

# Arithmeticity of groups $\mathbb{Z}^n \rtimes \mathbb{Z}$

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## Abstract

We study when the group  $\mathbb{Z}^n \rtimes_A \mathbb{Z}$  is arithmetic, where  $A \in \mathrm{GL}_n(\mathbb{Z})$  is hyperbolic and semisimple. We begin by giving a characterization of arithmeticity phrased in the language of algebraic tori, building on work of Grunewald–Platonov. We use this to prove several more concrete results that relate the arithmeticity of  $\mathbb{Z}^n \rtimes_A \mathbb{Z}$  to the reducibility properties of the characteristic polynomial of  $A$ . Our tools include algebraic tori, representation theory of finite groups, Galois theory, and the inverse Galois problem.

## 1 Introduction

This paper focuses on the following question.

**Question 1.** Fix  $A \in \mathrm{GL}_n(\mathbb{Z})$ . When is the semidirect product  $\Gamma_A := \mathbb{Z}^n \rtimes_A \mathbb{Z}$  an arithmetic group?

Recall that a group  $\Gamma$  is called *arithmetic* if it embeds in an algebraic group  $G$  defined over  $\mathbb{Q}$  with image commensurable to  $G(\mathbb{Z})$ .

**Standing assumption.** We restrict focus to the generic case when  $A$  is *hyperbolic* (no eigenvalues on the unit circle) and *semisimple* (diagonalizable over  $\mathbb{C}$ ).

With the standing assumption, Question 1 can be answered in terms of the eigenvalues of  $A$  as follows. Let  $\chi(A) = \mu_1^{n_1} \cdots \mu_m^{n_m}$  be the characteristic polynomial, decomposed into irreducible factors over  $\mathbb{Q}$ . Choose a root  $\lambda_i$  of  $\mu_i$ , and view it as an element in (the  $\mathbb{Z}$ -points of) the algebraic torus  $R_{\mathbb{Q}(\lambda_i)/\mathbb{Q}}(G_m)$ , where  $G_m$  is the multiplicative group and  $R_{K/\mathbb{Q}}(\cdot)$  denotes the restriction of scalars.

**Theorem A** (Arithmeticity criterion). *Fix  $A \in \mathrm{GL}_n(\mathbb{Z})$  hyperbolic and semisimple, with characteristic polynomial  $\chi(A) = \mu_1^{n_1} \cdots \mu_m^{n_m}$  and fix eigenvalues  $\lambda_1, \dots, \lambda_m$ , as above. View  $\lambda = (\lambda_1, \dots, \lambda_m)$  as an element of  $T(\mathbb{Z})$ , where  $T$  is the algebraic torus  $\prod_{i=1}^m R_{\mathbb{Q}(\lambda_i)/\mathbb{Q}}(G_m)$ . Let  $S \subset T$  be the Zariski closure of the subgroup generated by*

$\lambda$ . Then  $\mathbb{Z}^n \rtimes_A \mathbb{Z}$  is arithmetic if and only if the rank of  $S(\mathbb{Z})$  as an abelian group is 1.

An important component of the proof of Theorem A is an argument of Grunewald–Platonov [GP98] that addresses the arithmeticity question for  $\Gamma = \mathcal{O}_K \rtimes_\lambda \mathbb{Z}$ , where  $\mathcal{O}_K$  is the ring of integers in a number field  $K$ , and  $\mathbb{Z}$  acts on  $\mathcal{O}_K$  by multiplication by a unit  $\lambda \in \mathcal{O}_K^\times$ . To relate the work of Grunewald–Platonov to the general case, we use the conjugacy classification of hyperbolic, semisimple elements of  $\mathrm{GL}_n(\mathbb{Z})$  in terms of number fields and ideal classes.

While Theorem A gives a complete answer to Question 1, from a practical viewpoint it is not completely satisfactory because taking the Zariski closure adds a layer of computational difficulty. It would be better if the arithmeticity criterion in Theorem A were phrased directly in terms of the eigenvalues or characteristic polynomial of the matrix  $A$ . To illustrate this point, the reader might try to use Theorem A to determine if  $\Gamma_A$  is arithmetic for the two matrices below (this can be done in an ad hoc way, but we give a more systematic approach below).

$$A_1 = \begin{pmatrix} 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \quad A_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \quad (1)$$

We prove several results that refine Theorem A in special cases. Our most complete result concerns the case when all the eigenvalues of  $A$  are real.

**Theorem B** (Improved arithmeticity criterion: totally real case). *Fix  $A \in \mathrm{GL}_n(\mathbb{Z})$  hyperbolic and semisimple, with  $\chi(A) = \mu_1^{n_1} \cdots \mu_m^{n_m}$  and  $\lambda = (\lambda_1, \dots, \lambda_m)$  as in the statement of Theorem A. Assume that all of the eigenvalues of  $A$  are real. Then the following are equivalent.*

- (i) *The group  $\mathbb{Z}^n \rtimes_A \mathbb{Z}$  is arithmetic.*
- (ii) *After replacing  $A$  by  $A^k$  for some  $k \geq 1$ , the  $\lambda_i$  are all powers of a unit  $\epsilon \in \mathcal{O}_L^\times$  in a real quadratic extension  $L/\mathbb{Q}$ .*

Condition (ii) of Theorem B implies in particular that each of the  $\mu_i$  have degree 2, and so a power of  $A$  is conjugate to a block diagonal matrix with  $2 \times 2$  blocks. As a simple application, the matrix  $A_1$  in (1) has  $\chi(A_1) = x^4 - 4x^2 + 1$ , which is irreducible over  $\mathbb{Q}$ , but  $\chi(A_1^2) = (x^2 - 4x + 1)^2$ , so we conclude by Theorem B that  $\Gamma_{A_1}$  is arithmetic. For the other matrix  $A_2$  in (1),  $\chi(A_2) = x^5 - x^3 - 2x^2 + 1$  has non-real roots, so Theorem B does not apply to this example.

The author does not know of an analogue of Theorem B when  $A \in \mathrm{GL}_n(\mathbb{Z})$  has complex eigenvalues. However, Theorem B motivates Question 3 below. Before stating it, we need a definition.

**Definition 2.** We say that  $A \in \mathrm{GL}_n(\mathbb{Z})$  is *irreducible* if its characteristic polynomial  $\chi(A)$  is irreducible over  $\mathbb{Q}$ ; otherwise we say  $A$  is reducible. We say that  $A$  is *fully irreducible* if  $A^k$  is irreducible for each  $k \geq 1$ .

Note that  $A$  is reducible if and only if it is conjugate in  $\mathrm{GL}_n(\mathbb{Z})$  to a block diagonal matrix  $(A_1, A_2) \in \mathrm{GL}_{n_1}(\mathbb{Z}) \times \mathrm{GL}_{n_2}(\mathbb{Z})$ , where  $n_1, n_2 \geq 1$  (here one should remember our standing assumption that  $A$  is semisimple).

According to Theorem B, if  $A \in \mathrm{GL}_n(\mathbb{Z})$  is fully irreducible and its eigenvalues are real, then  $\Gamma_A$  is arithmetic if and only if  $n = 2$ . This points us toward the following question.

**Question 3.** For which  $n \geq 2$ , does there exist a fully irreducible  $A \in \mathrm{GL}_n(\mathbb{Z})$  so that  $\Gamma_A = \mathbb{Z}^n \rtimes_A \mathbb{Z}$  is arithmetic?

We develop techniques that address Question 3, and use them to prove the next two theorems, which display an interesting contrast in behavior.

**Theorem C** (Fully irreducible, arithmetic examples in high dimension). *For each  $k \geq 1$ , there exists  $n \geq k$  and a fully irreducible  $A \in \mathrm{GL}_n(\mathbb{Z})$  so that  $\mathbb{Z}^n \rtimes_A \mathbb{Z}$  is arithmetic.*

**Theorem D** (No irreducible, arithmetic examples in prime dimension). *Fix a prime  $p \geq 5$ . There does not exist a hyperbolic, irreducible  $A \in \mathrm{GL}_p(\mathbb{Z})$  so that  $\mathbb{Z}^p \rtimes_A \mathbb{Z}$  is arithmetic.*

For example, the matrix  $A_2$  in (1) is irreducible, so  $\Gamma_{A_2}$  is not arithmetic by Theorem D.

For each  $n$  as in Theorem C, our proof shows that there are infinitely many commensurability classes of arithmetic groups  $\mathbb{Z}^n \rtimes_A \mathbb{Z}$  with  $A \in \mathrm{GL}_n(\mathbb{Z})$  fully irreducible. Theorem C becomes easier if we replace “fully irreducible” by “irreducible” since then one can use examples similar to the matrix  $A_1$  in (1), which is irreducible but not fully irreducible.

**Remark 4.** For a lattice  $\Gamma$  in a real, semisimple Lie group  $G$ , much is known about the arithmeticity question, especially from the work of Margulis on superrigidity [Mar91]. Margulis proved that any irreducible lattice is arithmetic if  $\mathrm{rank}_{\mathbb{R}}(G) \geq 2$ . He also proved that arithmeticity is characterized in terms of the commensurator  $\mathrm{Comm}(\Gamma)$ : a lattice  $\Gamma$  in a semisimple Lie group is arithmetic if and only if  $\Gamma$  has infinite index in  $\mathrm{Comm}(\Gamma)$ . The groups considered in this paper are lattices

in solvable Lie groups, and there are many differences between the solvable and semisimple cases. In particular, by work of Studenmund [Stu15, Thm. 1.2], a lattice in a solvable Lie group always has infinite index in its commensurator, independent of arithmeticity.

**Remark 5.** A matrix  $A \in \mathrm{GL}_n(\mathbb{Z})$  induces a linear automorphism of  $T^n \cong \mathbb{R}^n/\mathbb{Z}^n$ . The associated mapping torus  $E_A = \frac{T^n \times [0,1]}{(x,1) \sim (Ax,0)}$  has fundamental group  $\pi_1(E) \cong \mathbb{Z}^n \rtimes_A \mathbb{Z}$ , and  $E_A$  fibers as a  $T^n$ -bundle  $E_A \rightarrow S^1$  with monodromy  $A$ . Reducibility properties of  $A$  translate to reducibility of the bundle  $E_A \rightarrow S^1$  in an obvious way. For example, by Theorem B if  $\mathbb{Z}^n \rtimes_A \mathbb{Z}$  is arithmetic and  $A$  has real eigenvalues, then  $E_A$  has a finite cover  $E \rightarrow E_A$  whose induced bundle  $E \rightarrow S^1$  decomposes as a fiberwise product of  $T^2$  bundles. In particular, arithmeticity puts a strong constraint on the topology of the bundle when the eigenvalues are real. This topological interpretation was one of the original motivations for this paper.

**Remark 6.** Question 1 is a variant of—and is motivated by—an open problem in the study of hyperbolic 3-manifolds, where one considers bundles  $E_\phi \rightarrow S^1$  with fiber a surface  $\Sigma$  and pseudo-Anosov monodromy  $\phi \in \pi_0 \mathrm{Homeo}(\Sigma)$ . In this setting, Thurston proved that  $E_\phi$  admits a complete hyperbolic metric (unique by Mostow rigidity), and one can ask for a characterization of those  $\phi$  for which  $\pi_1(E_\phi)$  is arithmetic (in  $\mathrm{PSL}_2(\mathbb{C})$ ). This question seems to be wide open, except for a computer-assisted computation of Bowditch–Maclachlan–Reid [BMR95] that gives a complete list of the arithmetic monodromies when  $\Sigma = T^2 \setminus \{\mathrm{pt}\}$  is a punctured torus.

**Techniques.** A central theme in the proofs of Theorems B–D is that various problems (such as arithmeticity of  $\Gamma_A$ , full irreducibility of  $A$ , or computing the rank of  $S(\mathbb{Z})$  appearing in Theorem A) can be translated into problems about algebraic tori and their character groups. The character group  $X(T) := \mathrm{Hom}(T, G_m)$  of an algebraic torus  $T$  carries an action of the Galois group of the splitting field of  $T$ , and this enables the use of Galois theory and representation theory to find examples with certain properties or prove that certain examples don't exist. Our proof of Theorem C relies on the existence of number fields with Galois group  $\mathrm{Gal}(P/\mathbb{Q})$  isomorphic to the symmetric group and complex conjugation acting as a transposition. The existence of these number fields is ensured by known instances of the inverse Galois problem. The proof of Theorem D uses the classification of transitive permutation groups of prime degree and the representation theory of metacyclic groups.

The main novelty of this paper is in the variety of techniques used to study Questions 1 and 3. These techniques, while well-known, connect algebraic groups and number theory to the study of solvable lattices in a new way.

**Section outline.** Sections 2, 3, and 5 contain background material: §2 on the group theory of  $\mathbb{Z}^n \rtimes_A \mathbb{Z}$ ; §3 on the conjugacy classification for hyperbolic, semisimple

elements of  $\mathrm{GL}_n(\mathbb{Z})$ ; and §5 on algebraic tori. Theorems A and B are proved in §4 and §6, respectively. The final section §7 contains proofs of Theorems C and D as well as an example illustrating Theorem C.

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## 2 Group theory of $\mathbb{Z}^n \rtimes_A \mathbb{Z}$

In this section we collect some well-known facts about the groups  $\Gamma_A = \mathbb{Z}^n \rtimes_A \mathbb{Z}$ , their isomorphism classes, and their finite-index subgroups.

**Lemma 7** (Isomorphism classes). *Fix hyperbolic matrices  $A, B \in \mathrm{GL}_n(\mathbb{Z})$ . Then  $\Gamma_A \cong \Gamma_B$  if and only if  $A$  is conjugate in  $\mathrm{GL}_n(\mathbb{Z})$  to one of  $B, B^{-1}$ .*

*Proof.* We write elements of  $\Gamma_A$  as pairs  $(x, i) \in \mathbb{Z}^n \times \mathbb{Z}$  with multiplication

$$(x, i)(y, j) = (x + A^i y, i + j).$$

If  $B = CAC^{-1}$ , it is easy to check that  $(x, i) \mapsto (Cx, i)$  defines an isomorphism  $\Gamma_A \rightarrow \Gamma_B$ . If  $B^{-1} = CAC^{-1}$ , then we conclude that  $\Gamma_A \cong \Gamma_B$  using the above argument together with the fact that  $\Gamma_B \cong \Gamma_{B^{-1}}$  for any  $B$ . The latter isomorphism is easy to see from the point-of-view of mapping tori since the map  $T^n \times [0, 1] \rightarrow T^n \times [0, 1]$  defined by  $(\theta, t) \mapsto (\theta, 1 - t)$  descends to a homeomorphism  $E_B \cong E_{B^{-1}}$ .

For the converse, suppose that  $\Phi : \mathbb{Z}^n \rtimes_A \mathbb{Z} \rightarrow \mathbb{Z}^n \rtimes_B \mathbb{Z}$  is an isomorphism. First we show that  $\Phi(\mathbb{Z}^n) = \mathbb{Z}^n$ . Suppose that  $(x, i) \in \Phi(\mathbb{Z}^n)$ . Then also  $(Bx, i) \in \Phi(\mathbb{Z}^n)$  because  $\Phi(\mathbb{Z}^n)$  is normal in  $\Gamma_B$ , and so  $(Bx - x, 0) = (Bx, i)(x, i)^{-1}$  is also in  $\Phi(\mathbb{Z}^n)$ . The vector  $y := Bx - x$  is nonzero because  $B$  is hyperbolic. Now since  $\Phi(\mathbb{Z}^n)$  is abelian,

$$(0, 0) = (x, i)(y, 0)(x, i)^{-1}(y, 0)^{-1} = (B^i y - y, 0).$$

This implies that  $i = 0$ , again since  $B$  is hyperbolic. Hence  $\Phi$  restricts to  $C : \mathbb{Z}^n \rightarrow \mathbb{Z}^n$  for some  $C \in \mathrm{GL}_n(\mathbb{Z})$ .

Next write  $\Phi(0, 1) = (z, j)$ . Computing  $\Phi$  on  $(0, 1)(x, 0)(0, -1)$  in two ways, we find that  $B^j Cx = CAx$  for all  $x \in \mathbb{Z}^n$ , which implies that  $B^j = CAC^{-1}$ . Here  $j = \pm 1$  because  $\Phi(-C^{-1}z, 1) = (0, j)$ , which implies that  $(-C^{-1}z, 1)$  has a  $j$ -th root, so  $j$  divides 1.  $\square$

**Commensurability.** Recall that groups  $\Gamma_1, \Gamma_2$  are *commensurable* if there is a group  $\Gamma_3$  that embeds as a finite-index subgroup  $\Gamma_3 \hookrightarrow \Gamma_i$  for  $i = 1, 2$ .

It is easy to show that any finite-index subgroup of  $\mathbb{Z}^n \rtimes_A \mathbb{Z}$  is abstractly isomorphic to  $L \rtimes_{A^k} \mathbb{Z}$ , where  $L \subset \mathbb{Z}^n$  is an  $A^k$ -invariant sublattice.

We say that  $\Gamma_{A_1}$  and  $\Gamma_{A_2}$  are *fiberwise commensurable* if there exists  $A_i$ -invariant lattices  $L_i \subset \mathbb{Z}^n$  so that  $L_1 \rtimes_{A_1} \mathbb{Z} \cong L_2 \rtimes_{A_2} \mathbb{Z}$ . Using Lemma 7, it is equivalent to say that the action  $A_1 \curvearrowright L_1$  is isomorphic to either  $A_2 \curvearrowright L_2$  or  $A_2^{-1} \curvearrowright L_2$ . (We use the terminology *fiberwise commensurable* because this definition is the group-theoretic version of the existence of a common fiberwise cover for the mapping tori  $E_{A_1}$  and  $E_{A_2}$ .)

Fiberwise commensurability can be defined generally for semi-direct products, but it has a special property for the groups we're studying.

**Lemma 8.** *Fix  $A_1, A_2 \in \mathrm{GL}_n(\mathbb{Z})$  hyperbolic. If  $\Gamma_{A_1}$  and  $\Gamma_{A_2}$  are fiberwise commensurable, then  $\Gamma_{A_1}$  embeds as a finite-index subgroup of  $\Gamma_{A_2}$  (and vice versa).*

*Proof.* First observe that if  $L \subset \mathbb{Z}^n$  is an  $A$ -invariant lattice, then  $\Gamma_A$  is a finite-index subgroup of  $L \rtimes_A \mathbb{Z}$  (note that the other containment is obvious). To see this, choose  $c \gg 0$  so that  $c\mathbb{Z}^n \subset L \subset \mathbb{Z}^n$ . Then  $c\mathbb{Z}^n$  is also  $A$ -invariant, and  $\Gamma_A = \mathbb{Z}^n \rtimes_A \mathbb{Z} \cong c\mathbb{Z}^n \rtimes_A \mathbb{Z}$  is a finite-index subgroup of  $L \rtimes_A \mathbb{Z}$ .

Consequently, if  $\Gamma_{A_1}, \Gamma_{A_2}$  are fiberwise commensurable with  $L_1, L_2 \subset \mathbb{Z}^n$  as in the definition, then one obtains an inclusion of finite-index subgroups

$$\Gamma_{A_1} \hookrightarrow L_1 \rtimes_{A_1} \mathbb{Z} \cong L_2 \rtimes_{A_2} \mathbb{Z} \hookrightarrow \Gamma_{A_2}. \quad \square$$

### 3 Number-theoretic construction of integer matrices

In this section we recall the conjugacy classification of semisimple, hyperbolic elements of  $\mathrm{GL}_n(\mathbb{Z})$ . This is needed for the proofs of Theorems A, B, D. As a consequence of the classification, if  $A \in \mathrm{GL}_n(\mathbb{Z})$  has eigenvalues  $\lambda_1, \dots, \lambda_m$  with multiplicities  $n_1, \dots, n_m$  as in the statement of Theorem A, then one can construct a finitely-generated abelian group  $M \subset \bigoplus \mathbb{Q}(\lambda_i)^{n_i}$  that is invariant under the diagonal action of  $\lambda = (\lambda_1, \dots, \lambda_m)$  so that  $\mathbb{Z}^n \rtimes_A \mathbb{Z}$  is isomorphic to  $M \rtimes_\lambda \mathbb{Z}$  (in the simplest example,  $M = \bigoplus \mathcal{O}_i^{n_i}$ , where  $\mathcal{O}_i \subset \mathbb{Q}(\lambda_i)$  is the ring of integers).

The focus on this section is the following result that classifies hyperbolic, semisimple matrices  $A$  whose characteristic polynomial  $\chi(A)$  is fixed.

**Theorem 9** (Latimer–MacDuffee, Wallace, Husert). *Fix  $d_1, n_1, \dots, d_m, n_m \geq 1$  and set  $n = \sum_{i=1}^m d_i n_i$ . For each  $i$ , fix an algebraic unit  $\lambda_i$  with minimal polynomial  $\mu_i$*

of degree  $d_i$ . Assume that no roots of  $\mu_i$  lie on the unit circle and that  $\mu_i \neq \mu_j$  for  $i \neq j$ . Then there is a bijection

$$\left\{ \begin{array}{l} \text{conjugacy classes of} \\ \text{semisimple } A \in \mathrm{GL}_n(\mathbb{Z}) \\ \text{with } \chi(A) = \mu_1^{n_1} \cdots \mu_m^{n_m} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{module classes of finitely-generated} \\ \text{full } \mathbb{Z}[\lambda_1] \times \cdots \times \mathbb{Z}[\lambda_m]\text{-modules} \\ M \subset \mathbb{Q}(\lambda_1)^{n_1} \oplus \cdots \oplus \mathbb{Q}(\lambda_m)^{n_m} \end{array} \right\}.$$

The term “full” is defined in the second bullet below. For the meaning of the equivalence relation “module classes” we refer the reader to [HKF17, §1.1]; this notion will not play an explicit role in what follows. Theorem 9 is due in various forms to Latimer–MacDuffee [LM33], Wallace [Wal84], and Huser [HKF17]. See [HKF17, Thm. 1.4] for the statement given above.

We explain how the bijection works, starting with two special cases.

- When  $m = 1$  and  $n_1 = 1$  the theorem classifies  $A \in \mathrm{GL}_n(\mathbb{Z})$  with a fixed irreducible characteristic polynomial  $\mu$  (none of whose roots lie on the unit circle). If  $\lambda$  is a root of  $\mu$ , then there is a bijection

$$\left\{ \begin{array}{l} \text{conjugacy classes} \\ \text{of } A \in \mathrm{GL}_n(\mathbb{Z}) \\ \text{with } \chi(A) = \mu \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{Ideal classes of} \\ \text{nonzero fractional ideals} \\ I \subset \mathbb{Q}(\lambda) \text{ of } \mathbb{Z}[\lambda] \end{array} \right\}.$$

See [Wal84, Thm. 2]. For the bijection, in one direction, given  $A \in \mathrm{GL}_n(\mathbb{Z})$ , by basic linear algebra, one can find an eigenvector  $Aw = \lambda w$  such that  $w \in \mathbb{Q}(\lambda)^n$ . If we write  $w = (w_1, \dots, w_n)$  in coordinates, then

$$I = \mathrm{span}_{\mathbb{Z}}\{w_1, \dots, w_n\} \subset \mathbb{Q}(\lambda)$$

is a fractional ideal of  $\mathbb{Z}[\lambda]$  (the equation  $Aw = \lambda w$  gives a way to rewrite  $\lambda w_i$  as a  $\mathbb{Z}$ -linear combination of  $w_1, \dots, w_n$ ). For the other direction, given  $I \subset \mathbb{Q}(\lambda)$ , choose a  $\mathbb{Z}$ -basis  $w_1, \dots, w_n \in I$ . Then  $\lambda w_1, \dots, \lambda w_n \in I$  is also a  $\mathbb{Z}$ -basis for  $I$ . Take  $A \in \mathrm{GL}_n(\mathbb{Z})$  the matrix of the transformation  $I \rightarrow I$  taking  $w_i$  to  $\lambda w_i$  (with respect to the  $w_i$ -basis). See [Wal84] for more details.

- When  $m = 1$  and  $n_1 \geq 1$ , the theorem classifies semisimple  $A \in \mathrm{GL}_n(\mathbb{Z})$  with  $\chi(A) = \mu^{n_1}$  with  $\mu$  irreducible over  $\mathbb{Q}$  (the degree of  $\mu$  is  $d_1$  and  $n = d_1 n_1$ ). If  $\lambda$  is a root of  $\mu$ , then there is a bijection

$$\left\{ \begin{array}{l} \text{conjugacy classes of} \\ \text{semisimple } A \in \mathrm{GL}_n(\mathbb{Z}) \\ \text{with } \chi(A) = \mu^{n_1} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{Module classes of} \\ \text{finitely-generated, full} \\ \mathbb{Z}[\lambda]\text{-modules } M \subset \mathbb{Q}(\lambda)^{n_1} \end{array} \right\}.$$

A  $\mathbb{Z}[\lambda]$ -module  $M \subset \mathbb{Q}(\lambda)^{n_1}$  is called *full* if its  $\mathbb{Q}$ -span is all of  $\mathbb{Q}(\lambda)^{n_1}$ . For the bijection, given  $A \in \mathrm{GL}_n(\mathbb{Z})$ , choose linearly independent vectors

$w^{(1)}, \dots, w^{(n_1)} \in \mathbb{Q}(\lambda)^n$  such that  $Aw^{(j)} = \lambda w^{(j)}$  for each  $1 \leq j \leq n_1$ . Next form an  $(n \times n_1)$ -matrix whose  $(i, j)$ -entry is  $w_i^{(j)}$ , the  $i$ -th coordinate of  $w^{(j)}$ . Then the rows  $w_i = (w_i^{(1)}, \dots, w_i^{(n_1)}) \in \mathbb{Q}(\lambda)^{n_1}$  generate a full  $\mathbb{Z}[\lambda]$ -module  $M = \mathbb{Z}\{w_1, \dots, w_n\}$ . Conversely, given a full  $\mathbb{Z}[\lambda]$ -module  $M \subset \mathbb{Q}(\lambda)^{n_1}$ , choose a basis  $M = \mathbb{Z}\{w_1, \dots, w_n\}$ , and take  $A$  to be the matrix of multiplication by  $\lambda$  on  $M$  with respect to the given basis. See [HKF17, §1.2] for more details.

For the general case of Theorem 9, the bijection works similarly. Given semisimple  $A \in \mathrm{GL}_n(\mathbb{Z})$  with  $\chi(A) = \mu_1^{n_1} \cdots \mu_m^{n_m}$ , conjugate  $A$  to a block diagonal matrix  $(A_1, \dots, A_m)$ , where  $\chi(A_i) = \mu_i^{n_i}$ . Then repeat the construction of the preceding paragraph for each  $A_i$  to get a full  $\mathbb{Z}[\lambda_i]$ -module  $M_i \subset \mathbb{Q}(\lambda_i)^{n_i}$ . Then set  $M = M_1 \oplus \cdots \oplus M_m$ . The construction in the reverse direction is also similar to what was discussed above.

We summarize the above discussion with the following corollary, which will be used later.

**Corollary 10.** *Given semisimple, hyperbolic  $A \in \mathrm{GL}_n(\mathbb{Z})$  with  $\chi(A) = \mu_1^{n_1} \cdots \mu_m^{n_m}$ , there exists a full submodule  $M \subset \mathbb{Q}(\lambda_1)^{n_1} \oplus \cdots \oplus \mathbb{Q}(\lambda_m)^{n_m}$ , where  $\lambda_i$  is a root of  $\mu_i$ , so that the action of  $A$  on  $\mathbb{Z}^n$  is isomorphic (as  $\mathbb{Z}[\mathbb{Z}]$ -modules) to the diagonal action of  $(\lambda_1, \dots, \lambda_m)$  on  $M \cong \mathbb{Z}^n$ .*

## 4 Proof of Theorem A

Theorem A is proved in two steps that are carried out in §4.1 and §4.2.

Step 1. We show that if  $A, B \in \mathrm{GL}_n(\mathbb{Z})$  are semisimple with the same characteristic polynomial, then  $\Gamma_A$  is arithmetic if and only if  $\Gamma_B$  is arithmetic. This allows us to reduce the proof of Theorem A to the case  $\Gamma_\lambda = \bigoplus \mathcal{O}_i^{n_i} \rtimes_\lambda \mathbb{Z}$ , where  $\lambda$  and  $n_i$  are as in the statement of Theorem A and  $\mathcal{O}_i \subset \mathbb{Q}(\lambda_i)$  is the ring of integers.

Step 2. We solve the arithmeticity problem for  $\Gamma_\lambda = \bigoplus \mathcal{O}_i^{n_i} \rtimes_\lambda \mathbb{Z}$ . The case when  $m = 1$  and  $n_1 = 1$ , i.e.  $\Gamma = \mathcal{O}_K \rtimes_\lambda \mathbb{Z}$  for some  $\lambda \in \mathcal{O}_K^\times$ , was solved by Grunewald–Platonov [GP98], and we adapt their argument to the general case.

### 4.1 Characteristic polynomial and fiberwise commensurability

Fix  $A \in \mathrm{GL}_n(\mathbb{Z})$  with  $\lambda = (\lambda_1, \dots, \lambda_m)$  and  $n_1, \dots, n_m$  as in the statement of Theorem A. By Corollary 10, the action of  $A$  on  $\mathbb{Z}^n$  has the same characteristic polynomial as the action of  $\lambda = (\lambda_1, \dots, \lambda_m)$  on  $M = \bigoplus \mathcal{O}_i^{n_i}$ , where  $\mathcal{O}_i \subset \mathbb{Q}(\lambda_i)$  is the ring of integers. By the following proposition,  $\mathbb{Z}^n \rtimes_A \mathbb{Z}$  is arithmetic if and only if  $\Gamma_\lambda = M \rtimes_\lambda \mathbb{Z}$  is arithmetic.



**Lemma 11** (Arithmeticity depends only on characteristic polynomial). *Fix semisimple, hyperbolic  $A, B \in \mathrm{GL}_n(\mathbb{Z})$ . If  $\chi(A) = \chi(B)$ , then*

- (i) *the groups  $\Gamma_A$  and  $\Gamma_B$  are fiberwise commensurable, and*
- (ii) *the group  $\Gamma_A$  is arithmetic if and only if  $\Gamma_B$  is arithmetic.*

*Proof.* First we note that (i) implies (ii): by Lemma 8, if  $\Gamma_A$  and  $\Gamma_B$  are fiberwise commensurable, then  $\Gamma_A$  is a finite-index subgroup of  $\Gamma_B$ , and vice versa. This implies (ii) since arithmeticity is obviously inherited by finite-index subgroups.

Proof of (i): To show  $\Gamma_A$  and  $\Gamma_B$  are fiberwise commensurable, write  $\chi = \mu_1^{n_1} \cdots \mu_m^{n_m}$  for the common characteristic polynomial. Choose a root  $\lambda_i$  of  $\mu_i$  for each  $i$ . From Theorem 9 and Corollary 10, there exist  $\prod \mathbb{Z}[\lambda_i]$ -modules  $M_A, M_B \subset \bigoplus \mathbb{Q}(\lambda_i)^{n_i}$  so that  $M_A \rtimes_{\lambda} \mathbb{Z} \cong \Gamma_A$  and  $M_B \rtimes_{\lambda} \mathbb{Z} \cong \Gamma_B$ . The intersection  $M_A \cap M_B$  is also a full  $\prod \mathbb{Z}[\lambda_i]$ -module, so  $(M_A \cap M_B) \rtimes_{\lambda} \mathbb{Z}$  is a finite index subgroup of both of  $M_A \rtimes_{\lambda} \mathbb{Z}$  and  $M_B \rtimes_{\lambda} \mathbb{Z}$ . This shows  $\Gamma_A, \Gamma_B$  are fiberwise commensurable, as desired.  $\square$

**Remark 12.** We note for later use that the following converse of Lemma 11 is also true: if  $\Gamma_A$  and  $\Gamma_B$  are fiberwise commensurable, then  $\chi(A) = \chi(B)$ . This follows quickly from the fact that if  $L \subset \mathbb{Z}^n$  is an  $A$ -invariant lattice, then the linear maps  $A : \mathbb{Z}^n \rightarrow \mathbb{Z}^n$  and  $A : L \rightarrow L$  have the same characteristic polynomial.

## 4.2 Quasi-split tori and arithmeticity

We prove Theorem A for  $\Gamma_{\lambda} = \bigoplus \mathcal{O}_i^{n_i} \rtimes_{\lambda} \mathbb{Z}$ , where  $\lambda = (\lambda_1, \dots, \lambda_m)$  and  $\mathcal{O}_i \subset \mathbb{Q}(\lambda_i)$  is the ring of integers. For notational simplicity, we set  $M = \bigoplus \mathcal{O}_i^{n_i}$  and  $K_i = \mathbb{Q}(\lambda_i)$ .

Consider the algebraic group

$$T = \prod R_{K_i/\mathbb{Q}}(G_m),$$

where  $G_m$  is the multiplicative group, and  $R_{K_i/\mathbb{Q}}(\cdot)$  is the restriction of scalars functor (see e.g. [PR94, §2.1.2]). Then  $T$  is defined over  $\mathbb{Q}$ , and

$$T(\mathbb{Q}) = K_1^{\times} \times \cdots \times K_m^{\times} \quad \text{and} \quad T(\mathbb{Z}) = \mathcal{O}_1^{\times} \times \cdots \times \mathcal{O}_m^{\times}.$$

In particular,  $\lambda \in T(\mathbb{Z})$ . Let  $S \subset T$  be the Zariski closure of the subgroup  $\langle \lambda \rangle \subset T$ . In Proposition 13 below we prove that  $\Gamma_{\lambda}$  is arithmetic if and only if  $S(\mathbb{Z})$  has rank 1. This will finish the proof of Theorem A.

Terminology: If  $\Lambda \subset S(\mathbb{Q})$  is commensurable with  $S(\mathbb{Z})$ , we say  $\Lambda$  is an *arithmetic subgroup of  $S$* .

**Proposition 13** (Arithmeticity criterion). *Fix  $\Gamma_{\lambda} = M \rtimes_{\lambda} \mathbb{Z}$  and  $T$  as above. Set  $\Lambda = \langle \lambda \rangle \subset T(\mathbb{Z})$ , and let  $S \subset T$  be the Zariski closure of  $\Lambda$ . Then  $\Gamma_{\lambda}$  is arithmetic if and only if  $\Lambda$  is an arithmetic subgroup of  $S$ .*

In the situation of Proposition 13,  $\Lambda \cong \mathbb{Z}$  is a subgroup of  $S(\mathbb{Z})$ , which is abelian, so  $\Lambda \subset S$  is an arithmetic subgroup if and only if  $\text{rank } S(\mathbb{Z}) = 1$ . Thus the conclusion of Proposition 13 gives us the desired conclusion for Theorem A.

In the case  $m = 1$  and  $n_1 = 1$ , i.e.  $\Gamma_\lambda = \mathcal{O}_K \rtimes_\lambda \mathbb{Z}$  with  $\lambda \in \mathcal{O}_K^\times$ , the proof of Proposition 13 is given in [GP98, Prop. 3.1]. The argument in the general case is a straightforward generalization, as we explain next. At times we refer to [GP98] for further details.

*Proof of Proposition 13.* The “if” statement is easy. If  $\Lambda \subset S$  is an arithmetic subgroup, then  $\Gamma_\lambda$  is an arithmetic subgroup in  $\prod R_{K_i/\mathbb{Q}}(G_a)^{n_i} \rtimes S$ , where  $G_a$  denotes the additive group.

Now we prove the “only if” statement. Assuming that  $\Gamma_\lambda$  is arithmetic, there exists a solvable algebraic group  $H$  defined over  $\mathbb{Q}$  and an embedding  $j : \Gamma_\lambda \hookrightarrow H(\mathbb{Q})$  whose image  $\Gamma = j(\Gamma_\lambda)$  is commensurable to  $H(\mathbb{Z})$ . By [GP98, Thm. 2.2], we can also assume that  $H$  is *reduced*; see [GP98, §2] for the definition.

Note: in general a solvable group  $\Gamma$  may be realized in as a lattice in different  $H$  (this is already true for  $\Gamma = \mathbb{Z}$ ). The proof identifies  $H$  by determining its unipotent radical and maximal torus (at least in the case that  $H$  is reduced).

*Claim 1.* The group  $j(\Lambda)$  is an arithmetic subgroup of its Zariski closure  $S_0 \subset H$ .

Here we are identifying  $\Lambda = \langle \lambda \rangle$  with the obvious  $\mathbb{Z}$  subgroup of  $\Gamma_\lambda = M \rtimes_\lambda \mathbb{Z}$ .

*Proof of Claim 1.* First observe that  $\Gamma \cap S_0$  is an arithmetic subgroup of  $S_0$  (this follows easily from the definition of arithmeticity), so it suffices to show that  $j(\Lambda) = \Gamma \cap S_0$ . Note that  $S_0$  is abelian. In addition  $\Lambda \subset \Gamma_\lambda$  is a maximal abelian subgroup because no coordinate of  $\lambda = (\lambda_1, \dots, \lambda_m)$  is a root of unity. Since  $\Gamma \cap S_0$  is an abelian subgroup of  $\Gamma$  and contains  $j(\Lambda)$ , maximality of  $j(\Lambda)$  implies that  $\Gamma \cap S_0 = j(\Lambda)$ , proving Claim 1.

Ultimately, we will identify  $S_0 \subset H$  with  $S \subset T$ . In order to do this, we use the unipotent radical  $U(H) \subset H$ . As before  $\Gamma \cap U(H)$  is an arithmetic subgroup of  $U(H)$ .

*Claim 2.* The group  $\Gamma \cap U(H)$  is equal to  $j(M)$ .

Before explaining the claim, recall that the *fitting subgroup* of a polycyclic group  $\Gamma$  is the unique maximal normal nilpotent subgroup (for  $\Gamma_\lambda = M \rtimes_\lambda \mathbb{Z}$ , the fitting subgroup is  $M$ ).

*Proof of Claim 2.* The fact that  $H$  is reduced implies that  $\Gamma \cap U(H)$  is the fitting subgroup of  $\Gamma$  (see [GP98, Lem. 2.1]), and the fitting subgroup of  $\Gamma$  is equal to  $j(M)$ .

Next we identify  $U(H)$ , using that  $j(M) = \Gamma \cap U(H)$  is an arithmetic subgroup. Choosing a  $\mathbb{Z}$ -basis for  $M = \bigoplus \mathcal{O}_i^{n_i}$  gives a  $\mathbb{Q}$ -basis for  $M_{\mathbb{Q}} := \bigoplus K_i^{n_i}$  and also

for the Lie algebra  $\mathfrak{u}(H)(\mathbb{Q})$ , yielding an isomorphism  $M_{\mathbb{Q}} \cong U(H)(\mathbb{Q})$  as in the diagram below.

$$\begin{array}{ccc} M & \xrightarrow{j} & U(H)(\mathbb{Q}) \\ \downarrow & & \downarrow \exp^{-1} \\ M_{\mathbb{Q}} & \xrightarrow{\Theta} & \mathfrak{u}(H)(\mathbb{Q}) \end{array}$$

The exponential map here is an isomorphism; see [Seg83, §6.A]. Let  $\beta : T(\mathbb{Q}) \rightarrow \mathrm{GL}_n(\mathbb{Q})$  be the morphism induced by the action of  $\prod K_i^{\times}$  on  $M_{\mathbb{Q}} \cong \mathbb{Q}^n$  (using the chosen basis for  $M$ ). Via  $\Theta$ , we can identify this with the adjoint action of  $j(\Lambda)$  on  $\mathfrak{u}(H)$ :

$$\begin{array}{ccc} \Lambda & \xrightarrow{j} & j(\Lambda) \subset S_0 \\ \downarrow & & \downarrow \text{adjoint} \\ T(\mathbb{Q}) & \xrightarrow{\beta} & \mathrm{GL}_n(\mathbb{Q}) \end{array}$$

Define  $S_1 = \text{adjoint}(S_0)$ . Then  $\text{adjoint}(j(\Lambda)) \subset S_1$  is Zariski dense, and  $\text{adjoint}(j(\Lambda))$  is an arithmetic subgroup of  $S_1$  because  $j(\Lambda) \subset S_0$  is an arithmetic subgroup [Bor66, Thm. 6]. Since  $\beta$  (as multiplication of  $\prod K_i^{\times}$  on  $\bigoplus K_i^{n_i}$ ) is injective and the diagram commutes, we may identify  $S_1$  with a subgroup of  $T$ . Under this identification,  $\Lambda \subset S_1$  is an arithmetic subgroup. Since  $S \subset T$  is the Zariski closure of  $\Lambda$ , we conclude  $S = S_1$ , so  $\Lambda \subset S$  is an arithmetic subgroup. This completes the proof of the proposition and the proof of Theorem A.  $\square$

## 5 Algebraic tori, character groups, and integral points

In this section we recall some facts about algebraic tori that will be needed in Sections 6 and 7. Our main reference is [PR94].

**Algebraic tori and their character groups.** An *algebraic torus* over  $\mathbb{Q}$  is an algebraic group that is isomorphic to  $(G_m)^r$  over  $\bar{\mathbb{Q}}$ , where  $r = \dim T$ . The *character group*  $X(T) := \mathrm{Hom}(T, G_m)$  of  $T$  is isomorphic to  $\mathbb{Z}^r$  and has the structure of a  $\mathbb{Z}[\mathbb{G}]$ -module, where  $\mathbb{G} = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$  is the absolute Galois group. The functor  $T \mapsto X(T)$  defines a contravariant equivalence of categories between (algebraic tori defined over  $\mathbb{Q}$ ) and (finitely-generated free-abelian  $\mathbb{Z}[\mathbb{G}]$ -modules) [PR94, Thm. 2.1]. For example, a surjection  $X(T) \twoheadrightarrow X(S)$  between character groups is induced by an embedding  $S \hookrightarrow T$ , and conversely.

The action of  $\mathbb{G}$  on  $X(T)$  defines a homomorphism  $\rho : \mathbb{G} \rightarrow \mathrm{GL}_r(\mathbb{Z})$ . The fixed field of  $\ker(\rho) \subset \mathbb{G}$  is a finite Galois extension  $P/\mathbb{Q}$ , which is the *splitting field* of  $T$ . It is the smallest field that satisfies the following equivalent properties (see [PR94, §2.1.7]):

- (i) The torus  $T$  is  $K$ -split, i.e. there is an isomorphism  $T \cong (G_m)^r$  defined over  $K$ .
- (ii) Every character  $T \rightarrow G_m$  is defined over  $K$ .

The image  $\text{Im}(\rho) \cong \mathbb{G}/\ker(\rho)$  is isomorphic to  $\text{Gal}(P/\mathbb{Q})$ . In particular, if  $P$  is the splitting field for  $T$ , then  $\text{Gal}(P/\mathbb{Q})$  acts faithfully on  $X(T)$ .

Recall that a map of algebraic groups  $T_1 \rightarrow T_2$  is an *isogeny* if it is surjective with finite kernel. If  $T_1, T_2$  are tori, there is an isogeny between them if and only if  $X(T_1) \otimes \mathbb{Q} \cong X(T_2) \otimes \mathbb{Q}$  as  $\mathbb{Q}[\mathbb{G}]$ -modules. In particular, isogeny of algebraic tori is an equivalence relation [PR94, §2.1.7].

**Quasi-split tori.** A torus is called *quasi-split* if it is a finite product of tori of the form  $R_{K/\mathbb{Q}}(G_m)$ . The field norm  $N_{K/\mathbb{Q}} : K \rightarrow \mathbb{Q}$  defines a character  $N : R_{K/\mathbb{Q}}(G_m) \rightarrow G_m$  that's defined over  $\mathbb{Q}$ , and its kernel is called the *norm torus*, denoted  $R_{K/\mathbb{Q}}^1(G_m)$ . The norm torus has  $\mathbb{Q}$ -rank 0, and its  $\mathbb{Z}$ -points is equal to the units in  $K$  of norm 1, denoted  $\mathcal{O}_K^1$ .

Quasi-split tori are characterized by the property that their character groups are permutation modules. If  $T = R_{K/\mathbb{Q}}(G_m)$  and  $P$  is the Galois closure of  $K/\mathbb{Q}$ , then  $X(T) \cong \mathbb{Z}[G/H]$ , where  $G = \text{Gal}(P/\mathbb{Q})$  and  $H = \text{Gal}(P/K)$ . Furthermore, the character group  $X(T^1)$  of the norm torus  $T^1 \subset T$  is isomorphic to the quotient  $\mathbb{Z}[G/H]/\mathbb{Z}$  by the trivial sub-representation [PR94, §2.1.7].

A decomposition of  $\mathbb{Q}[G/H]$  as a  $G$ -representation leads to a decomposition of  $T = R_{K/\mathbb{Q}}(G_m)$ , up to isogeny. More precisely, choose a decomposition  $\mathbb{Q}[G/H] \cong \bigoplus X_{i,\mathbb{Q}}$  into irreducible representations of  $G$ , and define  $X_i$  as the image of  $\mathbb{Z}[G/H]$  under the projection  $\mathbb{Q}[G/H] \rightarrow X_{i,\mathbb{Q}}$ . This leads to a commutative diagram

$$\begin{array}{ccc}
 X(T) & \xrightarrow{\phi} & \bigoplus X_i \\
 \downarrow & & \downarrow \\
 X(T) \otimes \mathbb{Q} & \xrightarrow{\cong} & \bigoplus X_{i,\mathbb{Q}}
 \end{array}$$

From the diagram, one finds that  $\phi$  is injective and has finite cokernel. It follows that  $T$  is isogenous to  $\prod T_i$ , where  $T_i$  is the torus with  $X(T_i) \cong X_i$ .

**Integer points of a torus.** In order to apply the arithmeticity criterion Proposition 13, we need to be able to determine the rank of  $S(\mathbb{Z})$  as an abelian group for an arbitrary torus  $S$ . This rank is given by the following formula, which generalizes Dirichlet's unit theorem [PR94, §4.5]

$$\text{rank } S(\mathbb{Z}) = \text{rank}_{\mathbb{R}}(S) - \text{rank}_{\mathbb{Q}}(S). \quad (2)$$

Recall that the  $\mathbb{Q}$ -rank is the dimension of the largest  $\mathbb{Q}$ -split subtorus, and similarly for  $\mathbb{R}$ -rank. Next we explain how to compute the  $\mathbb{R}$ - and  $\mathbb{Q}$ -ranks of  $S$  in

terms of  $X(S)$ . First observe that if  $L$  is the splitting field of  $S$ , then  $\text{rank}_{\mathbb{Q}}(S) = \text{rank } X(S)^{\text{Gal}(L/\mathbb{Q})}$  since a character is defined over  $\mathbb{Q}$  if and only if it is fixed by the action of  $\text{Gal}(L/\mathbb{Q})$ .

**Lemma 14.** *Fix an algebraic torus  $S$  defined over  $\mathbb{Q}$ , and denote its splitting field by  $L$ . Let  $\tau \in \text{Gal}(L/\mathbb{Q})$  be complex conjugation. Then the  $\mathbb{R}$ -rank of  $S$  is equal to the rank of  $X(S)^\tau$  as an abelian group.*

*Proof.* First we identify  $S$  as a torus over  $\mathbb{R}$ . Consider the action of  $\text{Gal}(\mathbb{C}/\mathbb{R}) = \langle \tau \rangle$  on  $X(S)$ . Any free-abelian  $\mathbb{Z}[\mathbb{Z}/2\mathbb{Z}]$ -module decomposes as a direct sum of copies of the trivial representation  $\mathbb{Z}$ , the sign representation  $\mathbb{Z}_-$ , and the group ring  $\mathbb{Z}[\mathbb{Z}/2\mathbb{Z}]$  (see e.g. [CR06, §74]). Hence, there is a decomposition

$$X(S) \cong \mathbb{Z}[\text{Gal}(\mathbb{C}/\mathbb{R})]^\alpha \oplus \mathbb{Z}^\beta \oplus \mathbb{Z}_-^\gamma,$$

for some  $\alpha, \beta, \gamma \geq 0$ . The rank of  $X(S)^\tau$  is equal to  $\alpha + \beta$ . Using the correspondence between  $\mathbb{Z}[\text{Gal}(\mathbb{C}/\mathbb{R})]$ -modules and tori defined over  $\mathbb{R}$ , we conclude that there is an isomorphism

$$S \cong R_{\mathbb{C}/\mathbb{R}}(G_m)^\alpha \times G_m^\beta \times R_{\mathbb{C}/\mathbb{R}}^1(G_m)^\gamma$$

defined over  $\mathbb{R}$ . (Compare with [PR94, §2.2.4].) The  $\mathbb{R}$ -ranks of  $R_{\mathbb{C}/\mathbb{R}}(G_m)$  and  $R_{\mathbb{C}/\mathbb{R}}^1(G_m)$  are 1 and 0, respectively. Thus  $\text{rank}_{\mathbb{R}}(S) = \alpha + \beta$ . This is equal to  $\text{rank } X(S)^\tau$ , as computed above, which proves the lemma.  $\square$

## 6 Proof of Theorem B

We begin with the easier direction.

(ii) implies (i). By assumption, after replacing  $A$  with a power, we have  $\chi(A) = \frac{\mu_1^{n_1} \cdots \mu_m^{n_m}}{\mu_1^{n_1} \cdots \mu_m^{n_m}}$ , where  $\mu_i$  is a degree-2 polynomial. Denoting  $\lambda_i$  a root of  $\mu_i$ , we are assuming that there exists a quadratic extension  $L/\mathbb{Q}$  and a (fundamental) unit  $\epsilon \in \mathcal{O}_L^\times$  so that  $\lambda_i = \epsilon^{\ell_i}$  for some  $\ell_i \neq 0$ .

We want to show that  $\Gamma_A$  is arithmetic. Our replacement of  $A$  with  $A^k$  at the beginning does not change arithmeticity of  $\Gamma_A$  because arithmeticity is inherited by finite-index supergroups [GP99, Thm. D]. By Corollary 10 and Lemma 11, it suffices to show arithmeticity for the group

$$(\mathcal{O}_L^{n_1} \oplus \cdots \oplus \mathcal{O}_L^{n_m}) \rtimes_{(\epsilon^{\ell_1}, \dots, \epsilon^{\ell_m})} \mathbb{Z}, \quad (3)$$

which is fiberwise commensurable to  $\Gamma_A$ . We show the group in (3) is arithmetic using Proposition 13. Setting  $S = R_{L/\mathbb{Q}}^1(G_m)$  and  $T = S^m$ .

We want to identify the Zariski closure  $\Delta$  of  $\langle (\epsilon^{\ell_1}, \dots, \epsilon^{\ell_m}) \rangle \subset T$ . It suffices to determine its character group  $X(\Delta)$ , which is the quotient of  $X(T)$  by the subgroup

$N$  of characters that vanish on  $\Delta$ , or equivalently, on  $(\epsilon^{\ell_1}, \dots, \epsilon^{\ell_m})$ . To determine  $N$ , let  $\phi \in X(S) \cong \mathbb{Z}$  be a generator. If  $p_i : S^m \rightarrow S$  is the  $i$ -th projection map and  $\phi_i = \phi \circ p_i$ , then by a simple computation one finds

$$N = \{\phi_1^{a_1} \cdots \phi_m^{a_m} \mid a_1 \ell_1 + \cdots + a_m \ell_m = 0\} \cong \mathbb{Z}^{m-1}.$$

Then  $X(\Delta) \cong \mathbb{Z}$ , and by comparing the action of  $\text{Gal}(L/\mathbb{Q})$  on  $X(\Delta)$  and  $S$ , we conclude that  $\Delta \cong S$ . Thus  $\text{rank } \Delta(\mathbb{Z}) = \text{rank } S(\mathbb{Z}) = 1$ . Proposition 13 implies that the group in (3) is arithmetic, and so  $\Gamma_A$  is also arithmetic.

(i) implies (ii). Fix  $A \in \text{GL}_n(\mathbb{Z})$  such that  $\Gamma_A$  is arithmetic, and assume all the eigenvalues of  $A$  are real. We want to show that there is  $k \geq 0$  and a quadratic extension  $L/\mathbb{Q}$  so that the eigenvalues of  $A^k$  are powers of a fundamental unit  $\epsilon \in \mathcal{O}_L^\times$ .

We use arithmeticity of  $\Gamma_A$  to obtain information about the eigenvalues of  $A$ . Write  $\chi(A) = \mu_1^{n_1} \cdots \mu_m^{n_m}$  and  $\lambda = (\lambda_1, \dots, \lambda_m)$  with  $\lambda_i$  a root of  $\mu_i$ . Set  $K = \mathbb{Q}(\lambda_i)$ , denote the ring of integers  $\mathcal{O}_i \subset K_i$ , and set  $M = \bigoplus \mathcal{O}_i^{n_i}$ . By Corollary 10 and Lemma 11, arithmeticity of  $\Gamma_A$  implies arithmeticity of  $\Gamma_\lambda = M \rtimes_\lambda \mathbb{Z}$ .

Set  $T_i = R_{K_i/\mathbb{Q}}(G_m)$  and  $T = \prod T_i$ . Write  $\lambda = (\lambda_1, \dots, \lambda_m) \in \prod T_i(\mathbb{Z}) = T(\mathbb{Z})$ . Observe that  $\lambda^2 \in T^1(\mathbb{Z})$ , where  $T^1 = \prod R_{K_i/\mathbb{Q}}^1(G_m) \subset T$ . To see this, consider the short exact sequence (discussed in §5)

$$0 \rightarrow R_{K_i/\mathbb{Q}}^1(G_m) \hookrightarrow R_{K_i/\mathbb{Q}}(G_m) \xrightarrow{N} G_m \rightarrow 0.$$

Restricting attention to the  $\mathbb{Z}$ -points, since the norm of an algebraic unit is  $\pm 1 \in G_m(\mathbb{Z})$ , we conclude that  $\lambda_i^2 \in R_{K_i/\mathbb{Q}}^1(G_m)$ . We replace  $\lambda$  by  $\lambda^2$  (which corresponds to replacing  $A$  by  $A^2$ ).

Let  $S \subset T^1$  be the Zariski closure of  $\langle \lambda \rangle$ . Observe that  $\dim(S) = 1$  because

$$1 = \text{rank } S(\mathbb{Z}) = \text{rank}_{\mathbb{R}}(S) - \text{rank}_{\mathbb{Q}}(S) = \text{rank}_{\mathbb{R}}(S) = \dim(S)$$

The first equality holds by Proposition 13 because  $\Gamma_\lambda$  is arithmetic; the second equality is Equation (2); the third equality holds because  $S \subset T^1$  and  $\text{rank}_{\mathbb{Q}}(T^1) = 0$  (see §5); the final equality holds because  $T$  and hence  $S$  is defined over  $\mathbb{R}$ , by the assumption on eigenvalues of  $A$ .

By definition  $\dim(S) = 1$  means that  $X(S) \cong \mathbb{Z}$ . Since the splitting field  $L/\mathbb{Q}$  of  $S$  embeds in  $\text{Aut}(X(S)) \cong \{\pm 1\}$ , the group  $\text{Gal}(L/\mathbb{Q})$  is either trivial or  $\mathbb{Z}/2\mathbb{Z}$ . If  $\text{Gal}(L/\mathbb{Q})$  were trivial, then  $S$  would split over  $\mathbb{Q}$ , so we conclude  $\text{Gal}(L/\mathbb{Q}) = \mathbb{Z}/2\mathbb{Z}$ . Thus  $L/\mathbb{Q}$  is a real quadratic extension.

Next we show that the embedding  $S \hookrightarrow T^1$  factors through a diagonal embedding

$$S \hookrightarrow S \times \cdots \times S \hookrightarrow T_1^1 \times \cdots \times T_m^1.$$

This is achieved by studying the surjection  $f : X(T^1) \rightarrow X(S)$  induced by the inclusion  $S \hookrightarrow T^1$ . For each  $i$ , let  $P_i$  be the Galois closure of  $K_i/\mathbb{Q}$ . The splitting field  $P$  of  $T$  is the smallest Galois extension of  $\mathbb{Q}$  containing all the  $P_i$ . Denote  $G = \text{Gal}(P/\mathbb{Q})$  and  $H_i = \text{Gal}(P/K_i)$ . Then  $X(T_i) \cong \mathbb{Z}[G/H_i]$  and  $X(T^1) \cong \mathbb{Z}[G/H_i]/\mathbb{Z}$  (as discussed in §5). Denote  $G' := \text{Gal}(L/\mathbb{Q})$ . Since  $L \subset P$ , there is a surjection  $G \rightarrow G'$  given by restricting an automorphism of  $P$  to  $L$ .

The map  $f : X(T^1) \rightarrow X(S)$  is a  $\mathbb{Z}[G]$ -module map. For each  $i$ , it restricts to the summand  $X(T_i^1) \subset X(T^1)$  giving a map

$$f_i : X(T_i^1) \cong \mathbb{Z}[G/H_i]/\mathbb{Z} \rightarrow \mathbb{Z}[G']/\mathbb{Z} \cong X(S).$$

Such a map is determined by  $f([eH_i])$ . There are two cases to consider, depending on whether or not  $H_i$  is a subgroup of  $\ker(G \rightarrow G')$ . If  $H_i \not\subset \ker(G \rightarrow G')$ , then the only  $\mathbb{Z}[G]$ -module map  $\mathbb{Z}[G/H_i]/\mathbb{Z} \rightarrow \mathbb{Z}[G']/\mathbb{Z}$  is the zero map. If  $H_i \subset \ker(G \rightarrow G')$ , then for each  $n \in \mathbb{Z} \cong \mathbb{Z}[G']/\mathbb{Z}$ , there is a unique equivariant map with  $f([eH_i]) = n$ .

Observe that the map  $f_i$  is nonzero for each  $i$ . To see this, suppose that  $f_i = 0$  for some  $i$ . Then  $S \subset T_1 \times \cdots \times T_m$  is contained in

$$T_1 \times \cdots \times T_{i-1} \times 0 \times T_{i+1} \times \cdots \times T_m,$$

but this forces  $\lambda_i = 0$ , which leads to a contradiction. Then each  $f_i$  is nontrivial, and surjects onto  $\ell_i X(S)$  for some  $\ell_i \in \mathbb{Z} \setminus \{0\}$ . Note that  $\ell_i X(S)$  is isomorphic to  $X(S)$  as a  $\mathbb{Z}[G]$  module, so the surjection  $X(T_i^1) \rightarrow \ell_i X(S)$  corresponds to an embedding  $S \hookrightarrow T_i^1$ .

Thus, our surjective map  $f : X(T^1) \rightarrow X(S)$  factors through surjections

$$X(T_1^1) \oplus \cdots \oplus X(T_m^1) \twoheadrightarrow X(S) \oplus \cdots \oplus X(S) \twoheadrightarrow X(S).$$

The first map is given by  $(\phi_1, \dots, \phi_m) \mapsto (f_1(\phi_1), \dots, f_m(\phi_m))$  and the second map is given by  $(\psi_1, \dots, \psi_m) \mapsto \psi_1 + \cdots + \psi_m$ . The latter surjection gives an embedding  $S \hookrightarrow S \times \cdots \times S$ . Observe that  $S(\mathbb{Z}) \cong \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$  since  $S = R_{L/\mathbb{Q}}^1(G_m)$  with  $L/\mathbb{Q}$  a real quadratic extension (the torsion subgroup of  $\mathcal{O}_L^\times$  consists of roots of unity, so is  $\{\pm 1\}$  in the totally real case). Choose a fundamental unit  $\epsilon \in S(\mathbb{Z})$ . After replacing  $\lambda$  by  $\lambda^2$ , we can assume that  $\lambda_i \in S(\mathbb{Z})$  is a power of  $\epsilon$  for each  $i$ .

In summary, we've shown that if we replace  $A$  with  $A^2$ , then the eigenvalues are all units in a quadratic extension  $L/\mathbb{Q}$ , and if we replace  $A$  with  $A^4$ , then the eigenvalues are all powers of the fundamental unit  $\epsilon \in \mathcal{O}_L^\times$ . This finishes the proof of “(i) implies (ii)” and completes the proof of Theorem B.  $\square$

## 7 Monodromies with complex eigenvalues

In §6 we related the arithmeticity of  $\Gamma_A = \mathbb{Z}^n \rtimes_A \mathbb{Z}$  to the reducibility of  $\chi(A)$  when the eigenvalues of  $A \in \text{GL}_n(\mathbb{Z})$  are real. Now we remove this restriction on

eigenvalues, and focus on Question 3. We prove Theorem C in §7.1 and Theorem D in §7.2.

## 7.1 Proof of Theorem C

To prove Theorem C, given  $k \geq 1$ , we find  $n \geq k$ , a degree- $n$  extension  $K/\mathbb{Q}$ , and  $\lambda \in \mathcal{O}_K^\times$  so that (1) the action of  $\lambda$  on  $\mathcal{O}_K \cong \mathbb{Z}^n$  is fully irreducible, and (2) the group  $\Gamma_\lambda = \mathcal{O}_K \rtimes_\lambda \mathbb{Z}$  is arithmetic. Our first step is to give a sufficient condition for the action of  $\lambda$  on  $K$  (and hence  $\mathcal{O}_K$ ) to be (fully) irreducible.

**Lemma 15** (Irreducibility test). *Let  $K$  be a number field, and fix  $\lambda \in \mathcal{O}_K^\times$ . Denote  $P$  the Galois closure of  $K/\mathbb{Q}$ . Set  $T = R_{K/\mathbb{Q}}(G_m)$ , and let  $S \subset T$  be the Zariski closure of  $\langle \lambda \rangle$ .*

- (a) *The action of  $\lambda$  on  $K$  is irreducible if and only if  $K = \mathbb{Q}(\lambda)$ .*
- (b) *Assume that  $X(S) \otimes \mathbb{Q}$  is irreducible as a representation of  $\text{Gal}(P/\mathbb{Q})$ . If the action of  $\lambda^k$  on  $K$  is reducible for some  $k \geq 1$ , then there exists a  $\mathbb{Z}[\text{Gal}(P/\mathbb{Q})]$  permutation module  $\mathbb{Z}[Y]$  so that the surjection  $X(T) \twoheadrightarrow X(S)$ , induced by the inclusion  $S \hookrightarrow T$ , factors through surjective maps*

$$X(T) \twoheadrightarrow \mathbb{Z}[Y] \twoheadrightarrow X(S)$$

*of  $\mathbb{Z}[\text{Gal}(P/\mathbb{Q})]$ -modules.*

*Proof of Lemma 15.* (a) If  $K = \mathbb{Q}(\lambda)$ , then since  $\mathbb{Q}(\lambda) \cong \mathbb{Q}[t]/(\mu)$ , where  $\mu$  is the minimal polynomial of  $\lambda$ , then  $\mu$  (which is the characteristic polynomial of the action of  $\lambda$  on  $K$  with respect to any identification  $K \cong \mathbb{Q}^n$ ) is irreducible because  $K$  is a field. Conversely, if  $L := \mathbb{Q}(\lambda)$  is properly contained in  $K$ , then by the primitive element theorem, we can write  $K = L(\epsilon)$  for some  $\epsilon \in K$ . Then  $\lambda$  preserves the decomposition  $K \cong L \oplus \epsilon L \oplus \cdots \oplus \epsilon^d L$ , showing the action is reducible.

(b) Using part(a), if  $\lambda^k$  acts reducibly, take  $L$  with  $\lambda^k \in L \subsetneq K$ . Then  $\lambda^k \in T' \subset T$ , where  $T' = R_{L/\mathbb{Q}}(G_m)$ . The assumption that  $X(S) \otimes \mathbb{Q}$  is irreducible implies that  $\lambda^k$  is Zariski dense in  $S$ , so we have inclusions  $S \subset T' \subset T$ . Correspondingly, there are surjections  $X(T) \twoheadrightarrow X(T') \twoheadrightarrow X(S)$ . Since  $T'$  is quasi-split,  $X(T')$  is a permutation module, as desired.  $\square$

Next we find algebraic tori  $S$  with  $\dim(S)$  large and  $\text{rank } S(\mathbb{Z}) = 1$ .

**Proposition 16** (High-dimensional tori with small  $\mathbb{R}$ -rank). *For every  $k \geq 1$ , there exists an algebraic torus  $S$  defined over  $\mathbb{Q}$  so that  $\text{rank}_{\mathbb{Q}}(S) = 0$ ,  $\text{rank}_{\mathbb{R}}(S) = 1$ , and  $\dim(S) = k$ .*



*Proof.* This is easy for  $k = 1, 2$ . For  $k = 1$  one may take  $S = R_{L/\mathbb{Q}}^1(G_m)$ , where  $L/\mathbb{Q}$  is a real quadratic extension. For  $k = 2$ , consider  $S = R_{L/\mathbb{Q}}^1(G_m)$ , where  $L/\mathbb{Q}$  is an imaginary cubic extension.

Assume  $k \geq 3$  (the following construction will also work for  $k = 2$ ). First we build a  $\mathbb{Z}[S_{k+1}]$ -module  $X$  so that a transposition  $\tau \in S_{k+1}$  acts in a special way. Then we use the inverse Galois problem to show there is a torus  $S$  with  $X(S) = X$ .

Consider the  $\mathbb{Q}[S_{k+1}]$  module  $X_{\mathbb{Q}} = \mathbb{Q}^k \otimes \mathbb{Q}_-$ , where  $\mathbb{Q}^k$  is the standard representation and  $\mathbb{Q}_-$  is the sign representation. Choose a module map  $\mathbb{Q}[S_{k+1}] \twoheadrightarrow X_{\mathbb{Q}}$ , and let  $X \subset X_{\mathbb{Q}}$  be the image of  $\mathbb{Z}[S_{k+1}] \subset \mathbb{Q}[S_{k+1}]$ . The module  $X$  has a finite-index submodule isomorphic to  $\mathbb{Z}^k \otimes \mathbb{Z}_-$ . Then for any transposition  $\tau \in S_{k+1}$ ,

$$X \cong \mathbb{Z}[\mathbb{Z}/2\mathbb{Z}] \oplus (\mathbb{Z}_-)^{k-2} \text{ as } \mathbb{Z}[\langle \tau \rangle] \text{ modules.} \quad (4)$$

*Claim.* There exists a Galois extension  $P/\mathbb{Q}$  so that (i)  $\text{Gal}(P/\mathbb{Q}) \cong S_{k+1}$  and (ii) under this isomorphism, complex conjugation corresponds to a transposition  $\tau \in S_{k+1}$ .

The claim is a special case of the inverse Galois problem. While this problem is open in general, it has been solved for symmetric groups [KM01, Prop. 2].

Now we use the claim. The claim is equivalent to the existence of a representation

$$\rho : \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \twoheadrightarrow S_{k+1} \subset \text{GL}(X)$$

with complex conjugation mapping to  $\tau$ . The fixed field of  $\ker(\rho)$  is the desired Galois extension  $P/\mathbb{Q}$ . Under the correspondence between  $\mathbb{Z}[\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})]$ -modules and algebraic tori (see §5), this implies that there is a torus  $S$  with  $X(S) \cong X$ .

The  $\mathbb{Q}$ -rank of  $S$  is 0 because  $X(S) \cong X$  does not contain a trivial sub-representation. In addition  $\text{rank}_{\mathbb{R}}(S) = 1$  by Equation (4) and Lemma 14. This proves the proposition.  $\square$

Now we finish the proof of Theorem C. Given  $k \geq 2$ , take  $S$  as in Proposition 16. By Equation (2),  $\text{rank } S(\mathbb{Z}) = 1$ . Fix an infinite order element  $\lambda \in S(\mathbb{Z})$ . The surjection  $\mathbb{Z}[S_{k+1}] \twoheadrightarrow X$  from the proof of Proposition 16 induces an embedding  $S \hookrightarrow R_{P/\mathbb{Q}}(G_m)$  where  $P/\mathbb{Q}$  is a Galois extension with  $\text{Gal}(P/\mathbb{Q}) = S_{k+1}$ . Let  $T \subset R_{P/\mathbb{Q}}(G_m)$  be a minimal quasi-split torus  $T = R_{K/\mathbb{Q}}(G_m)$  that contains  $S$ . The group  $\mathcal{O}_K \rtimes_{\lambda} \mathbb{Z}$  is arithmetic by Proposition 13. In addition the action of  $\lambda$  on  $\mathcal{O}_K$  is fully irreducible by Lemma 15. This proves that there is fully irreducible  $A \in \text{GL}_n(\mathbb{Z})$  so that  $\Gamma_A$  is arithmetic, where  $n = [K : \mathbb{Q}] = \dim(T) \geq \dim(S) = k$ . This proves Theorem C.  $\square$

**Example 17.** We give an example that demonstrates how Lemma 15 can be further used to determine  $n = [K : \mathbb{Q}]$  precisely.

We show that there is fully irreducible  $A \in \mathrm{GL}_{10}(\mathbb{Z})$  so that  $\Gamma_A$  is arithmetic. First let  $P/\mathbb{Q}$  be a Galois extension with  $\mathrm{Gal}(P/\mathbb{Q}) = S_5$  and complex conjugation a transposition, and let  $S \subset R_{P/\mathbb{Q}}(G_m)$  be a torus constructed as in the proof of Theorem C. Choose  $K \subset P$  so that  $\mathrm{Gal}(P/K) \cong A_4$ . As an  $S_5$ -representation,  $\mathbb{Q}[S_5/A_4]$  has character

conjugacy classes of $S_5$	( )	(12)	(12)(34)	(123)	(1234)	(12345)	(12)(345)
character of $\mathbb{Q}[S_5/A_4]$	10	0	2	4	0	0	0

By computing inner product of characters one finds that  $\mathbb{Q}^4 \otimes \mathbb{Q}_-$  appears in  $\mathbb{Q}[S_5/A_4]$  with multiplicity 1. Then  $S$  is contained in  $T = R_{K/\mathbb{Q}}(G_m) \subset R_{P/\mathbb{Q}}(G_m)$ . We want to show that  $T$  is a minimal quasi-split torus containing  $S$ . We prove this using Lemma 15(b).

We want to show that there is no  $H \subset S_5$  so that (i) there is a surjection  $\mathbb{Q}[S_5/A_4] \twoheadrightarrow \mathbb{Q}[S_5/H]$  and (ii)  $\mathbb{Q}^4 \otimes \mathbb{Q}_-$  is a subrepresentation of  $\mathbb{Q}[S_5/H]$ . Condition (i) implies  $H \subset S_5$  has index  $\leq 10$ , since  $\mathbb{Q}[S_5/A_4]$  has dimension 10. There is no need to consider the case  $[S_5 : H] = 10$ . The subgroups satisfying  $[S_5 : H] < 10$  are  $A_5$ ,  $S_4$ , and the affine group  $C_5 \rtimes C_4$ . It is straight-forward to check that  $\mathbb{Q}[S_5/H]$  does not contain  $\mathbb{Q}^4 \otimes \mathbb{Q}_-$  as a subrepresentation for each of these subgroups  $H$ .

**Remark 18** (Infinitely many examples up to commensurability). By the result of [KM01] used above, there are infinitely many different  $P$  with  $\mathrm{Gal}(P/\mathbb{Q}) = S_{k+1}$  and complex conjugation acting as a transposition. Repeating the above construction with different fields  $P$  then leads to infinitely many non-commensurable examples (this follows from the discussion of §2, 4).

There is another way to obtain infinitely many examples. If  $\dim S \geq 2$ , then there is a surjection  $\mathbb{Z}[G] \twoheadrightarrow X(S) \oplus X(S) \cong X(S \times S)$  (here it is relevant that every complex representation of  $S_n$  is defined over  $\mathbb{Q}$ , so each irreducible  $\mathbb{Q}[G]$  module appears in  $\mathbb{Q}[G]$  with multiplicity equal to its dimension). Then if we repeat the argument above (now taking  $T \subset R_{P/\mathbb{Q}}(G_m)$  a minimal quasi-split torus containing  $S \times S$ ), the different embeddings  $S \hookrightarrow S \times S$  lead to non-commensurable examples. Compare with [GP99, Prop. 5.3].

## 7.2 Proof of Theorem D

Fix an irreducible  $A \in \mathrm{GL}_p(\mathbb{Z})$ . Let  $\lambda$  be an eigenvalue of  $A$ , and set  $K = \mathbb{Q}(\lambda)$ . By Lemma 11, to show that  $\Gamma_A = \mathbb{Z}^p \rtimes_A \mathbb{Z}$  is not arithmetic, it suffices to show that  $\Gamma_\lambda = \mathcal{O}_K \rtimes_\lambda \mathbb{Z}$  is not arithmetic. We proceed by considering a series of cases.

The Galois closure  $P$  of  $K/\mathbb{Q}$  is either totally real or totally imaginary.

**Case:  $P$  totally real.** In this case,  $\Gamma_\lambda$  is non-arithmetic by [GP98, proof of Thm. 1.3]. It can also be seen from the proof of Theorem B, since the proof shows that if

$K$  is totally real and  $\mathcal{O}_K \rtimes_{\lambda} \mathbb{Z}$  is arithmetic, then  $K$  contains a quadratic subfield. But this is impossible if  $[K : \mathbb{Q}]$  is an odd prime.

**Case:  $P$  totally imaginary.** The Galois group  $G = \text{Gal}(P/\mathbb{Q})$  is a transitive subgroup of  $S_p$ . We use two theorems that restrict  $G$ ; see [Neu74, Thms. 1 and 3].

- (Burnside) If  $G \subset S_p$  is transitive and not solvable, then  $G$  is 2-transitive.
- (Galois) If  $G \subset S_p$  is transitive and solvable, then  $G \subset \mathbb{Z}/p\mathbb{Z} \rtimes (\mathbb{Z}/p\mathbb{Z})^{\times}$ .

In Galois's theorem, the permutation action of  $\mathbb{Z}/p\mathbb{Z} \rtimes (\mathbb{Z}/p\mathbb{Z})^{\times}$  is by affine transformations on  $\{0, 1, \dots, p-1\} \cong \mathbb{Z}/p\mathbb{Z}$ , i.e. given  $s \in (\mathbb{Z}/p\mathbb{Z})^{\times}$  and  $r \in \mathbb{Z}/p\mathbb{Z}$ , define a permutation  $x \mapsto sx + r$ .

We consider separately the cases  $G$  solvable or not.

**Case:  $G$  not solvable.** Set  $H = \text{Gal}(P/K)$ , and let  $\tau \in G \subset S_p$  be the element that acts by complex conjugation. If  $k$  denotes the number of fixed points of  $\tau$  acting on  $G/H$ , then as  $\tau$ -modules

$$\mathbb{Z}[G/H]/\mathbb{Z} \cong \mathbb{Z}[\mathbb{Z}/2\mathbb{Z}]^{\frac{p-k}{2}} \oplus \mathbb{Z}^{k-1}. \quad (5)$$

Note that  $k \geq 1$  because  $p = |G/H|$  is odd. To see the isomorphism (5) concretely, write  $G/H = \{y_1, \dots, y_p\}$  with  $y_1, \dots, y_{p-k}$  permuted in pairs, and  $y_{p-k+1}, \dots, y_p$  fixed. Then  $[y_1], \dots, [y_{p-1}]$  forms a basis for  $\mathbb{Z}[G/H]/\mathbb{Z}$ , and the action of  $\tau$  is apparent from this description.

Set  $T = R_{K/\mathbb{Q}}^1(G_m)$ . By Lemma 14 and equation (5),  $\text{rank } T(\mathbb{Z}) = \frac{p-k}{2} + (k-1)$ . Then  $p \geq 5$  and  $k \geq 1$  implies that  $\text{rank } T(\mathbb{Z}) \geq 2$ . Now we use the following lemma.

**Lemma 19.** *Let  $G$  be any group, and let  $H \subset G$  be a subgroup. If  $G$  acts 2-transitively on  $G/H$ , then  $\mathbb{C}[G/H] \cong \mathbb{C} \oplus V$ , and  $V$  is irreducible.*

Lemma 19 is an exercise in [Ser77, Exercise 7.2], whose solution we sketch. To show  $V$  is irreducible, show  $\langle \chi_V, \chi_V \rangle = 1$ , where  $\langle \cdot, \cdot \rangle$  is the usual inner product on characters. Since  $\chi_V = 1 - \chi_{\mathbb{C}[G/H]}$ , this reduces to showing that  $\langle \chi_{\mathbb{C}[G/H]}, \chi_{\mathbb{C}[G/H]} \rangle = 2$ . Note that  $\mathbb{C}[G/H] = \text{Ind}_H^G(\mathbb{C})$  is induced from the trivial representation of  $H$ , so by Frobenius reciprocity,

$$\langle \chi_{\mathbb{C}[G/H]}, \chi_{\mathbb{C}[G/H]} \rangle = \langle \text{Ind}_H^G(1), \chi_{\mathbb{C}[G/H]} \rangle = \langle 1, \text{Res}_H^G \chi_{\mathbb{C}[G/H]} \rangle,$$

and the right-hand side is the number of fixed points of  $H$  acting on  $G/H$ .

The lemma implies that  $X(T) \cong \mathbb{Z}[G/H]/\mathbb{Z}$  is irreducible as a  $\mathbb{Z}[G]$ -module, which implies that  $T$  has no nontrivial sub-torus. Hence  $\langle \lambda \rangle \subset T$  is Zariski dense for every infinite order  $\lambda \in T(\mathbb{Z})$ . Since  $\text{rank } T(\mathbb{Z}) \geq 2$ , this implies  $\mathcal{O}_K \rtimes_{\lambda} \mathbb{Z}$  is non-arithmetic by Proposition 13.

**Case:  $G$  solvable.** By Galois's theorem,  $G$  is a subgroup of  $\mathbb{Z}/p\mathbb{Z} \rtimes (\mathbb{Z}/p\mathbb{Z})^\times$ . Write  $C_\ell = \mathbb{Z}/\ell\mathbb{Z}$ . Observe that  $C_p \subset G$  since  $G$  transitive implies that  $p$  divides  $|G|$  which implies that  $G$  has an element of order  $p$  (Cauchy's theorem). Then  $G$  has the form  $C_p \rtimes H$ , where  $H \cong C_q$  for some  $q$  dividing  $p-1$  (recall that  $C_p^\times \cong C_{p-1}$ ). The group  $G = C_p \rtimes H$  has a presentation

$$G = \langle r, s \mid r^p = 1 = s^q, rs = sr^a \rangle, \quad (6)$$

where  $r$  and  $s$  generate  $C_p$  and  $H$  respectively, and  $a \in \{1, \dots, p-1\} \cong C_p^\times$  is an element of order  $q$ .

We will argue as in the non-solvable case that  $\mathbb{Z}[G/H]/\mathbb{Z}$  is an irreducible  $\mathbb{Z}[G]$ -module. If  $|H| = p-1$  (this is the largest  $H$  can be), then  $G$  acts 2-transitively on  $G/H$  and we can apply Lemma 19 in the same way. However, if  $|H| < p-1$ , then  $G$  is not 2-transitive, and a different argument is needed.

**Proposition 20.** *Let  $G = C_p \rtimes H$ , where  $H \subset C_p^\times \cong C_{p-1}$  has order  $q < p-1$ . Then  $\mathbb{Q}[G/H] \cong \mathbb{Q} \oplus V$ , and  $V$  is irreducible.*

*Proof.* First we identify  $\mathbb{C}[G/H]$  as a  $\mathbb{C}[G]$ -module. The group  $G$  is metacyclic, and its representations over  $\mathbb{C}$  are well-known; see e.g. [CR06, §47].

The group  $G$  has  $q + \frac{p-1}{q}$  irreducible representations over  $\mathbb{C}$ . There are  $q$  one-dimensional representations, namely those that factor through the abelianization  $G \twoheadrightarrow H \cong C_q$ . There are  $\frac{p-1}{q}$  irreducible  $q$ -dimensional representations, each induced from an irreducible representation of  $C_p$ . For  $0 \leq i \leq p-1$ , let  $L_i$  denote the 1-dimensional complex vector space spanned by (the formal symbol)  $\ell_i$ , and make  $L_i$  a representation of  $C_p = \langle r \rangle$  with action  $r(\ell_i) = \zeta^i \cdot \ell_i$ , where  $\zeta = e^{2\pi i/p}$  is a primitive  $p$ -th root of unity. Write

$$L_i^G := \mathbb{C}[G] \otimes_{\mathbb{C}[H]} L_i$$

for the induced representation. Then  $L_i^G$  is irreducible for  $1 \leq i \leq p-1$ , and  $L_1^G, \dots, L_{(p-1)/q}^G$  are distinct irreducible representations of  $G$ .

*Claim.* There is an isomorphism of  $\mathbb{C}[G]$ -modules

$$\mathbb{C}[G/H] \cong \mathbb{C} \oplus L_1^G \oplus \dots \oplus L_{(p-1)/q}^G.$$

*Discussion of the claim.* The claim can be proved by computing the character of the representation on each side. This is a straightforward computation, so we omit the details. However, below we will use the explicit computation of the character  $\chi_i$  of  $L_i^G$ , so we record it here.

Fix  $1 \leq i \leq \frac{p-1}{q}$ . It is easy to compute  $\chi_i(s^j) = 0$  for each  $1 \leq j \leq q-1$ . Note that  $\{s^j r^i : 0 \leq i \leq p-1\}$  forms a conjugacy class (one can show this directly using the

presentation (6)). Then it remains to compute  $\chi_i(r^k)$  for  $1 \leq k \leq p-1$ . Consider the basis  $\{1 \otimes \ell_i, s \otimes \ell_i, \dots, s^{q-1} \otimes \ell_i\}$  for  $L_i^G$ . In this basis, an easy computation shows that the action of  $r \in G$  has matrix

$$\begin{pmatrix} \zeta^i & & & & \\ & \zeta^{ia} & & & \\ & & \zeta^{ia^2} & & \\ & & & \ddots & \\ & & & & \zeta^{ia^{q-1}} \end{pmatrix}$$

Then  $\chi_i(r^k) = (\zeta^{ik}) + (\zeta^{ik})^a + \dots + (\zeta^{ik})^{a^{q-1}}$ . Observe that  $\chi_i(r^k) = \chi_i(r^{ka})$ , which agrees with the fact that the conjugacy classes of elements of  $\{1, r, \dots, r^{p-1}\}$  is the same as orbits of the  $H$ -action on  $C_p$ . This concludes the discussion of the claim.

With the setup above, we can prove the proposition. From the computation of  $\chi_i$ , we deduce that the characters  $\chi_1, \dots, \chi_{(p-1)/q}$  form a single orbit under the natural action of  $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ . Then the sum of these characters is  $\mathbb{Q}$ -valued, and there is a smallest  $m \geq 1$  so that

$$m(\chi_1 + \dots + \chi_{(p-1)/q})$$

is realized as a character of  $G$ -representation defined over  $\mathbb{Q}$ , and this representation is irreducible over  $\mathbb{Q}$ . Here  $m$  is *Schur index*; see [Isa76, Cor. 10.2]. If  $V$  is a complement to  $\mathbb{Q} \subset \mathbb{Q}[G/H]$  (which exists because  $\mathbb{Q}[G]$  is semisimple), then  $\chi_V = \chi_1 + \dots + \chi_{(p-1)/q}$  (this is the straightforward part of the claim above, whose proof was omitted). This implies that  $m = 1$  and that  $V$  is irreducible.  $\square$

Now we can conclude as in the non-solvable case, replacing Lemma 19 with Proposition 20: The proposition implies that  $T = R_{K/\mathbb{Q}}^1(G_m)$  has no nontrivial sub-torus. Hence to show  $\mathcal{O}_K \rtimes_{\lambda} \mathbb{Z}$  is non-arithmetic, it suffices to show that  $\text{rank } T(\mathbb{Z}) \geq 2$ ; by Equation (2), this is equivalent (since  $\text{rank}_{\mathbb{Q}}(T) = 0$ ) to showing that  $\text{rank}_{\mathbb{R}}(T) \geq 2$ . To show  $\text{rank}_{\mathbb{R}}(T) \geq 2$  we use Lemma 14.

Let  $\tau \in G$  be complex conjugation. Since  $P$  is totally imaginary,  $\tau$  is nontrivial. Every involution in  $G = C_p \rtimes H$  is conjugate into  $H$ , and  $H \subset C_{p-1}$  has a unique element of order 2, which acts on  $C_p = \langle r \rangle$  by  $r \mapsto r^{-1}$ . Then the action of  $\tau$  on  $G/H = \{eH, rH, \dots, r^{p-1}H\}$  has a single fixed point, so  $\tau$  acts on  $\mathbb{Z}[G/H]/\mathbb{Z}$  in the same way that  $\mathbb{Z}/2\mathbb{Z}$  acts on  $\mathbb{Z}[\mathbb{Z}/2\mathbb{Z}]^{(p-1)/2}$ . Hence  $X(T)^{\tau}$  has rank  $(p-1)/2 \geq 2$ . This shows  $\text{rank}_{\mathbb{R}}(T) \geq 2$ , as desired.

This completes the proof of Theorem D.  $\square$

**Remark 21.** Note that Theorem D is not true for the primes  $p = 2, 3$ . This is obvious for  $p = 2$ . For  $p = 3$ , if  $K/\mathbb{Q}$  has 1 real embedding, then the Galois closure  $P$  of  $K/\mathbb{Q}$  has  $\text{Gal}(P/\mathbb{Q}) = S_3$ . Then  $T = R_{K/\mathbb{Q}}^1(G_m)$  has  $\text{rank } T(\mathbb{Z}) = 1$ , so for any infinite order  $\lambda \in T(\mathbb{Z})$ , the group  $\mathcal{O}_K \rtimes_{\lambda} \mathbb{Z}$  is arithmetic. Furthermore, the action

of  $\lambda$  on  $K$  is irreducible by Lemma 15 since there are no intermediate subfields  $\mathbb{Q} \subset K' \subset K$ .

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