the image of ρ under $\sigma = \tau_1 \cdots \tau_{n-1}$ is not dominant regular. Indeed in ϵ_i-coordinates, $\sigma(\rho) = (0, n-1, n-2, \ldots, 2, 1)$, which is not dominant regular since the coefficient on α_1 is 0. This implies $C(G, \mathbb{C}) \le n-2$.

It remains to show $C(G, \mathbb{C}) \ge n-2$, i.e. if $\sigma \in W$ can be expressed as a word in S of length $\ell \le n-2$, then $\sigma(\rho)$ is dominant regular. Recall above that the τ_i act on ϵ_i-coordinates as signed permutations, so the coordinates of $\sigma(\rho) = (y_1, \ldots, y_n)$ are a signed permutation of the coordinates of $\rho = (n-1, \ldots, 1, 0)$. In order to show $\sigma(\rho) > 0$, we need to show each of the sums $y_1 + \cdots + y_k$ is positive for $k \ne n-1$ and also that $y_1 + \cdots + y_{n-1} - y_n$ is positive.

We first consider two special cases from which the general case follows.

Special case 1. Suppose that σ is a word in $S \setminus \{\tau_{n-1}, \tau_n\}$. In ϵ_i-coordinates $\tau_1, \ldots, \tau_{n-2}$ act as permutations without sign changes that fix the last coordinate, so $\sigma(\rho) = (y_1, \ldots, y_{n-1}, 0)$, where $(y_1, \ldots, y_{n-1})$ is a permutation of $(n-1, \ldots, 1)$. In particular, $y_1, \ldots, y_{n-1}$ are all positive, and it follows that $\sigma(\rho)$ is regular dominant.

Special case 2. Suppose that σ is a word in $S \setminus \{\tau_1\}$. Then $\sigma(\rho) = (n-1, y_2, \ldots, y_n)$, where $(y_2, \ldots, y_n)$ is a signed permutation of $(n-2, \ldots, 1, 0)$.

Since τ_n is the only element of S that changes any sign, in order for j (the $(n-j)$-th coordinate of ρ) to appear with a negative sign in $\sigma(\rho)$, the length of σ must be at least j (this follows immediately from the τ_i action on the coordinates; note, for example, that the sign of j in $\tau_n\tau_{n-2}\cdots\tau_{n-j}(\rho)$ is negative). Similarly, in order for each of the coefficients $j_1, \ldots, j_m$ of ρ to appear with negative signs in $\sigma(\rho)$, the length of σ must be at least $j_1 + \cdots + j_m$ (again this follows from the τ_i action; note that each τ_i only moves one coefficient to the right at a time). Let $j_1, \ldots, j_m$

be the coefficients of ρ that become negative in $\sigma(\rho)$. Then $j_1+\cdots+j_m\le n-2$ because σ has length $\le n-2$. Hence for $1\le i\le n$,

$$y_1+\cdots+y_i\ge(n-1)-(j_1+\cdots+j_m)\ge(n-1)-(n-2)>0.$$

It follows that the coefficient of α_i in $\sigma(\rho)$ is positive for each i, possibly with the exception of $i=n-1$ (c.f. (1)). By the same reasoning, the coefficient of α_{n-1} is also positive: let $j_1,\ldots,j_m$ be the coefficients of ρ that are negative in $\sigma(\rho)$, and suppose $y_n=j_{m+1}$. Moving j_{m+1} to the n-th coordinate requires a word of length j_{m+1} (e.g. $\tau_{n-1}\tau_{n-2}\cdots\tau_{n-j}$), so as above $j_1+\cdots+j_{m+1}\le n-2$, and so, similar to the above,

$$y_1+\cdots+y_{n-1}-y_n\ge(n-1)-(j_1+\cdots+j_{m+1})>0.$$

General case. Suppose σ is any word in $\tau_1,\ldots,\tau_n$ of length $\le n-2$. Then τ_i does not appear in σ for some $1\le i\le n-1$, and we can write $\sigma=\sigma_1\sigma_2$, where σ_1 is a word in $\{\tau_1,\ldots,\tau_{i-1}\}$ and σ_2 is a word in $\{\tau_{i+1},\ldots,\tau_n\}$. For $j\le i$, the coefficient of α_j in $\sigma(\rho)$ and $\sigma_1(\rho)$ agree, and for $j\ge i+1$, the coefficient of α_j in $\sigma(\rho)$ and $\sigma_2(\rho)$ agree, so $\sigma(\rho)$ is dominant regular by the previous two cases. ∎

Proposition 2.3. *Fix $n\ge4$, and let $G=\mathrm{SO}_{n,n}$. If V is an irreducible representation, then $C(G,V)\ge C(G,\mathbb{C})$.*

Proof. Let λ be the highest weight of V. According to [FH91, §19.2], λ can be expressed as an integral linear combination $\lambda=\sum_{k=1}^n a_k\phi_k$, where $a_k\ge0$ and

$$\phi_k=\begin{cases}\epsilon_1+\cdots+\epsilon_k & k\le n-2\\ (\epsilon_1+\cdots+\epsilon_{n-1}-\epsilon_n)/2 & k=n-1\\ (\epsilon_1+\cdots+\epsilon_n)/2 & k=n.\end{cases}\tag{2}$$

If $\sigma\in W$, then $\sigma(\rho+\lambda)=\sigma(\rho)+\sum_k a_k\sigma(\phi_k)$. We proceed by studying when $\sigma(\phi_k)$ is dominant. To show $C(G,V)\ge C(G,\mathbb{C})=n-2$, it suffices to show that if $\sigma\in W^q$ for $q\le n-2$, then $\sigma(\phi_k)\ge0$ for each $1\le k\le n$. Then for any highest weight $\lambda=\sum a_k\phi_k$, we conclude that $\sigma(\rho+\lambda)=\sigma(\rho)+\sum a_k\,\sigma(\phi_k)$ is dominant regular because $\sigma(\phi_k)\ge0$ and $\sigma(\rho)>0$ (Proposition 2.2).

We consider separately cases $1\le k\le n-2$ and $k=n-1,n$. In either case the argument is similar to the corresponding step in the proof of Proposition 2.2.

Fix $1\le k\le n-2$ and write $\phi=\phi_k$. In ϵ_i-coordinates, $\phi=(1,\ldots,1,0,\ldots,0)$. Next we bound from below the minimum word length of σ needed for $\sigma(\phi)<0$, and we will find that there is no σ of length $\le n-2$.

First observe that the only way to act by elements of S to make a coefficient of ϕ negative is to move that coefficient to the right (using a word like $\tau_{n-2}\cdots\tau_i$), and then apply τ_n. Therefore, fixing $\ell<k/2$, any word σ such that $\sigma(\phi)$ has $\ell+1$ negative coordinates has length at least

$$(n-k)+\cdots+(n-k+\ell)=n(\ell+1)-\left[\frac{(k+1)k}{2}-\frac{(k-\ell)(k-\ell-1)}{2}\right].\tag{3}$$

$$\phi = (1, \ldots, 1, \underbrace{1, \ldots, 1}_{\ell+1}, \underbrace{0, \ldots, 0}_{n-k})$$

Figure 1: To make $\ell+1$ positive coefficients of ϕ negative requires a word whose length is at least the quantity in (3).

$$(\underbrace{1, \ldots, 1}_{\ell}, \underbrace{1, \ldots, 1}_{k-2\ell-1}, \underbrace{0, \ldots, 0}_{n-k}, \underbrace{-1, \ldots, -1}_{\ell+1})$$

Figure 2: We can make this vector non-dominant by moving $k-2\ell-1$ positive entries past $\ell+1$ negative entries. This requires a word whose length is at least the quantity in (4).

See Figure 1. After creating $\ell+1$ negative coefficients, to make a non-dominant vector, one needs to move sufficiently many positive entries to the right, passed the negative entries. Since we start with $k = \ell + (k-2\ell-1) + (\ell+1)$ positive entries, we must move $(k-2\ell-1)$ positive entries passed the $(\ell+1)$ negative entries. This requires a word of length at least

$$(k-2\ell-1)(\ell+1). \tag{4}$$

See Figure 2. Now we conclude. Suppose for a contradiction that σ has length $\leq n-2$ and that $\sigma(\phi) < 0$. Write $\sigma(\phi) = (y_1, \ldots, y_n)$, and let i be the smallest index so that the coefficient of α_i in $\sigma(\phi)$ is negative. If $i \neq n-1$, then this means $y_1 + \cdots + y_i < 0$. We will assume $i \neq n-1$; the case $i = n-1$ is similar (c.f. the proof of Proposition 2.2). The terms in this sum $y_1 + \cdots + y_i$ are all $+1, 0, -1$. By replacing σ with a shorter word, we can assume that the summands occur in decreasing order $1 + \cdots + 1 + 0 + \cdots + 0 + -1 + \cdots + -1$ (this follows from the description of the τ_i action and the fact that the coefficients of ϕ are decreasing). By minimality of our choice of i, if there are ℓ positive terms in the sum, then there are $\ell+1$ negative terms. Note then that $2\ell+1 \leq k$, so $\ell < k/2$.

Combining (3) and (4), if the leading coefficients of $\sigma(\phi)$ are

$$(1, \ldots, 1, 0, \ldots, 0, -1, \ldots, -1, \ldots),$$

then the length of σ is at least

$$n(\ell+1) - \left[\frac{(k+1)k}{2} - \frac{(k-\ell)(k-\ell-1)}{2}\right] + (k-2\ell-1)(\ell+1) = n(\ell+1) - \frac{3\ell^2+5\ell+2}{2}.$$

Since we're assuming σ has length $\leq n-2$, we must have $n(\ell+1) - \frac{3\ell^2+5\ell+2}{2} \leq n-2$. This inequality implies that

$$n \leq \frac{3\ell+5}{2}.$$

Since $\ell < k/2 \leq (n-2)/2$, this implies that $n < 4$. This is contrary to our hypothesis, so we conclude that there does not exist σ of length $\leq n-2$ so that $\sigma(\phi) < 0$.

The same analysis can be applied to ϕ_{n-1} and ϕ_n. The details are unilluminating and can easily be supplied by the interested reader, so we omit them here. ■

Proposition 2.3 is false for $n = 3$. In this case, $C(G, \mathbb{C}) = 1$, but there are V with $C(G, V) = 0$. For example, take V the irreducible representation with highest weight $m\phi_2$. Observe that, in ϵ_i-coordinates,

$$\tau_2(\rho + m\phi_2) = \left(2 + \frac{m}{2}, -\frac{m}{2}, 1 + \frac{m}{2}\right),$$

so the coefficient of α_2 in $\tau_2(\rho + m\phi_2)$ is $\frac{1}{2}\left((2 + \frac{m}{2}) - \frac{m}{2} - (1 + \frac{m}{2})\right) = 1 - \frac{m}{2}$, which is non-positive if $m \geq 2$. This implies that $C(G, V) = 0$, and Borel's theorem does not allow one to conclude, for example, that $H^1(\Gamma; V) = 0$ for a lattice $\Gamma < \mathrm{SO}_{3,3}(\mathbb{Z})$. However, $H^1(\Gamma; V)$ does vanish for any nontrivial V by a theorem of Margulis [Mar91, Ch. VII, Cor. 6.17].

The failure of Proposition 2.3 in the case $n = 3$ is related to the fact that $\mathrm{SO}_{3,3}$ is isogenous to SL_4. For SL_{n+1} one can compute that $C(G, \mathbb{C})$ is the smallest integer strictly less than $n/2$, but it's not true the $C(G, V) \geq C(G, \mathbb{C})$ for every irreducible representation. Indeed if one takes $V = \mathrm{Sym}^m(\mathbb{C}^{n+1})$, then $C(G, V) = 0$ for m sufficiently large. In this direction we remark that there are other known vanishing results for $H^*(\Gamma; V)$ beyond Borel's theorem. See [LS04, pg. 143].

Proposition 2.3 gives a lower bound on $C(G, V)$. We remark on the upper bound. Observe that if $\sigma = \tau_1 \cdots \tau_n$, then $\sigma(\phi) \leq 0$ because the coefficient of α_1 is non-positive. Since the coefficient of α_1 in $\sigma(\rho)$ is also non-positive, it follows that $\sigma(\rho + \lambda) \leq 0$ for every highest weight λ. This shows that $C(G, V) \leq n - 1$, and so

$$n - 2 \leq C(G, V) \leq n - 1$$

for any irreducible V. For any particular V one can determine which inequality is strict. For example $C(G, V) = n - 2$ when the highest weight of V is one of the basis vectors $\phi_1, \ldots, \phi_n$ (to show $C(G, V) \leq n - 2$, consider $\sigma = \tau_1 \cdots \tau_{n-1}$ if $1 \leq k \leq n - 1$ and $\sigma = \tau_1 \cdots \tau_{n-2}\tau_n$ for $k = n$). We leave further computations in this direction to the reader.

3. Computation for $\mathrm{Sp}_{2n}(\mathbb{R})$

In this section we carry out the analysis of §2 for Sp_{2n}. The goal is to prove the following proposition.

Proposition 3.1. *Fix $n \geq 3$, and let $G = \mathrm{Sp}_{2n}$. Then $C(G, V) = n - 1$ for each irreducible finite dimensional rational representation V of G.*

The outline of the argument is similar to the argument for Proposition 2.1. We explain the main differences and refer the reader to §2 when the details are similar. We start with the following information is from [Bou68, pg. 254-255].

- The simple roots are $\alpha_1 = \epsilon_1 - \epsilon_2$, $\ldots$, $\alpha_{n-1} = \epsilon_{n-1} - \epsilon_n$, and $\alpha_n = 2\epsilon_n$.
- The half the sum of positive roots is $\rho = \sum r_i\, \alpha_i$, where $r_i = \frac{(2n-i+1)i}{2}$ for $1 \leq i \leq n - 1$ and $r_n = \frac{n(n+1)}{4}$.
- The Weyl group $W = (\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n$. It acts as the signed permutation group of $\{\pm\epsilon_1, \ldots, \pm\epsilon_n\}$.

Let $\tau_i \in W$ be the reflection fixing the orthogonal complement of α_i. For $1 \le i \le n-1$, the reflection τ_i interchanges ϵ_i and ϵ_{i+1}, while τ_n only changes the sign on ϵ_n. The set $S = \{\tau_1, \ldots, \tau_n\}$ generates W. As in the $\mathrm{SO}_{n,n}$ case, $S \subset W^1$ and $\sigma \in W^q$ if and only if the S word-length of σ is q.

We record how the bases $(\epsilon_1, \ldots, \epsilon_n)$ and $(\alpha_1, \ldots, \alpha_n)$ are related: if $\sum x_i \alpha_i = \sum y_i \epsilon_i$, then

$$\begin{aligned} x_k &= y_1 + \cdots + y_k & k \le n-1 \\ x_n &= \tfrac{1}{2}(y_1 + \cdots + y_{n-1} + y_n) \end{aligned} \tag{5}$$

In ϵ_i-coordinates,

$$\rho = (n, n-1, \ldots, 2, 1).$$

Proposition 3.2. *If $G = \mathrm{Sp}_{2n}$, then $C(G, \mathbb{C}) = n - 1$.*

Proof. First observe that if $\sigma = \tau_1 \cdots \tau_n$, then $\sigma(\rho) = (-1, n, \ldots, 2)$ is not dominant regular. This shows $C(G, \mathbb{C}) \le n - 1$.

To show $C(G, \mathbb{C}) \ge n - 1$, let σ be a word in S of length $\le n - 1$. We will show $\sigma(\rho)$ is dominant regular.

Special case 1. First consider the case that σ is a word in $S \setminus \{\tau_n\}$. Since $\tau_1, \ldots, \tau_{n-1}$ act as permutations without changing sign, $\sigma(\rho) = (y_1, \ldots, y_{n-1}, 1)$, where $(y_1, \ldots, y_{n-1})$ are a permutation of $(n, \ldots, 2)$. Then $y_1 + \cdots + y_i > 0$ for each $1 \le i \le n$, which implies that $\sigma(\rho)$ is dominant regular.

Special case 2. Next consider the case that σ is a word in $S \setminus \{\tau_1\}$. Then $\sigma(\rho) = (n, y_2, \ldots, y_n)$, where $(y_2, \ldots, y_n)$ is a signed permutation of $(n-1, \ldots, 1)$.

Since τ_n is the only element of S that changes any sign, in order for j (the $(n-j+1)$-st coordinate of ρ) to appear with a negative sign in $\sigma(\rho)$, the length of σ must be at least j. Similarly, in order for each of the coefficients $j_1, \ldots, j_m$ of ρ to appear with negative signs in $\sigma(\rho)$, the length of σ must be at least $j_1 + \cdots + j_m$. Let $j_1, \ldots, j_m$ be the coefficients of ρ that become negative in $\sigma(\rho)$. Then $j_1 + \cdots + j_m \le n - 1$ because σ has length $\le n - 1$. Hence for $1 \le i \le n$,

$$y_1 + \cdots + y_i \ge n - (j_1 + \cdots + j_m) \ge n - (n-1) > 0.$$

This shows that $\sigma(\rho)$ is dominant regular.

General case. If σ has length $\le n - 1$, then there is some index $1 \le i \le n$ so that τ_i is not in σ. We covered the cases $i = 1$ and $i = n$ above, so we can assume $1 < i < n$. Then we can write $\sigma = \sigma_1 \sigma_2$, where σ_1 is a word in $\{\tau_1, \ldots, \tau_{i-1}\}$ and σ_2 is a word in $\{\tau_{i+1}, \ldots, \tau_n\}$. Then the coefficients of α_j in $\sigma_1(\rho)$ and $\sigma(\rho)$ agree for $j \le i$ and the coefficients of α_j in $\sigma_2(\rho)$ and $\sigma(\rho)$ agree for $j \ge i + 1$, so we again reduce to the previous cases to conclude that $\sigma(\rho)$ is dominant regular. ■

Proposition 3.3. *Let V be an irreducible representation. Then $C(G, V) = C(G, \mathbb{C})$.*

Proof. Let λ be the highest weight of V. According to [FH91, §17.2], λ can be expressed as an integral linear combination $\lambda = \sum_{k=1}^{n} a_k \, \phi_k$, where $a_k \ge 0$ and

$\phi_k = \epsilon_1 + \cdots + \epsilon_k$. If $\sigma \in W$, then $\sigma(\rho+\lambda) = \sigma(\rho) + \sum_k a_k\, \sigma(\phi_k)$. The proof of the proposition will follow by studying $\sigma(\phi_k)$.

First we explain why $C(G,V) \leq C(G,\mathbb{C}) = n-1$. Observe that for $\sigma = \tau_1 \cdots \tau_n$, the coefficient of α_1 in $\sigma(\phi_k)$ is 0 for each $1 \leq k \leq n$. In Proposition 3.2 we showed that the coefficient of α_1 in $\sigma(\rho)$ is negative, so it follows that $\sigma(\rho+\lambda) \leq 0$. Hence $C(G,V) \leq n-1$.

To show that $C(G,V) \geq n-1$, it suffices to show that if $\sigma \in W^q$ for $q \leq n-1$, then $\sigma(\phi_k) \geq 0$ for each $1 \leq k \leq n$. To simplify the notation, fix k and write $\phi = \phi_k$. In ϵ_i-coordinates $\phi = (1,\ldots,1,0,\ldots,0)$. Next we bound from below the minimum word length of σ needed for $\sigma(\phi) < 0$, and we will find that there is no σ of length $\leq n-1$.

First observe that the only way to act by elements of S to make a coefficient of ϕ negative is to move that coefficient to the right (using a word like $\tau_{n-1}\cdots\tau_i$) and the apply τ_n. Therefore, fixing $\ell < k/2$, any word σ such that $\sigma(\phi)$ has $\ell+1$ negative coordinates has length at least

$$(n-k+1)+\cdots+(n-k+1+\ell) = n(\ell+1) - \left[\frac{k(k-1)}{2} - \frac{(k-\ell-1)(k-\ell-2)}{2}\right]. \tag{6}$$

After creating $\ell+1$ negative coefficients, to make a non-dominant vector, one needs to move sufficiently many positive entries to the right, passed the negative entries. Since we start with $k = \ell + (k-2\ell+1) + (\ell+1)$ positive entries, we must move $(k-2\ell-1)$ positive entries passed the $(\ell+1)$ negative entries. This requires a word of length at least

$$(k-2\ell-1)(\ell+1). \tag{7}$$

Now we conclude. Suppose for a contradiction that σ has length $\leq n-1$ and that $\sigma(\phi) < 0$. Write $\sigma(\phi) = (y_1,\ldots,y_n)$, and let i be the smallest index so that the coefficient of α_i in $\sigma(\phi)$ is negative. Then $y_1+\cdots+y_i < 0$. The terms in the sum $y_1+\cdots+y_i$ are all $+1, 0, -1$. By replacing σ with a shorter word, we can assume that the summands occur in decreasing order. By minimality of our choice of i, if there are ℓ positive terms in the sum, then there are $\ell+1$ negative terms. Then $2\ell+1 \leq k$, so $\ell < k/2$.

Combining (6) and (7), if the leading coefficients of $\sigma(\phi)$ are

$$(1,\ldots,1,0,\ldots,0,-1,\ldots,-1,\ldots),$$

then the length of σ is at least

$$n(\ell+1) - \left[\frac{k(k-1)}{2} - \frac{(k-\ell-1)(k-\ell-2)}{2}\right] + (k-2\ell-1)(\ell+1) = n(\ell+1) - \frac{3\ell^2+3\ell}{2}.$$

Since we're assuming σ has length $\leq n-1$, we must have $n(\ell+1) - \frac{3\ell^2+3\ell}{2} \leq n-1$. This inequality implies that

$$n \leq \frac{3\ell+3}{2} - \frac{1}{\ell} \leq \frac{3\ell+3}{2}.$$

Since $\ell < k/2$, if $k \leq n-1$, this implies that $n < 3$, which contradicts the hypothesis. If $k = n$, then we can only conclude $n < 6$.

Assume now that $k = n$ and $n \leq 5$. Since $\ell < k/2 = n/2$ this implies that either $\ell = 1$ and $3 \leq n \leq 5$ or $\ell = 2$ and $n = 5$. The inequality $n \leq \frac{3\ell+3}{2} - \frac{1}{\ell}$ implies that $n \leq 2$ when $\ell = 1$ and it implies $n \leq 4$ when $\ell = 2$. In either case, this is a contradiction. Therefore, we conclude that if σ has length $\leq n - 1$, then $\sigma(\phi) \geq 0$. This completes the proof. ■

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