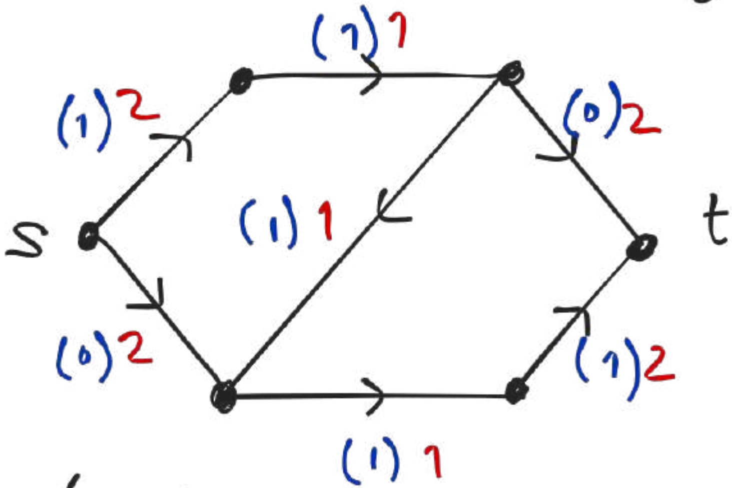
(up to symmetry either x, y lie in common Q_2 or are antipodal)

Max-flow problem

(another duality prob)

<u>Setup</u> A <u>network</u> is

- $G=(V,\vec{E})$ directed graph
- $s,t \in V$ source, terminus
- $c: E \to \mathbb{N}$ capacity



(think water through pipes)

A <u>flow</u> on network is $f: E \to \mathbb{N}$

s.t. $f(e) \leq c(e) \quad \forall e \in E$

and (conservation law) $\forall\, v \in V \setminus \{s,t\}$,

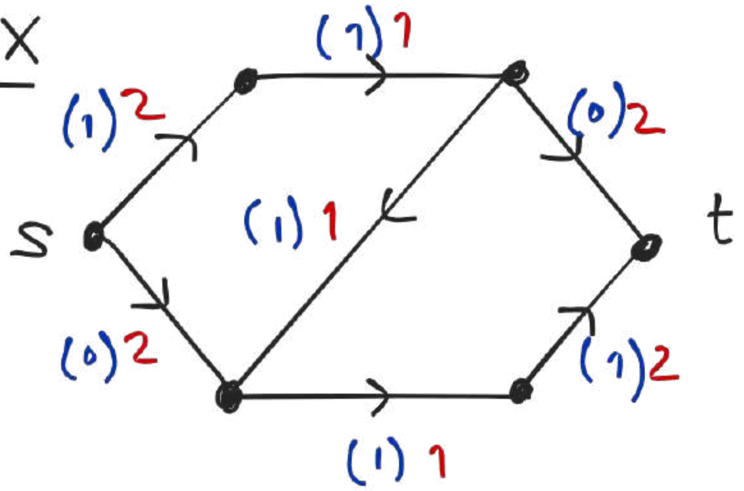
$$f^+(v) := \sum_{e \text{ into } v} f(e) = \sum_{e \text{ out of } v} f(e) =: f^-(v)$$

The <u>value</u> $\mathrm{val}(f)$ of f is $f^+(s)$.

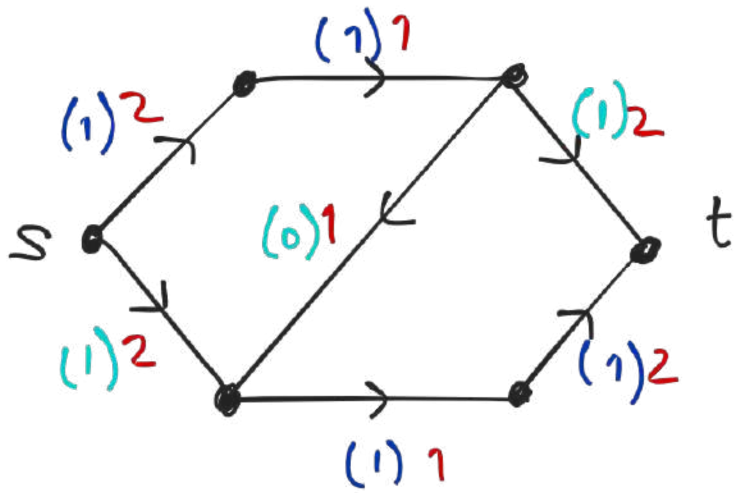
<u>Problem</u> Given a <u>network</u> what is the max value of flow on it?

<u>Exercise</u> $f^+(s) = f^-(t)$ (use conservation law)

<u>Ex</u>



flow not maximal



This flow is maximal. Could show with case work, but there's a better way!

A cut for network

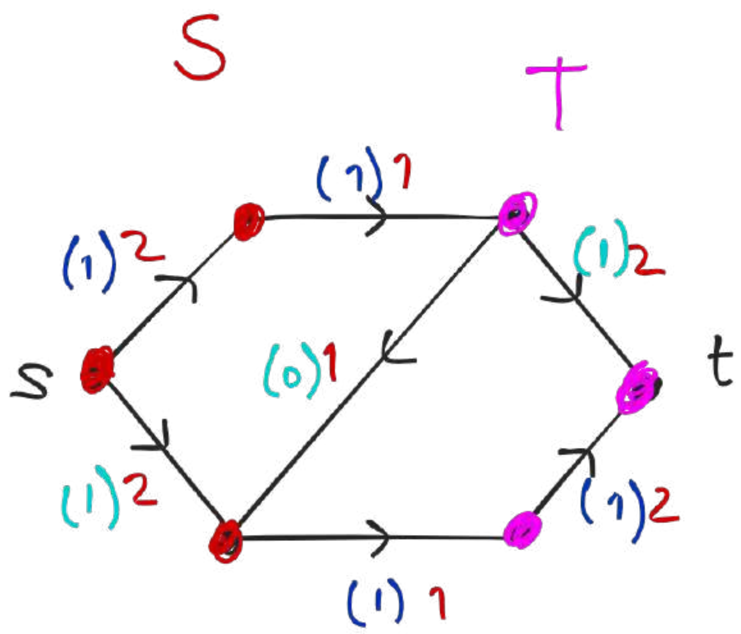
$G = (s, t \in V, \vec{E}, c)$ is a partition

$V = S \cup T$ w/ $s \in T$, $t \in T$.

The capacity of a cut is

$$c(S,T) := \sum_{e \in \vec{E} \text{ from } S \text{ to } T} c(e)$$

Ex



$$c(S,T) = 2$$

Thm (Ford-Fulkerson) For network $G = (s,t \in V, \vec{E}, c)$

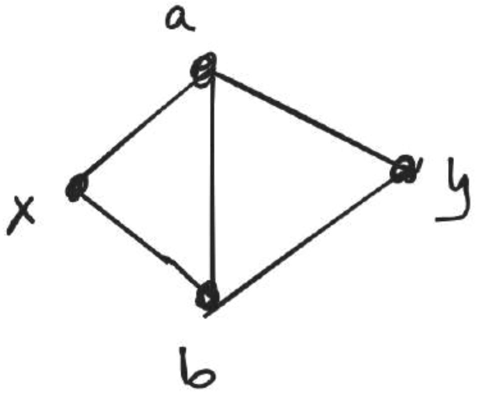
max value of flow on G = min capacity of a cut of G

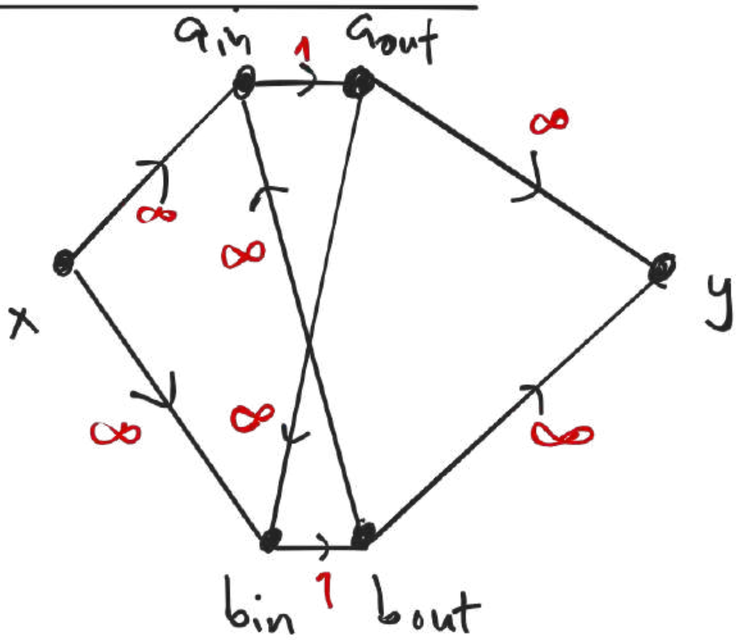
In particular, flow above is maximal.

Corollary (Menger's Thm)

$\kappa(x,y) = \lambda(x,y)$

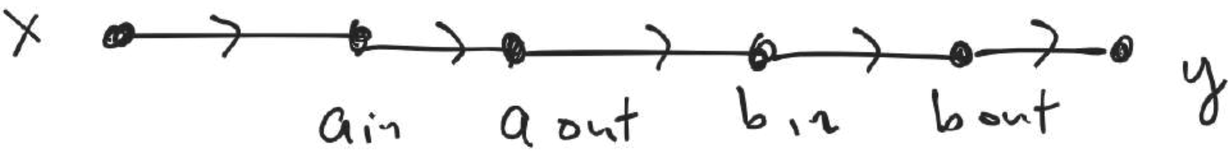
Translation to flow problem

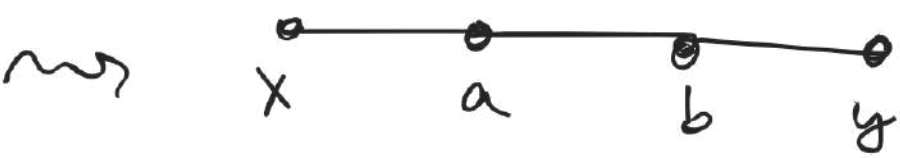




Main Observations

① A flow corresponds to collection of paths from x to y.





Paths disjoint b/c $v_{in} \xrightarrow{1} v_{out}$ has capacity 1 so each vertex can be used at most once.

$\#$ paths = value of flow.

② A cut (S,T) has finite capacity $\iff$ all (S,T)-edges are $v_{in} \longrightarrow v_{out}$

so $(S,T) \rightsquigarrow$ vertex cut U of G

$c(S,T) = |U|$.

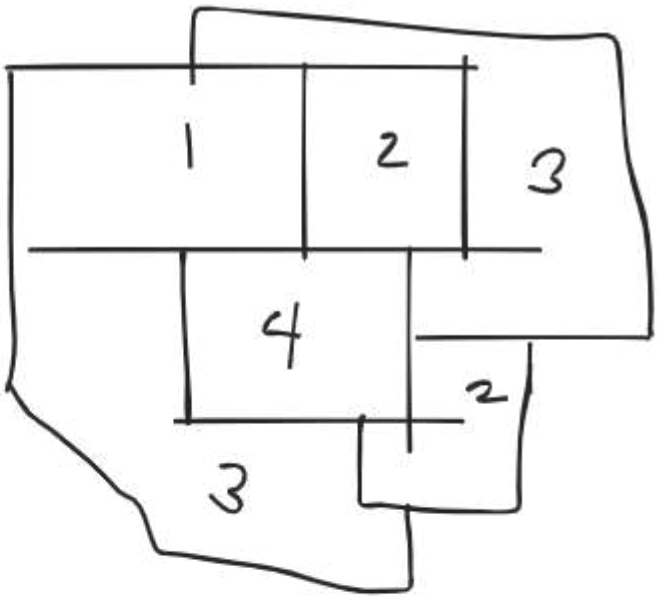
Thus

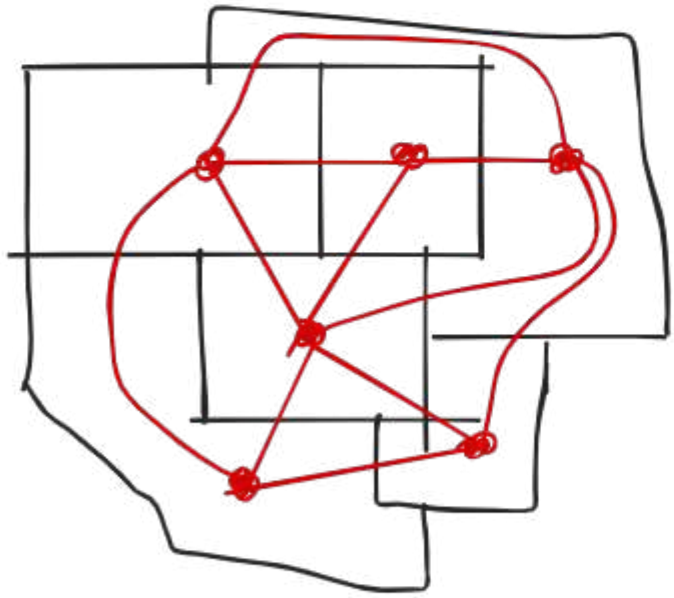
$$\kappa(x,y) = \text{max flow} = \text{min cut} = \lambda(x,y)$$

$\square$

Graph Coloring

Motivating problem: what's the fewest colors needed to color any map so that adjacent regions have different colors?





This is a graph theory problem.

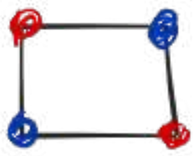
A <u>coloring</u> of a graph is a coloring of vertices s.t.

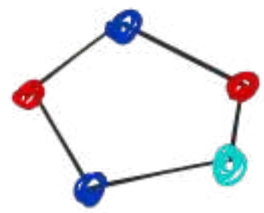
adjacent vertices have diff. color.

Chromatic number

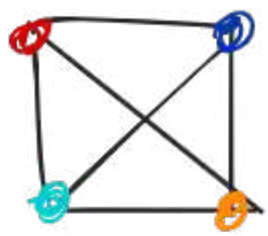
$\chi(G)$ = fewest colors in any coloring of G.

Ex. $\chi(C_4) = 2$ $\quad$ $\chi(C_5) = 3$





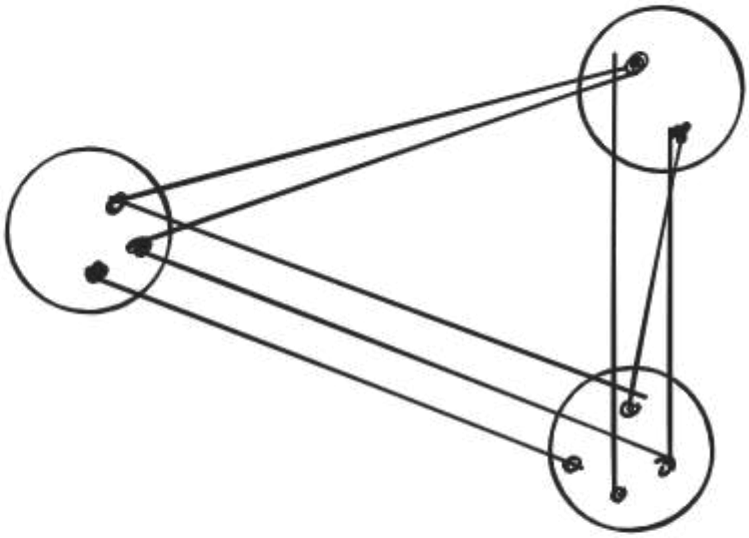
$\chi(K_n) = n$



Exercise: characterize graphs w/

- $\chi(G) = 1 \iff G$ has no edges
- $\chi(G) = 2 \iff G$ bipartite (and has ≥ 1 edge)

$\chi(G) = k$ means G has form



$k = 3$

<u>Ex</u> exam scheduling
vertices = classes
edge if share student.

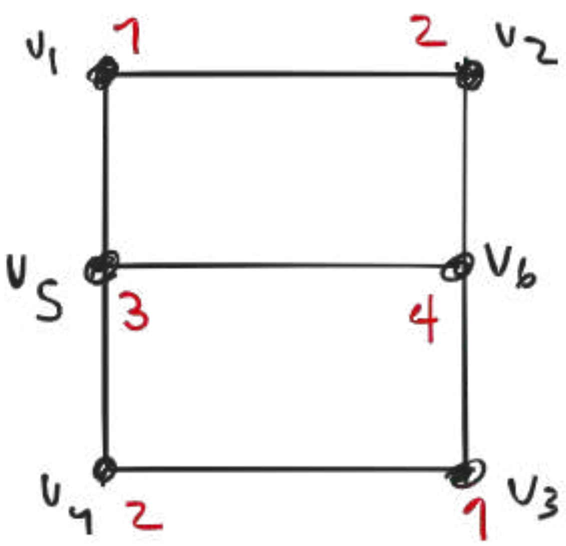
$\chi(G)$ = min # of different time blocks to schedule exams.

$\boxed{\chi \text{ upperbounds}}$

<u>Lemma</u> (Greedy coloring)

$\chi(G) \leq \Delta(G) + 1$. Δ = max vertex degree.

Proof



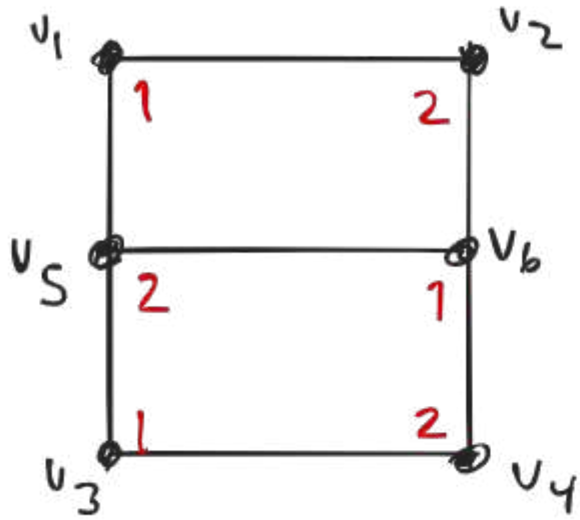
Use colors $1, \dots, \Delta(G)+1$.

Give algorithm: Color v_1 with 1

color v_i with smallest color not used by its neighbors.

This requires at most $\Delta(G)+1$ colors (worst case, $\deg(v_i) = \Delta(G)$ and all neighbors use $\Delta(G)$ colors.) $\square$

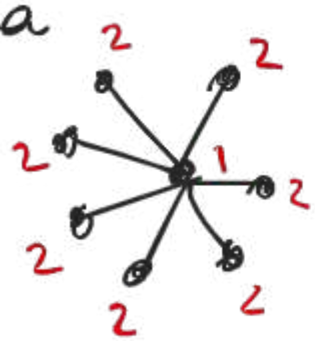
Rank Coloring



depends on numbering of vertices

Rmk Bound of lemma

can be far from sharp



sometimes sharp.

$\chi(K_n) = n \qquad \Delta(K_n) = n-1.$

$\chi(C_{2k+1}) = 3 \qquad \Delta(C_{2k+1}) = 2$

(not bipartite)

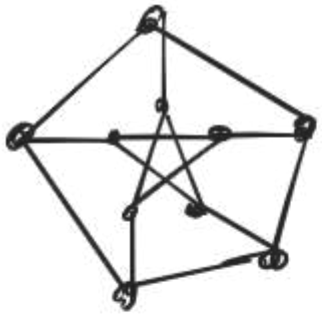
Thm (Brooks) Let G be connected

If $\chi(G) = \Delta(G)+1$, then

$G = K_n$ or C_{2k+1}.

Thus for any other graph $\chi(G) \le \Delta(G)$.

Eg $G =$



Thm $\Rightarrow \chi(G) \le 3$

G not bipartite

$\Rightarrow \chi(G) \ge 3.$

$\boxed{\chi \text{ lower bounds}}$

Easy observation: if $H < G$ subgraph then $\chi(G) \geq \chi(H)$.

eg if G contains K_n then

$$\chi(G) \geq n.$$

Q: Can $\chi(G)$ be large without G containing K_3 ??

Thm (Mycielski) Yes!

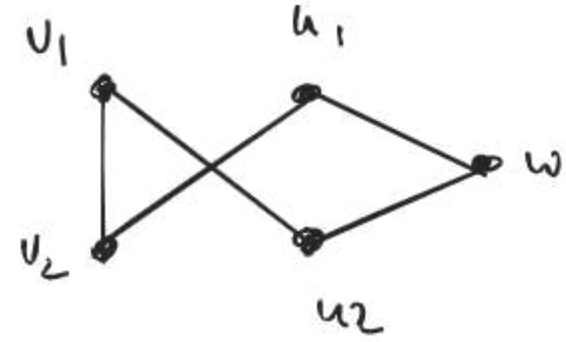
Mycielski construction input $G = (V, E)$

write $V = \{v_1, \dots, v_n\}$ Define G'

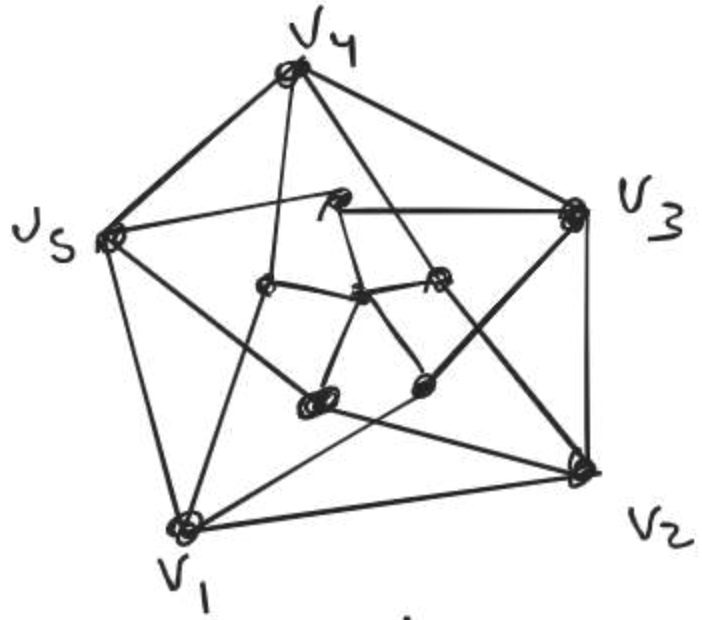
vertices $V \cup U \cup \{w\}$ $U = \{u_1, \dots, u_n\}$

edges E, $\{u_i, w\}$, $\{v_i, u_j\}$ if $\{v_i, v_j\} \in E$

Ex $G = K_2$ $G' = C_5$



G''



Grötzsch graph.

Claim

$\chi(G) = 2$ $\chi(G') = 3$ $\chi(G'') = 4$.

Thm (Mycielski) Let G be triangle free. w/ $\chi(G) = k$

Then G' is also triangle free

& $\chi(G') = k+1$

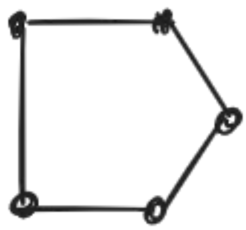
Remark.

Computing $\chi(G)$ is hard. There is no (known?) dual problem. There are upper bounds (max vertex degree) and lower bounds (subgraphs), but in general, these aren't sharp.

$\boxed{\text{Critical graphs}}$

$G=(V,E)$ is k-critical if G conn, $\chi(G)=k$, and $\chi(G\setminus e) < k$ $\forall e \in E$.

Ex odd cycles are 3-critical



$\chi(C_5)=3$

$\chi(P_4)=2$

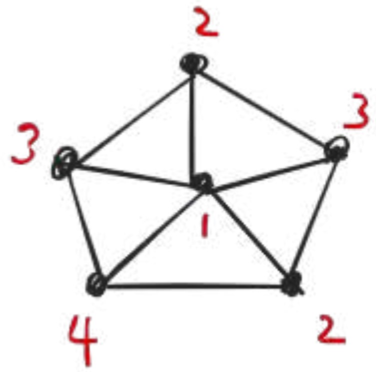
P_4 not 2-critical

$\chi($  $)=2$

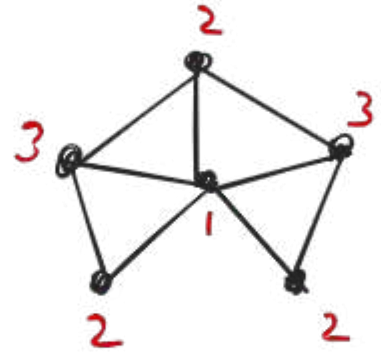
$\chi($  $)=2$

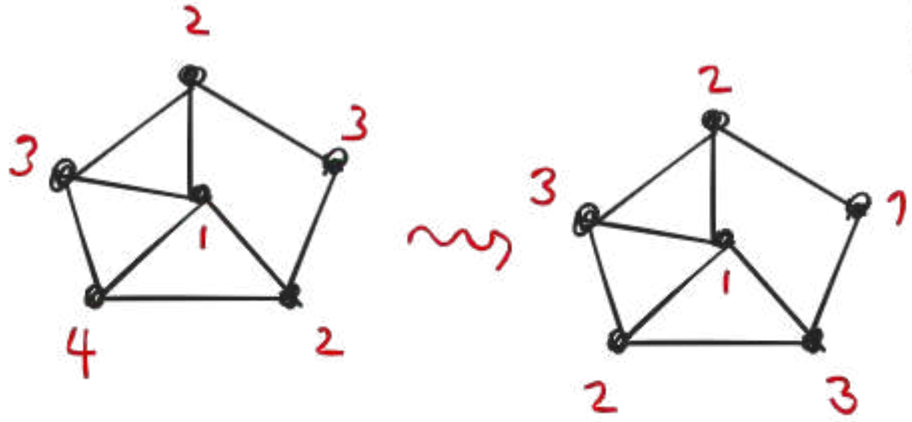
Exercise K_n is n-critical

Ex



is 4-critical





Properties

- G critical $\iff$ $\chi(H) < \chi(G)$ $\forall$ subgraphs $H < G$.

  (exercise)

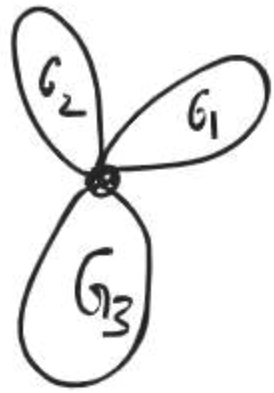
- Every G with $\chi(G) = k$ has a k-critical subgraph:

  among $H < G$ w/ $\chi(H) = k$ choose one w/ fewest edges.

- G critical $\Rightarrow$ G 2-connected

  ie $G \setminus \{v\}$ connected $\forall v \in V$.

<u>Proof</u> (Contrapositive)



$G \setminus v$ disconnected

$\chi(G) = \max \{\chi(G_i)\}$

$\Rightarrow \chi(G) = \chi(G_j)$ some j. $\square$

- G 3-critical $\Rightarrow$ $G = C_{2k+1}$.

<u>Pf</u> $\chi(G) = 3 \Rightarrow G$ not bipartite

$\Rightarrow G \supset C_{2k+1}$

$\chi(C_{2k+1})$ and G critical

$\Rightarrow G = C_{2k+1}$ $\square$

Brooks' Thm

$\chi(G)$ = fewest colors needed to colour $V(G)$ w/ no monochromatic edges.

Greedy coloring uses at most $\Delta(G)+1$ colors

<u>Thm</u> (Brooks) If $G \neq C_{2k+1}, K_n$ then $\chi(G) \leq \Delta(G)$

(ie can do better than greedy alg)

<u>Claim</u> Suffices to prove this for critical graphs

(<u>Recall</u> G critical $\iff$ $\chi(H) < \chi(G)$ $\forall$ subgraphs $H < G$)

<u>Pf of Claim:</u>
Given any G w/ $\chi(G) = k$

take $H < G$ k-critical.

<u>WTS</u> $\chi(G) \le \Delta(G)$

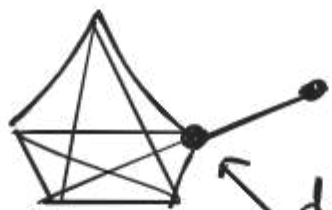
Brooks for critical graphs $\Longrightarrow$

- if $H \neq C_{2k+1}, K_n$ then

$$\chi(G) = \chi(H) \le \Delta(H) \le \Delta(G) \quad ✓$$

- if $H = C_{2k+1}$ or K_n then

$$\chi(G) = \chi(H) = \Delta(H) + 1 \le \Delta(G)$$



H

deg $\Delta(H)+1$.

<u>Proof Sketch of Brooks for critical</u>

Fix G k-critical

<u>WTS</u> either $G = C_{2k+1}, K_n$ or $\chi(G) \le \Delta(G)$

Step 1 Check that

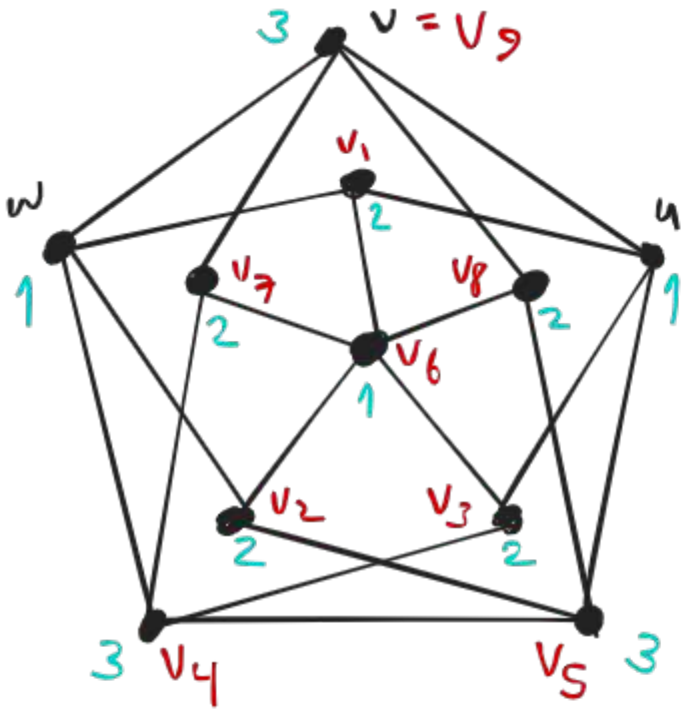
$k \leq 3 \implies G = C_{2k+1}$ or K_n

(eg last time: 3-critical $\Rightarrow G = C_{2k+1}$)

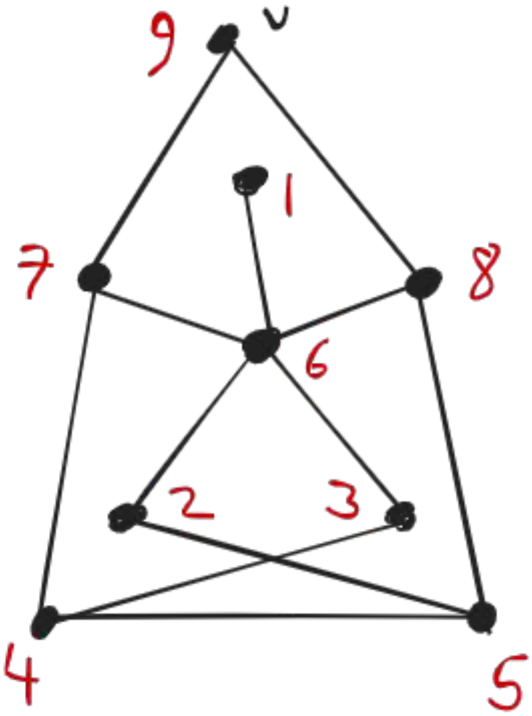
Thus we can assume $k \geq 4$

Step 2 G 3-connected or not

- G not 3-connected (last time: G has no vertex cut, so $\kappa(G) = 2$)
- G is 3-connected $\left(G \setminus S \text{ connected } \forall S \subset V \text{ with } |S| = 2 \right)$



Take path u, v, w w/ $\{u, w\} \in E$. order $V \setminus \{u, w\} = v_1, \dots, v_m = v$ decreasing dist to v in $G \setminus \{u, w\}$

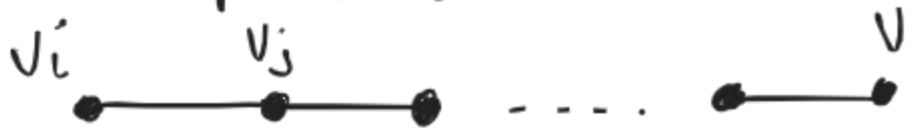


color u,w color 1.

color $v_1, \ldots, v_m$ in order using 1st available color from $\{1, \ldots, \Delta(G)\}$

Key observation: fix $i < m$

choose path v_i to v.



$j > i$

when we choose color of v_i, at most $\Delta(G) - 1$ neighbors already colored, so there is color available

When we get to v: u,w have same color so $\leq \Delta(G) - 1$ among neighbors. ✓

Q: Where did we use G 3-conn?

Ans: $G \setminus \{u, w\}$ connected so dist. to v makes sense

---

$\chi(G)$ and extremal problems

Q: Fix n, k. Among graphs $G = (V, E)$ with $|V| = n$ & $\chi(G) = k$, what is max/min value of $|E|$?

$$\left(\begin{array}{l} |E| = 0 \Rightarrow \chi = 1 \\ |E| = \binom{n}{2} \Rightarrow \chi = n \end{array} \right)$$

Prop (Minimizing edges)

$\chi(G) = k \Rightarrow |E| \geq \binom{k}{2}$.

<u>Proof</u> Color G with $1, \dots, k$

<u>Claim</u> For each $1 \le i, j \le k$ $\exists$ edge i — j.

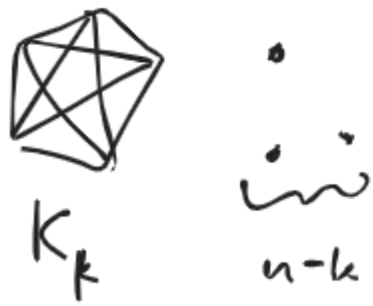
Claim $\Rightarrow$ $|E| \geq \binom{k}{2}$.

For definiteness, suppose there's no edge 1 — 2

Change each 2 to 1.

This is a coloring. $\Rightarrow \chi(G) \le k-1$.

$\ast$ $\square$

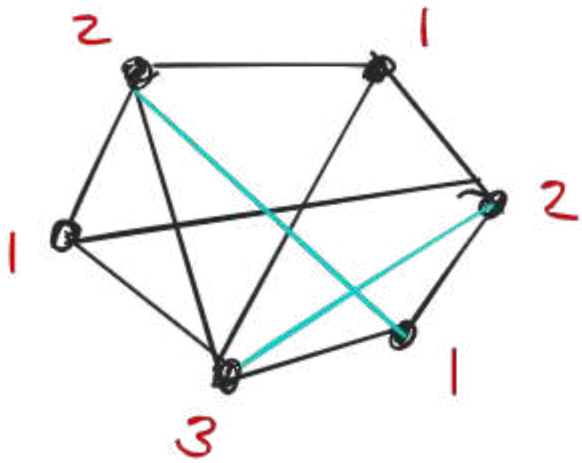
This is optimal eg. $G =$ 

Has $\binom{k}{2}$ edges

and $\chi(G) = k$.

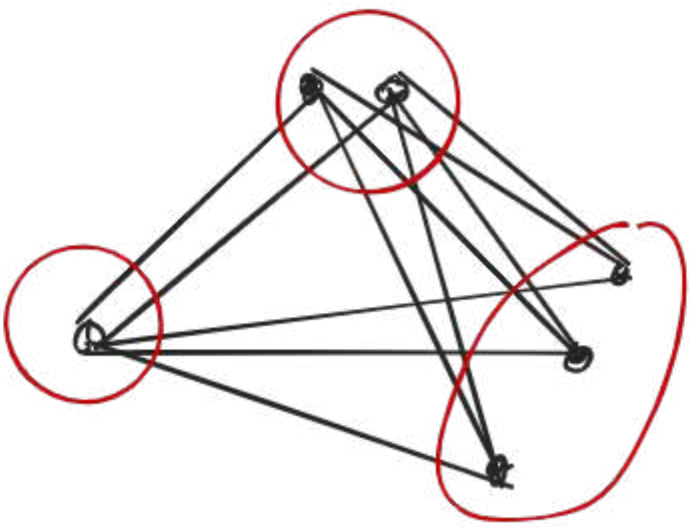
Maximizing edges

Observe: If G is k-colored can add edges between vertices of different color to get k-colored graph w/ more edges

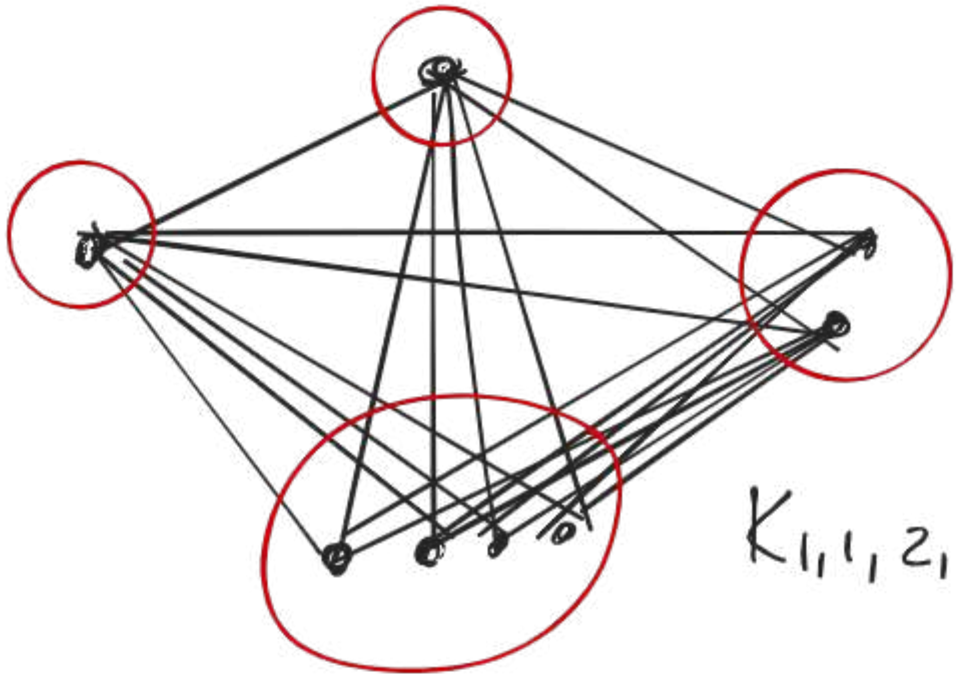


This leads us to consider:

Defn A complete multipartite graph



$K_{1,2,3}$



$$\chi(K_{n_1,\ldots,n_k}) = k$$

count edges ith group vert.
have deg $n-n_i$

$$2|E| = \sum_{i=1}^{k} (n-n_i)\cdot n_i$$

Which partition of n into k groups produces most edges?

Guess: partition into equal sizes.

Fix n, k. write $n = xk + y$

$0 \leq y < k$.

$T_{n,k}$:= complete k-partite graph

y groups of size $x+1$

$k-y$ groups of size x

$$\left((k-y)x + y(x+1) = kx + y = n. \right)$$

eg

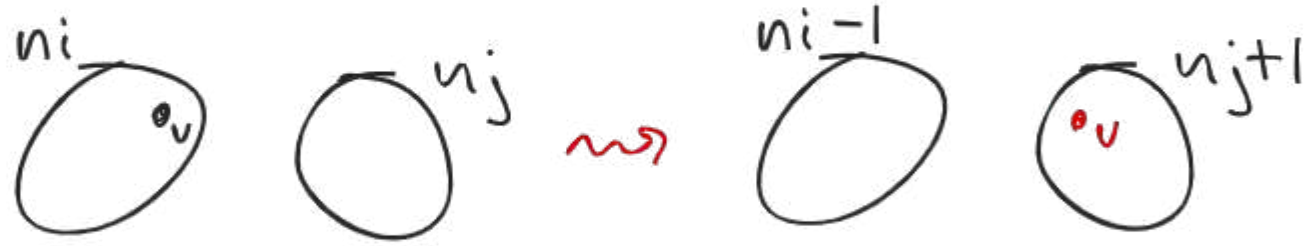
$$T_{14,3} = K_{4,5,5} \qquad 10 = 4+5+5$$

$$T_{12,5} = K_{2,2,2,3,3} \qquad 12 = 2+2+2+3+3$$

Prop Among k-chromatic graphs w/ n vertices $T_{n,k}$ has most edges.

<u>Proof</u> As observed, only need to consider graphs $K_{n_1,\dots,n_k}$

If $n_i - n_j \geq 2$, move vertex v from ith group to jth.



count degrees

$$(n - n_i + 1)(n_i - 1) + (n - n_j - 1)(n_j + 1)$$

$$= (n - n_i) n_i + (n - n_j) n_j + 2\underbrace{(n_i - n_j - 1)}_{\geq 1}$$

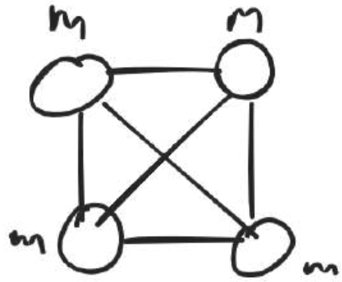
conclude

$K_{n_1 \cdots n_i - 1 \cdots n_j + 1 \cdots n_k}$

has more edge ... $\square$

$T_{n,k}$ called Turan graph.

Ramsey Theory



Last time: $T_{n,r}$

eg $T_{mr,r} = K_{\overbrace{m,m,\ldots,m}^{r}}$

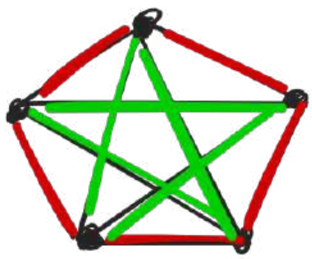
complete r-partite graph

$T_{n,r}$ extremal: most edges among n vertex graphs w/ $\chi = r$.

$T_{n,r}$ called Turan graphs

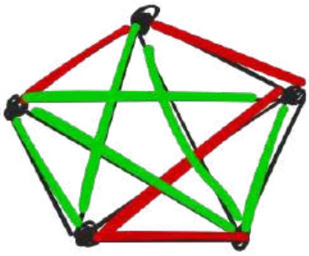
Today: different extremal problem

Warmup Problem given social network of n people.



no

Any two either friends or strangers. Does there exist 3 mutual friends or 3 mutual strangers?



Yes

Graph theory translation

2-color edges of K_n

what is smallest n s.t.

every 2 coloring of K_n has a monochromatic triangle?

<u>Prop</u> Every 2-coloring of K_6 has a monochromatic triangle. (so answer is 6)

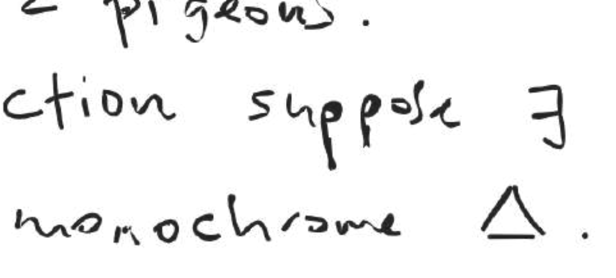
<u>Basic fact</u> (pigeonhole principle)

if put $n+1$ pigeons in n holes, one hole has 2 pigeons.

<u>Proof</u> By contradiction suppose $\exists$ coloring w/ no monochrome Δ.

Fix v.

wlog 3 of 5 edges incident to v are red. No red $\Delta \Rightarrow$ all edges betwn



x, y, z blue so x, y, z span

blue triangle $\cancel{\ }$. $\square$

<u>General Problem:</u> Given k, ℓ, compute

$R(k,\ell) :=$ smallest n s.t. any red/blue coloring

of edges of K_n has either red K_k

or blue K_ℓ.

<u>Ex</u> $R(3,3) = 6$ above

$\boxed{\text{Ex}}$ $R(2,\ell) = \ell$ for $\ell \geq 2$.

Take coloring of $K_{\ell-1}$ with only blue edges

$\Rightarrow R(2,\ell) \geq \ell$.

For coloring of K_ℓ, either exists

red edge ($= K_2$) or all edges blue

so have blue K_ℓ ✓

Exercise $R(k,\ell) = R(\ell,k)$

(given any graph can swap colors...)

Known computations of $R(k,k)$

$R(3,3)=6$, $R(4,4)=18$,

$$43 \le R(5,5) \le 48$$

Erdős: aliens $R(5,5)$ okay $R(6,6) \to$ war.

Our goal: ① upper/lower bounds

② give general context: order in chaos.

$\boxed{\text{Ramsey upper bounds}}$

Thm $R(k,\ell) \le R(k-1,\ell) + R(k,\ell-1)$

eg $R(3,3) \le R(2,3) + R(3,2)$

$= 3 + 3 = 6.$

made with ziteboard

$18 = R(4,4) \le R(3,4) + R(4,3)$

$\le R(2,4) + R(3,3) + R(3,3) + R(4,2)$

$= 4 + 6 + 6 + 4 = 20$

<u>Proof</u> Set $n = R(k-1, \ell) + R(k, \ell-1)$

<u>Fix</u> Any edge 2-coloring of K_n

<u>WTF</u>: red K_k or blue K_ℓ.

Fix vertex v.

Among $n-1$ edges

incident to v, let $R = \#$ red, $B = \#$ blue

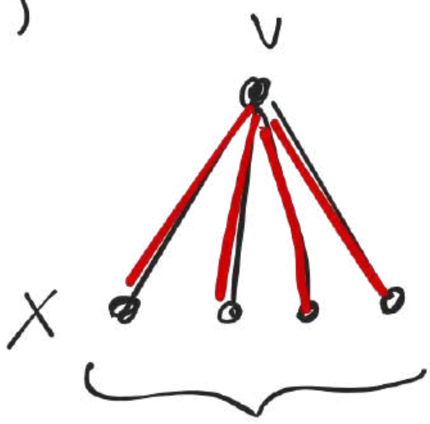
<u>Note</u> $R \le R(k-1, \ell) - 1$ & $B \le R(k, \ell-1) - 1$

$\Longrightarrow n-1 = R + B \le \underbrace{R(k-1,\ell) + R(k,\ell-1)}_{n} - 2$ $*$.

So either $R \geq R(k-1, \ell)$ or $B \geq R(k, \ell-1)$

(or both)

Say $R \geq R(k-1, \ell)$



The complete graph spanned by X has either $\underbrace{\text{red } K_{k-1}}_{\text{combine w/ } v \text{ to get red } K_k.}$ or $\underbrace{\text{blue } K_\ell}_{\text{done}}$

$\square$

Next time: lower bounds by "probabilistic method"

$\boxed{\text{Ramsey Theory \& arithmetic progressions}}$

Ramsey slogan: every very $\underbrace{\text{large system}}_{\text{2 color } E(K_n)}$ has a large well-$\underbrace{\text{organized subsystem}}_{\text{monochromatic } K_\ell}$

Defn An arithmetic progression is sequence of evenly spaced pos. integers

$a, a+s, a+2s, \ldots, a+(k-1)s.$

s is step size, k is length

eg 5, 12, 19, 26, 33, 40 $\quad s=7$
$k=6$

Thm (van der Waerden, 1929)

For every r, k $\exists$ $N(r,k)$ st.
for any $n \geq N(r,k)$ any r-coloring
of $\{1, \ldots, n\}$ has a monochromatic
progression of length k.

Revisit slogan

Ex $k=2$ any two #'s for
progression of length 2.
any r-coloring of $\{1, \ldots, r+1\}$
has two #s w/ same color
so take $N(r,2) = r+1$.

Ex $r=2$.

Claim Every 2-coloring of $\{1,\dots,9\}$ has monochromatic, length 3 progression so $N(2,3) \le 9$.

in fact $N(2,3) = 9$ since

1 2 3 4 5 6 7 8

has no monochrom length 3 prog.

Pf of Claim By contradiction suppose $\exists$ coloring w/ no mono 3 term prog.

wlog 5 is red

1 2 3 4 5 6 7 8 9

1, 9 not both red

Two cases: 1, 9 both blue or not.

Case 1

3, 7 not both red. wlog 3 blue.

$\Rightarrow$ 2 red $\Rightarrow$ 8 blue $\Rightarrow$ 7 red

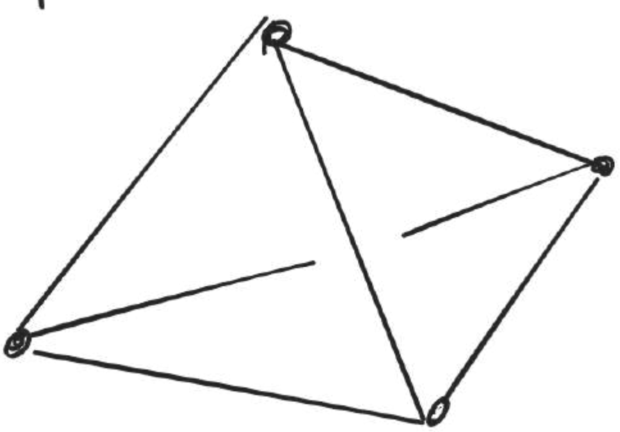
$\Rightarrow$ 6 blue $\Rightarrow$ 3, 6, 9 blue length progression $\divideontimes$.

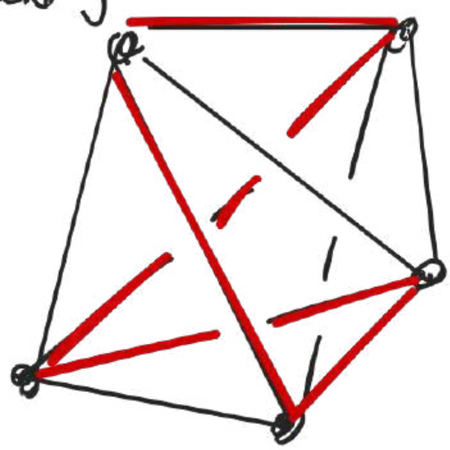
Case 2

1 2 3 4 5 6 7 8 9 $\square$

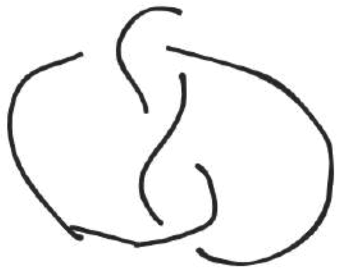
$\boxed{\text{Negami's Thm}}$

Take n points in $\mathbb{R}^3$ in general position. Get straight line embedding $K_n \subset \mathbb{R}^3$





look for knots

eg  

<u>Thm</u> (Negami) Let A be any knot $\exists\ N_A > 0$ s.t. every straight-line embedding of K_{N_A} in $\mathbb{R}^3$ contains A. (!)

Return to slogan

$\boxed{\text{Ramsey lower bound}}$

Last time

- $R(k,k) = \min \left\{ n \;\middle|\; \begin{array}{c} \text{every edge 2-color} \\ \text{of } K_n \text{ has} \\ \text{monochromatic } K_k \end{array} \right\}$

- gave inductive bound

$$R(k,l) \leq R(k-1,l) + R(k,l-1)$$

Cor (HW8) $R(k,k) \leq 2^{2k}$.

Thm For $k \geq 4$ $R(k,k) > 2^{k/2}$.

(so $R(k,k)$ grows exponentially)

- recall last time showed

$R(3,3) \geq 6$ by finding coloring of K_5 w/ no monochrome Δ's.

one way to pf thm is to find coloring of $K_{2^{k/2}}$ w/ no monochrome K_k

- we take different approach.

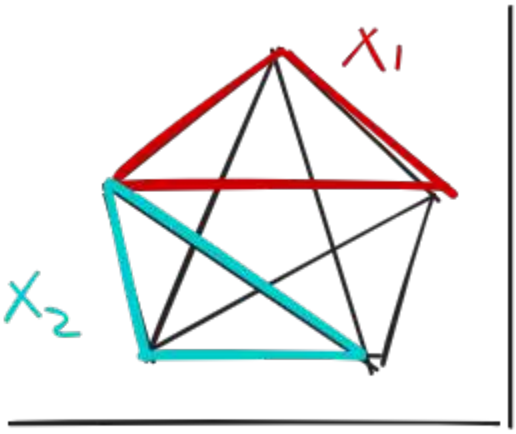
Fixing n, count/estimate # 2-colorings of $E(K_n)$ w/ monochromatic K_k. Show if $n \leq 2^{k/2}$, this is less that total # of colorings. $2^{\binom{n}{2}}$. Conclude $\exists$ coloring of $K_{2^{k/2}}$ w/ no monochr. K_k.

<u>Proof</u> Fix n, k.

want to count colorings of K_n w/ monochromatic K_k.

1 How many K_k subgraphs in K_n?

$\binom{n}{k}$. Set $m = \binom{n}{k}$ let $X_1, \dots, X_m$ the K_k's in K_n



(2) How many colorings w/ X_i monochromatic?

$$\underbrace{2}_{\substack{\text{red}\\\text{blue}}} \cdot \underbrace{2^{\binom{n}{2}-\binom{k}{2}}}_{\text{color } E(K_n \setminus K_k)}$$

$\Longrightarrow$

colorings where some X_i monochr is $\leq \binom{n}{k} \cdot 2 \cdot 2^{\binom{n}{2}-\binom{k}{2}}$

(This is an overcount.)

(3) want $\binom{n}{k} \cdot 2 \cdot 2^{\binom{n}{2}-\binom{k}{2}} < 2^{\binom{n}{2}}$

Equivalently want $\dfrac{\binom{n}{k} \cancel{2^{\binom{n}{2}}} \cdot 2}{2^{\binom{k}{2}}} < \cancel{2^{\binom{n}{2}}}$

$$\binom{n}{k} < 2^{\binom{k}{2}-1} = 2^{(k^2-k-2)/2}$$

observe $\binom{n}{k} = \dfrac{n(n-1)\cdots(n-k+1)}{k(k-1)\cdots 2 \cdot 1} < \dfrac{n^k}{2^{k-1}}$

Now if $n \leq 2^{k/2}$ then

$$\binom{n}{k} < \frac{n^k}{2^{k-1}} \leq 2^{k^2/2 - k - 1} = 2^{\frac{k^2-2k-2}{2}}$$

(RHS) is $\leq 2^{(k^2-k-2)/2}$

as long as $k^2 - 2k - 2 \leq k^2 - k - 2$

$\iff k \geq 4.$

Summary if $k \geq 4$ and $n \leq 2^{k/2}$

# of colorings of K_n w/ monochr K_k is < total # colorings $\square$.

$$\boxed{\text{Ramsey \& Fermat}}$$

Consider equation $X^n + Y^n = Z^n$

ask for nonzero integer solutions (disregard eg $(0,0,0)$ & $(X,0,X)$.)

$\underline{n=1}$ This is easy. (sum of int is int).

$\underline{n=2}$ when is sum of squares a square?

Pythagorean triples. There are many

eg $(X,Y,Z) = (3,4,5)$ or $(3\ell, 4\ell, 5\ell)$ any ℓ.

$\underline{\text{Thm}}$ (Fermat's last Thm, Wiles) 1637, 1995

For $n \geq 3$ $X^n + Y^n = Z^n$ has no nontrivial integer solutions

An easier problem (poll familiarity)

Recall $\mathbb{Z}/m\mathbb{Z} = \{0, 1, \ldots, m-1\}$

with addition, multiplication mod m

eg in $\mathbb{Z}/3\mathbb{Z}$ $1+1=2$, $1+2=0$

$1 \cdot 1 = 1$, $2 \cdot 2 = 1$.

Q: Does $X^n + Y^n = Z^n$ have solutions in $\mathbb{Z}/m\mathbb{Z}$?

eg $n=3$

- $m=3$ no solutions

  $1^2=1$, $2^2=1$ $\quad X^2+Y^2 \equiv 2$

  $z^2 \equiv 1$

- $m=7$ solutions!

  $1^2+1^2 = 2 = 3^2$

Thm (Schur) Fix $n \geq 1$. $\exists N$ s.t. if $p > N$ prime then $\exists$ solution to $X^n + Y^n = Z^n$ in $\mathbb{Z}/p\mathbb{Z}$.

Proved by Ramsey theory!

Prop (Schur) $\forall\, n, p \;\exists\, N(n,p)$ s.t. for $p \geq N(n,p)$ every n-coloring of $\{1, \dots, p\}$ contains monochromatic x, y, z w/ $x + y = z$.

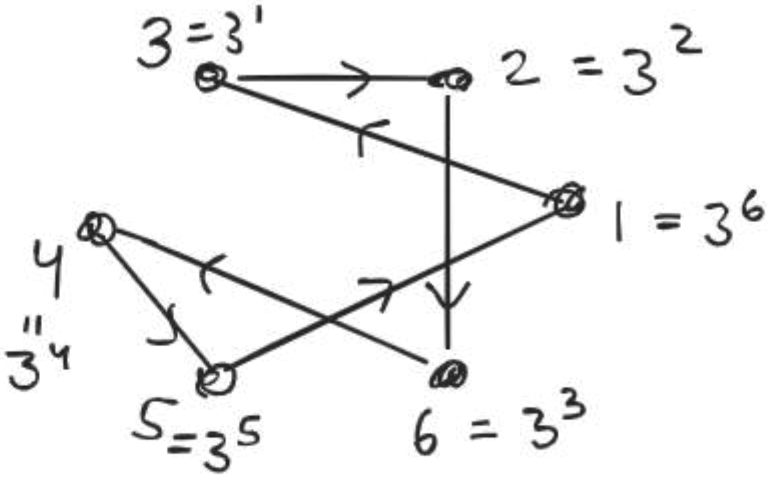
(Similar to Ramsey, van der Waerden
Ramsey slogan)

Proof of Schur's Thm (using prop)

(somewhat advanced - take it in)

Want $X, Y, Z \in \mathbb{Z}/p\mathbb{Z} \setminus \{0\}$
$$X^n + Y^n = Z^n. \quad \text{(for } p \text{ large)}$$

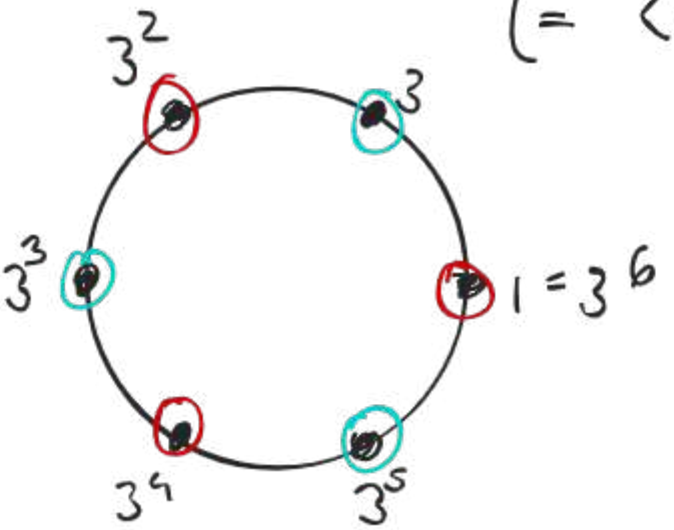
Fact Multiplicative group $(\mathbb{Z}/p\mathbb{Z})^\times = \{1, \dots, p-1\}$ is cyclic. eg $\langle 3 \rangle = (\mathbb{Z}/7\mathbb{Z})^\times \cong (\mathbb{Z}/6\mathbb{Z}, +)$



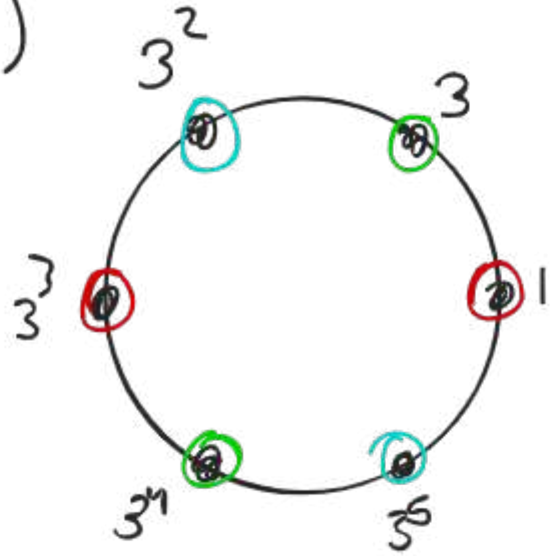
Consider $H \subset (\mathbb{Z}/p\mathbb{Z})^{\times} = \langle a \rangle$

Subgroup generated by nth powers.

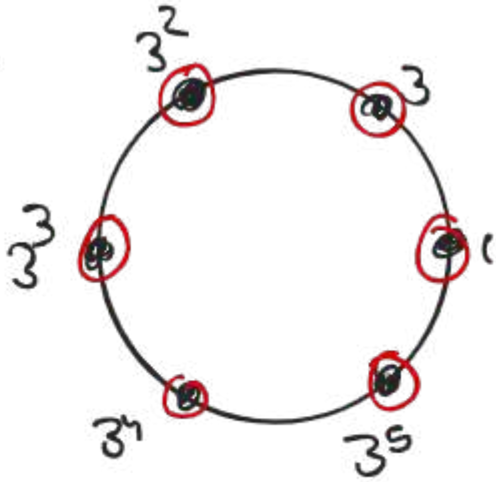
$(= \langle a^n \rangle)$



H for n=2



H for n=3



H for n=5

H splits $(\mathbb{Z}/p\mathbb{Z})^\times$ into at most n cosets. So get n-coloring of $\{1, \dots, p-1\}$

Schur $\Rightarrow$ for $p \gg 0$, $\exists$ $x, y, z \in \{1, \dots, p-1\}$ of same color.

with $x+y=z$.

Same color $\iff$ same coset εH, $\varepsilon \in (\mathbb{Z}/p\mathbb{Z})^\times$

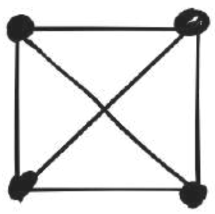
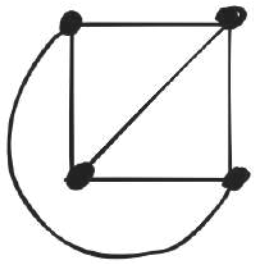
$x = \varepsilon X^n \quad y = \varepsilon Y^n \quad z = \varepsilon Z^n$

$x + y = z \Rightarrow \varepsilon X^n + \varepsilon Y^n = \varepsilon Z^n$

$\varepsilon \neq 0 \Rightarrow X^n + Y^n = Z^n$ in $\mathbb{Z}/p\mathbb{Z}$. $\square$

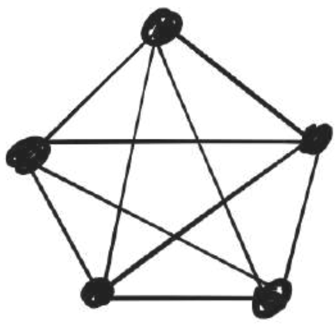
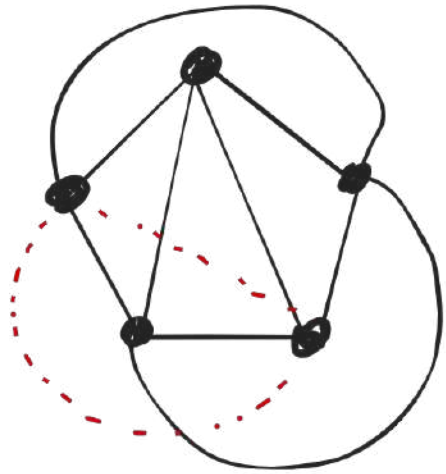
Planar graphs

$G=(V,E)$ is planar if it can be drawn in plane w/o edges crossing.
A drawing of G in $\mathbb{R}^2$ is called an embedding.

Ex's ①  

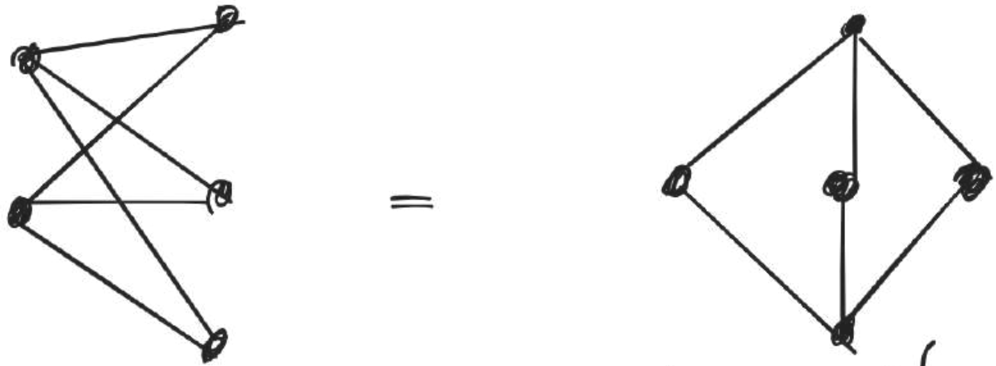
nonplanar embedding of K_4

planar emb of K_4

②  

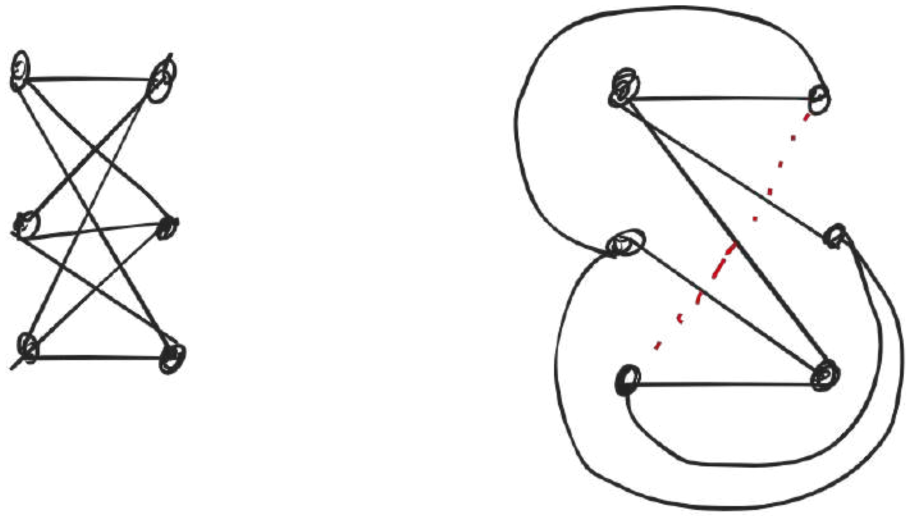
Seems nonplanar ... (?)

(3)



$K_{2,3}$ planar

(4)



Q: Given $G=(V,E)$, how to decide if G planar?

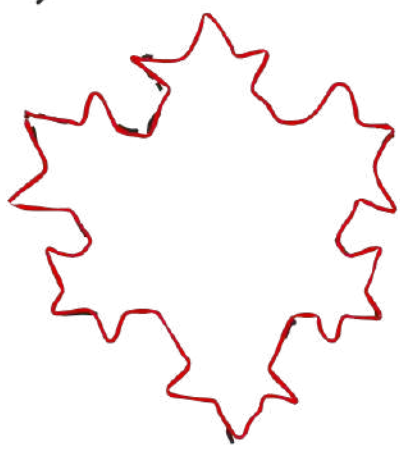
Eg underground subway ...

Thm K_5 & $K_{3,3}$ are not planar.

Rmk Need to understand what's special about $\mathbb{R}^2$. eg vs $T^2 \supset K_{3,3}$.

Euler's Formula

<u>Topological fact</u> any embedding of the circle in $\mathbb{R}^2$ splits $\mathbb{R}^2$ into two regions, bounded & unbounded (Jordan curve theorem)

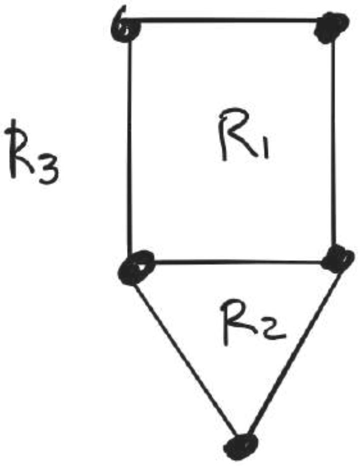


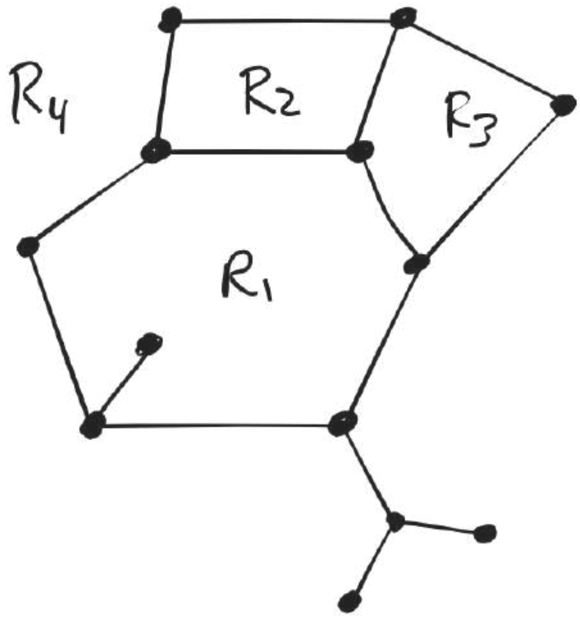
easy for polygonal curves

harder for arbitrary curves (Koch snowflake)

---

if $G=(V,E)\subset\mathbb{R}^2$ planar embedding, then G splits $\mathbb{R}^2$ into regions



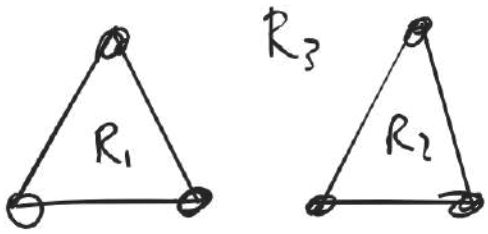


Consider $|V| - |E| + F$ # regions of $\mathbb{R}^2 \setminus G$

$|V|-|E|+F = 5-6+3 = 2$ | $|V|-|E|+F = 13-15+4 = 2$

Thm (Euler's Formula) For any connected planar graph $|V|-|E|+F=2$

Connected is necessary!



$6-6+3 = 3\ldots$

Proof by induction on $|E|-|V|$.

Base case $|E|-|V| = -1$. $\left(\begin{matrix} \text{ie} \\ |E| = |V|-1 \end{matrix}\right)$

$\Rightarrow$ G is a tree.

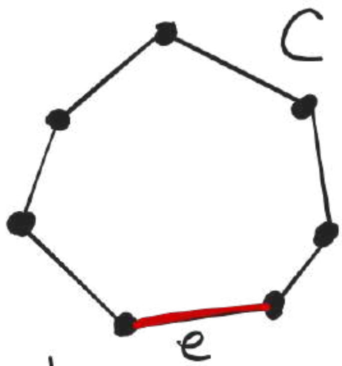
$\Rightarrow$ $\mathbb{R}^2 \setminus G$ has one component

(G has no cycle)

$|V|-|E|+F = 1+F = 2$

Induction step $|E|-|V| > -1$

$\Rightarrow$ G has cycle C.



$G' = G \setminus e$ connected

$|V(G')| = |V(G)|$

$|E(G')| = |E(G)|-1$

$F(G') = F(G)-1$

induction $2 = |V(G')| - |E(G')| + F(G')$

$= |V(G)| - (|E(G)|-1) + (F(G)-1)$

$= |V(G)| - |E(G)| + F(G)$ $\square$

Cor K_5 not planar.

Pf By contradiction suppose $K_5 \subset \mathbb{R}^2$.

Let $F = \#$ regions.

Step 1 $F \leq 6$

observe: every edge

is part of two regions (every edge in a cycle)

Every region has ≥ 3 sides $\Longrightarrow$

$$3F \leq \sum_{\substack{\text{regions} \\ R}} \text{sides}(R) = 2|E| = 2(10)$$

Step 2 Euler's formula:

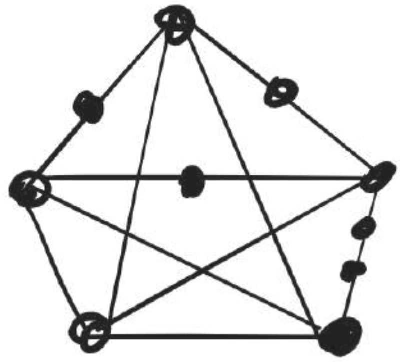
$$2 = |V| - |E| + F = 5 - 10 + F \leq 5 - 10 + 6 = 1$$

$\times$ $\square$

Similar: $K_{3,3}$ not planar.

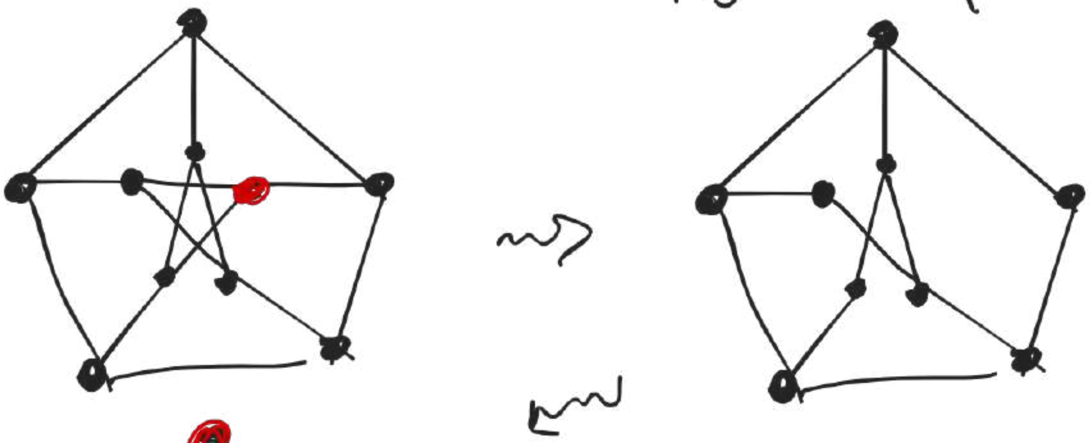
K_5 & $K_{3,3}$ simplest nonplanar graphs.

More nonplanar graphs: subdivisions

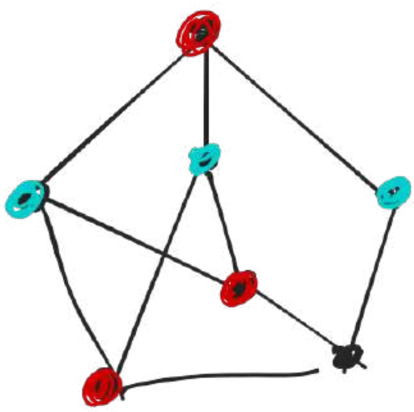
$G =$ 

Also not planar:

if $G \subset \mathbb{R}^2$ planar can delete vertices to get $K_5 \subset \mathbb{R}^2$ planar

Eg 

so Petersen not planar.

 $= K_{3,3}$

Thm (Kuratowski) TONCAS!

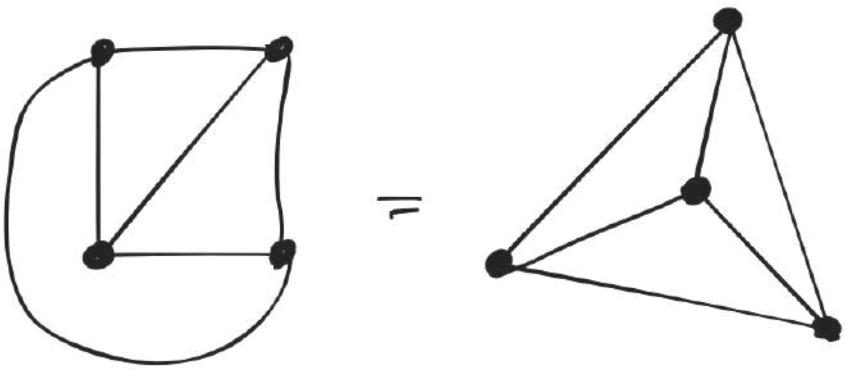
G planar $\iff$ G doesn't contain subdivision of K_5 or $K_{3,3}$.

Fary's Thm

Q: If G planar, does G have a "nice" planar embedding?

e.g. where all edges are straight lines?

Ex K_4



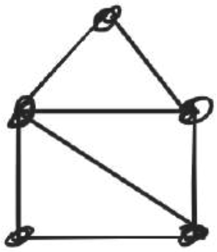
Thm (Fary) If G planar then it has a linear planar embedding.

__Defn__ A planar graph G is __maximal__ if $\exists$ G' with $G \subset G'$ and $V(G) = V(G')$

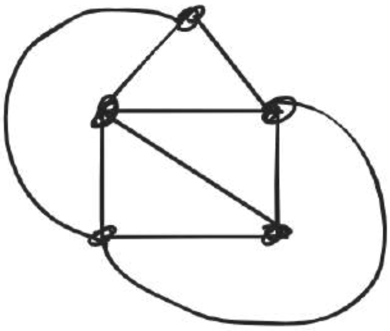
__Ex__

planar, not maximal

since



also planar



maximal since only edge to add gives K_5 (not planar)

__Prop__ Let $G = (V, E)$ planar. $|E| \geq 2$. TFAE

(1) G max planar

(2) $|E| = 3|V| - 6$

(3) components of $\mathbb{R}^2 \setminus G$ are triangles

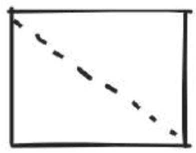
See example

__Proof of (1) $\Leftrightarrow$ (3):__

(1) $\Rightarrow$ (3): Contrapositive.

Components of $\mathbb{R}^2 \setminus G$ are n-gons for $n \geq 3$.

If have n-gon w/ $n \geq 4$ then can add edge



(3) $\Rightarrow$ (1): Contrapositive

$G \subsetneq G' \subset \mathbb{R}^2$ $\exists$ added edge that splits region of $\mathbb{R}^2 \setminus G$ ...

Euler's Formula

- Euler's Formula: $G = (V, E) \subset \mathbb{R}^2$ planar

$$\Rightarrow |V| - |E| + F = 2$$

↑ regions of $\mathbb{R}^2 \setminus G$

Consequences

① <u>Lemma</u> if $G = (V, E)$ planar and $|E| \geq 2$ then $|E| \leq 3|V| - 6$

with equality $\Leftrightarrow$ each component of $\mathbb{R}^3 \setminus G$ has 3 sides

$$\left(3F \leq \sum_{\text{region } R \text{ of } \mathbb{R}^2 \setminus G} \text{sides}(R) = 2|E| \right) + \text{Euler}$$

② For K_5: $|E| = 10$ & $3|V| - 6 = 9$

so K_5 not planar

③ <u>Prop</u> G planar $\Rightarrow$ G has a vertex of degree ≤ 5.

<u>Proof</u> By contradiction if

$\deg(v) \geq 6 \quad \forall\, v \in V$ then

$$6|V| \leq \sum \deg(v) = 2|E| \quad (\text{deg sum})$$

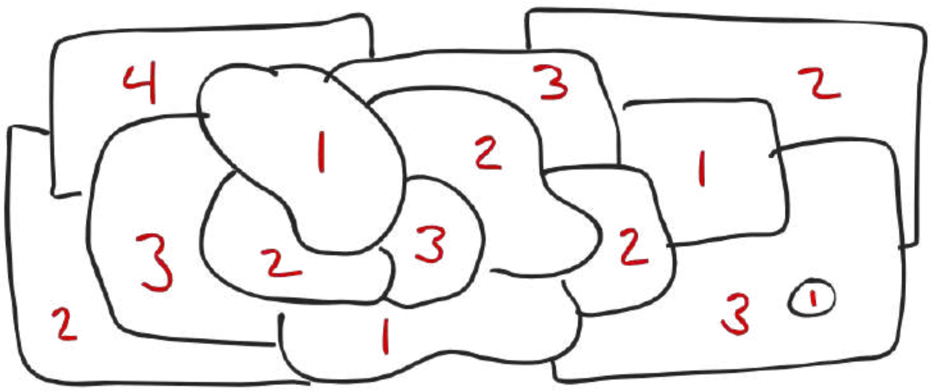
$$\underset{\text{lemma}}{\leq} 2\big(3|V| - 6\big) = 6|V| - 12$$

$\times$ ∎

(In fact, G has ≥ 4 vertices of deg ≤ 5.)

↳ Exercise

④ <u>Thm</u> (6-color Thm) Every map can be colored w/ 6 colors.



Proof of 6-color Thm Suffices to

Show planar graph has vertex 6-coloring. Use induction on $|V|$.

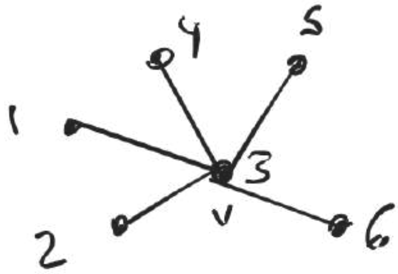
- Base case: $|V| \leq 6$

- Induction step: Fix $G=(V,E)$ planar w/ n vertices.

Prop $\Rightarrow \exists\ v \in V\ \deg(v) \leq 5$

Then $G \setminus v$ planar w/ $n-1$ vertices

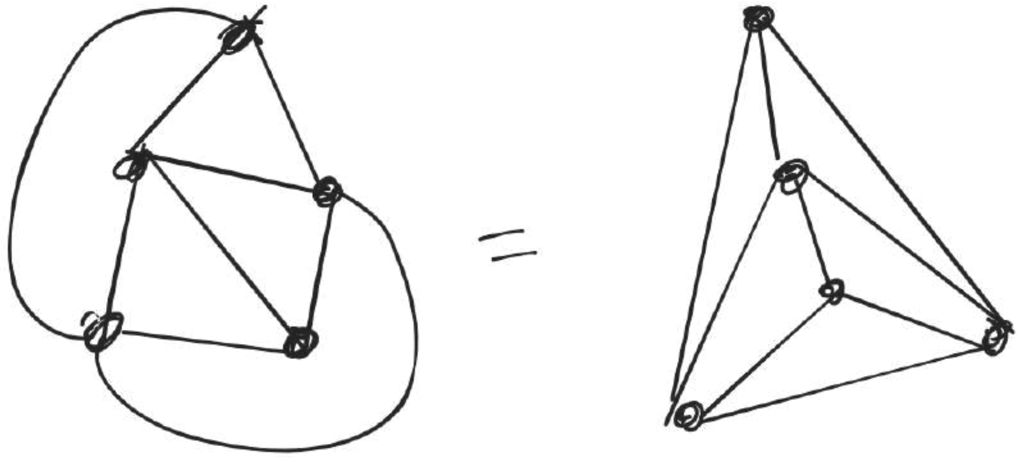
$\Rightarrow G \setminus v$ has a 6-coloring



$\Rightarrow G$ has a 6-coloring $\square$

5 color Thm (Arturo, Sameerah)

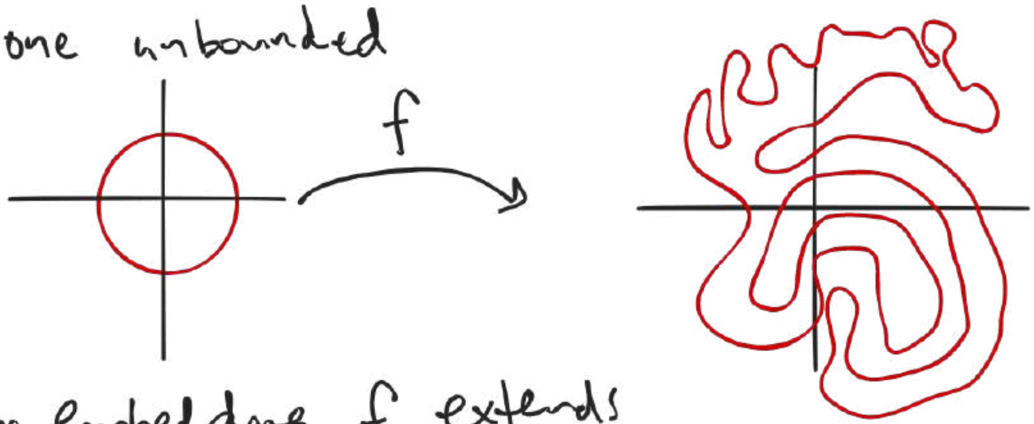
Fary's Thm If G planar then G has a linear embedding.

eg 

Key ingredients

① Jordan curve theorem

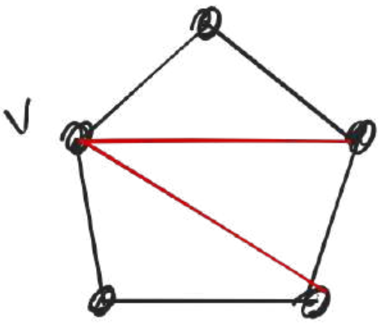
- An embedded circle in $\mathbb{R}^2$ divides $\mathbb{R}^2$ into two components one bounded one unbounded



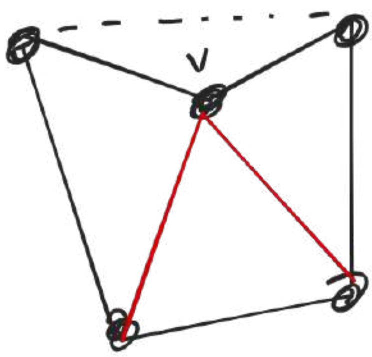
- Any embedding f extends to a topological equivalence $\mathbb{R}^2 \to \mathbb{R}^2$

(Jordan-Schönflies)

② linear embeddings of C_5



convex hull(V) = pentagon



convex hull(V) = quad

convex hull(V) = triangle

(consequence of)
Art gallery theorem:
every linear emb. of C_5 has vertex that can see entire interior. Perturb to get interior vertex w/ same property

(Romina, Timothy, Jacob)

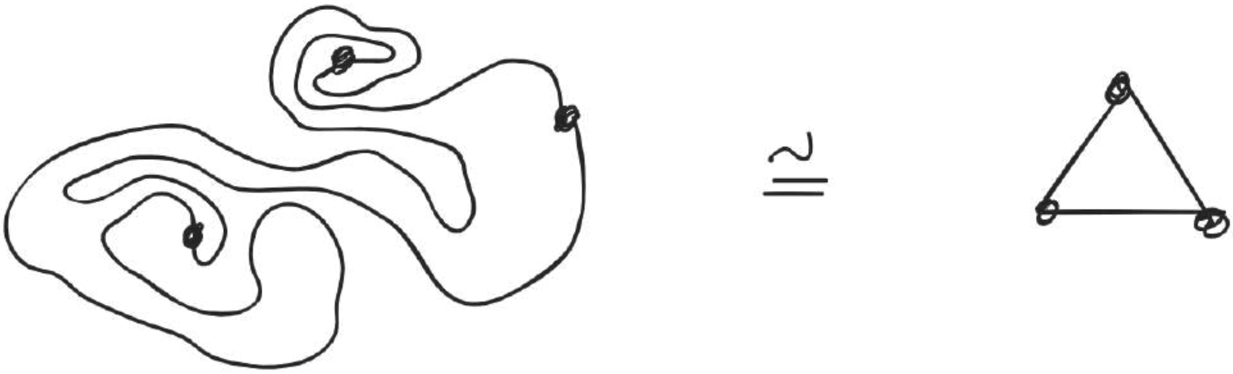
Proof sketch of Fary's Thm

Prove stronger statement: given planar $G \subset \mathbb{R}^2$ $\exists$ $h: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ (topological equivalence)

s.t. $h(G)$ is linear.

By induction on $|V|$.

- Base case: $|V| = 3$ (Jordan curve Thm)

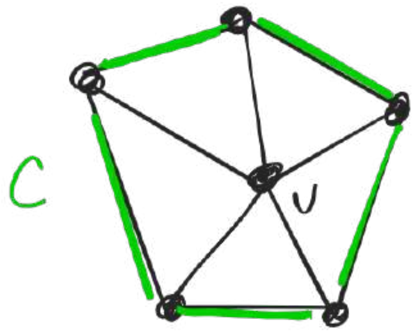


- Induction Step: Given $G \subset \mathbb{R}^2$ $|V(G)| = n$.

  wlog G maximal. (Subgraph of linear is linear)

  Above: G has ≥ 4 vertices of $\deg \leq 5$.

  $\Rightarrow$ one is "interior"



By induction applied to

$G' = G \setminus v$

$\exists$ homeo $h : \mathbb{R}^2 \to \mathbb{R}^2$

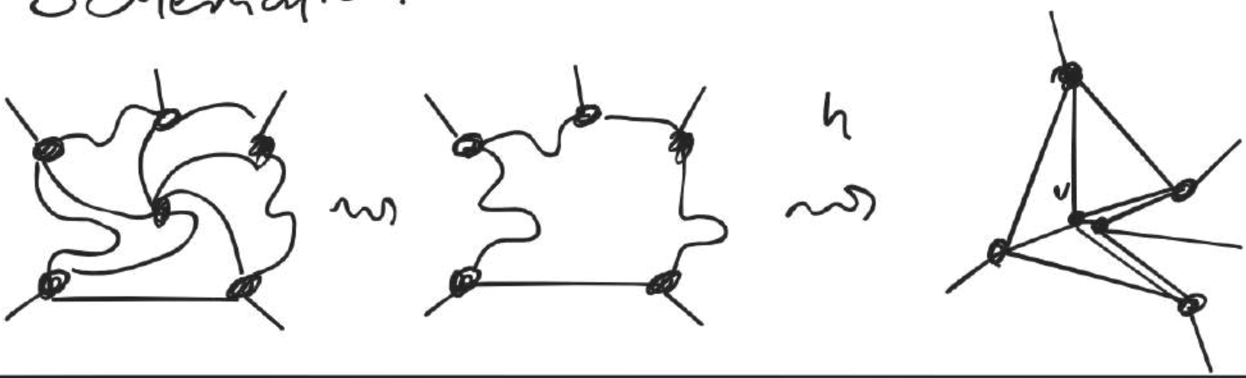
s.t. $h(G')$ linear.

Now $h(C)$ is a linear C_5

add v to interior using art gallery.

$\square$

Schematic:



Algorithmic Planar Embedding

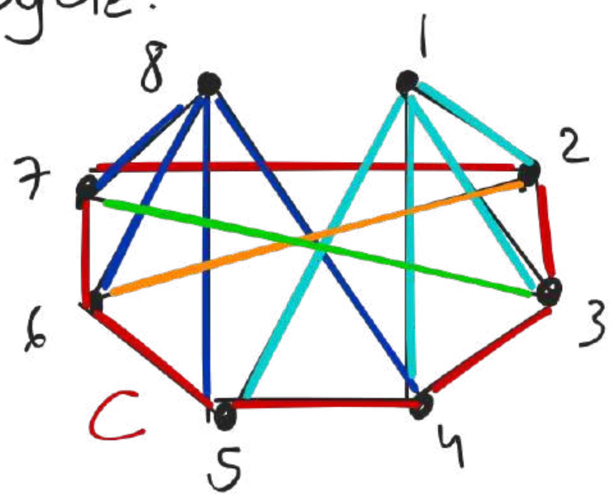
(1) decide if planar

(2) if planar construct emb

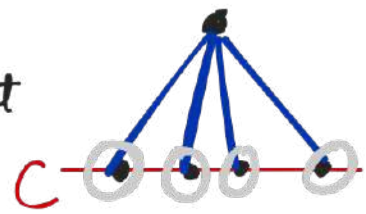
Conflict graph

G graph, $C \subset G$ cycle.

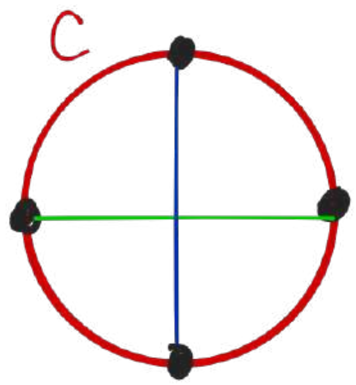
Fragments $\longleftrightarrow$ components of $G \setminus C$ or edges connecting vertices of C.



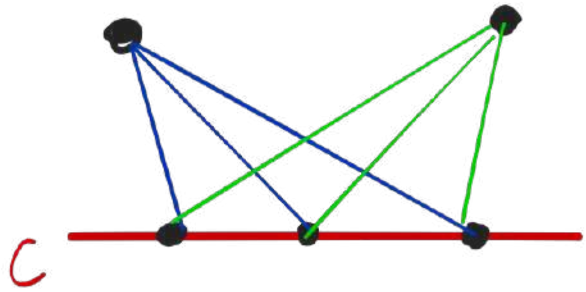
Points of contact of a fragment



Two fragments conflict if



or



edge fragments whose endpts link on C

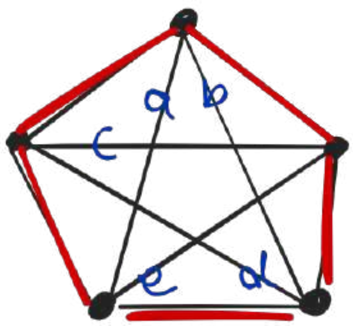
3 common points of contact

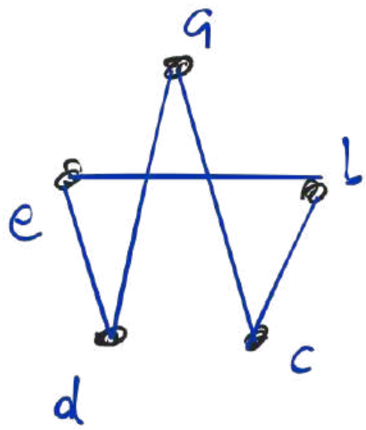
Conflict graph Γ of (G, C):

vertices $\leftrightarrow$ fragments, edges $\leftrightarrow$ conflicts.

For example above $\Gamma =$ 

<u>Ex</u>



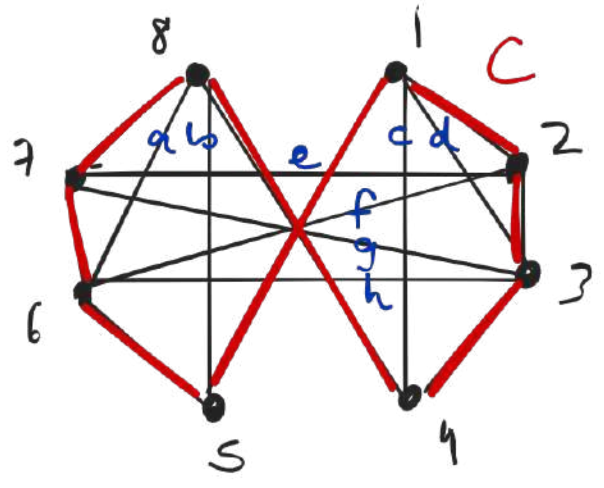
$\Gamma =$ 

<u>Observation</u> If G is planar then $\Gamma(G,C)$ is bipartite for each C

(for each conflict need to choose inside or outside)

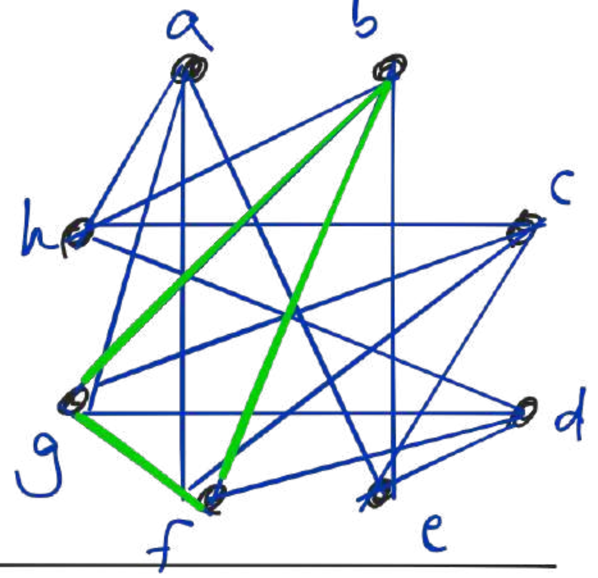
<u>Thm</u> (Tutte) TONCAS

G planar $\iff$ $\Gamma(G,C)$ bipartite $\forall$ cycles C.

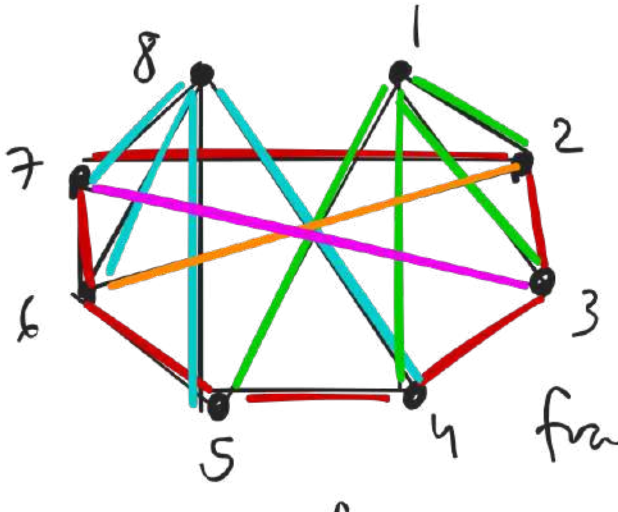


G not planar

$\Gamma(G,C)$ not bipartite.



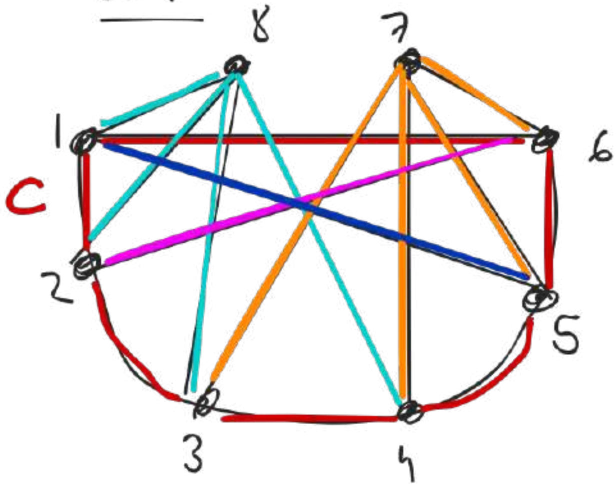
If G planar, can find planar embedding in "greedy" fashion

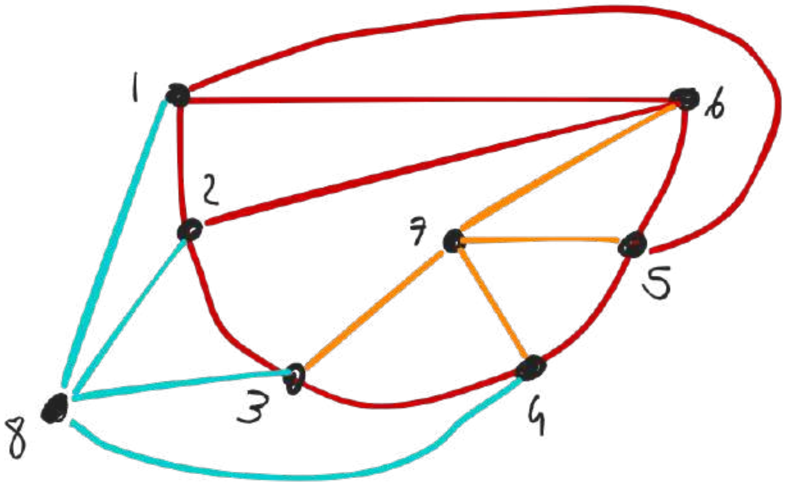


- pick a cycle. consider fragments
- successively add paths from fragment to embedded graph
- recompute fragments at each step. If G planar this terminates in planar embedding.

Algorithmic graph embedding

Ex

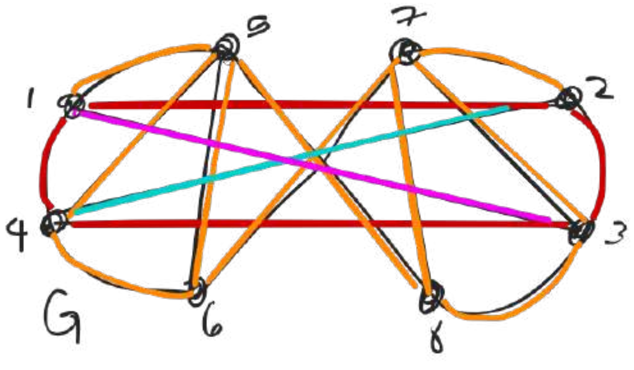


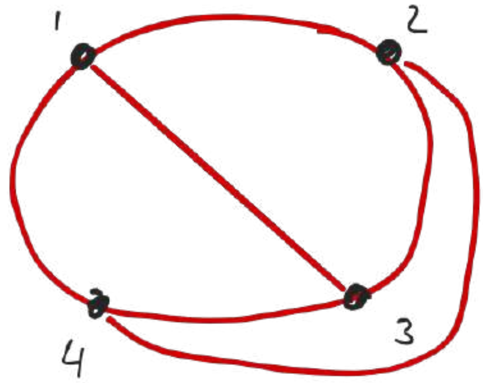


① Pick cycle C
consider fragments

② Successively choose
path in a fragment
and add (*) it to embedding

(*) need to attach fragment along region that
has all the vert. of attachment of the fragment.

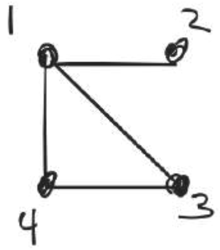




vertices of attachment of orange fragment = 1,2,3,4
There is no region containing all of these vertices
⟹ nonplanar

Spectral Graph Theory

$G = (V, E)$ $\quad V = \{v_1, \dots, v_n\}$



$$\underset{\substack{\uparrow \\ \text{Laplacian}}}{L} := \underbrace{\begin{pmatrix} 3 & & & \\ & 1 & & \\ & & 2 & \\ & & & 2 \end{pmatrix}}_{\text{degree matrix}} - \underbrace{\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}}_{\text{adjacency matrix}}$$

We are interested in <u>eigenvalues</u> of L.
and what they encode about G.

$$\boxed{\lambda \in \mathbb{C} \quad \text{s.t.} \quad Lx = \lambda x \quad \text{for some } x \in \mathbb{C}^n \text{ nonzero}}$$

<u>Some properties</u>

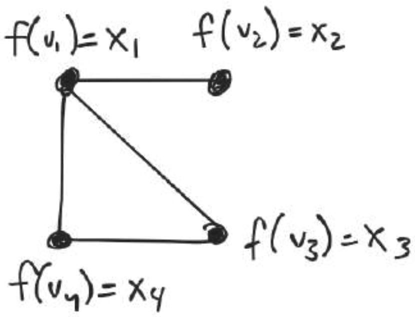
(1) eigenvalues are real because L is real, symmetric

(2) eigenvalues exist! <u>Spectral Thm</u> A symmetric real $n \times n$ matrix has n eigenvalues (w/ mult) $\lambda_1, \dots, \lambda_n$

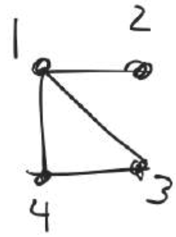
(3) eigenvalues are nonnegative (proof later)
$$0 \le \lambda_1 \le \dots \le \lambda_n$$

Identify $\mathbb{R}^n \longleftrightarrow$ function $V \xrightarrow{f} \mathbb{R}$

$$\mathbb{R}^4 \ni \underset{\substack{\| \\ (\pi,\ \sqrt{2},\ -1,\ \frac{3}{4})}}{(x_1, x_2, x_3, x_4)} \longleftrightarrow$$



$$\begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 3a - b - c - d \\ b - a \\ 2c - a - d \\ 2d - a - c \end{bmatrix}$$



$$L(f)(v_i) = \deg(v_i) f(v_i) - \sum_{v_i v_j \in E} f(v_j)$$

equivalently $$(Lx)_i = \deg(v_i) x_i - \sum_{v_i v_j \in E} x_j$$

Examples

① $G = K_n$
$$L = \begin{bmatrix} n-1 & -1 & \cdots & -1 \\ -1 & n-1 & & \vdots \\ \vdots & & \ddots & -1 \\ -1 & \cdots & -1 & n-1 \end{bmatrix}$$

multiplicity of λ as eigenvalue of L

is $\dim \ker (L - \lambda I)$

(in particular λ eigenval $\Leftrightarrow$ $L - \lambda I$ noninvertible
$\Leftrightarrow$ $\det(L - \lambda I) = 0$)

Observe
- 0 is eigenvalue, eigenvalue $(1, \dots, 1)$
(row sums are 0)
- $L - nI = \begin{pmatrix} -1 & \cdots & -1 \\ \vdots & & \vdots \\ -1 & \cdots & -1 \end{pmatrix}$ ker spanned by $(1, 0, \dots, \underset{i}{-1}, \dots, 0)$

$\Rightarrow n$ eigenvalue w/ mult $n-1$.

$$\lambda_1 = 0 \qquad \lambda_2 = \lambda_3 = \cdots = \lambda_n = n$$

② $G = C_n$

$$L = \begin{pmatrix} 2 & -1 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & -1 & 2 \end{pmatrix}$$

This is a circulant matrix

eigenvalues $2 - 2\cos\left(\frac{2\pi i k}{5}\right)$ $k = 1, 2, 3, 4, 5$

$\boxed{\lambda_1}$

<u>Thm</u> (i) $\lambda_1 = 0$.

(ii) $\underbrace{\dim \ker L}_{\text{multiplicity of } 0 \text{ as eigenvalue of } L.} = \#$ components of G

<u>Proof of (i):</u> Just need to give an eigenvector with eigenvalue 0.

$$0 = Lx \iff \underbrace{f(v_i) = \frac{1}{\deg(v_i)} \sum_{v_i v_j \in E} f(v_j)}_{\text{average value on neighbors}} \quad \forall i$$

$x = (x_1, \ldots, x_n) = (f(v_1), \ldots, f(v_n))$

If f constant, then $L(f) = 0$.

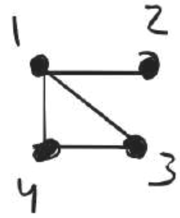
eg $\quad x = (1, \ldots, 1)$

<u>Proof of (ii)</u>

<u>Observation</u> $\quad Lx = 0 \iff x^t L x = 0$

$L = D - A$

<u>Example</u>



$$(x_1\,x_2\,x_3\,x_4)\begin{pmatrix}3&&&\\&1&&\\&&2&\\&&&2\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix} = 3x_1^2 + x_2^2 + 2x_3^2 + 2x_4^2$$

$$(x_1\,x_2\,x_3\,x_4)\begin{pmatrix}0&1&1&1\\1&0&0&0\\1&0&0&1\\1&0&1&0\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix} = 2\left[x_1x_2 + x_1x_3 + x_1x_4 + x_3x_4\right]$$

Generally

$$x^t L x = x^t D x - x^t A x$$

$$= \sum \deg(v_i)\, x_i^2 - \sum_{v_iv_j \in E} 2x_ix_j$$

$$= \sum_{v_iv_j\in E} x_i^2 - 2x_ix_j + x_j^2 = \sum_{v_iv_j\in E} (x_i - x_j)^2$$

$$0 = x^t L x = \sum_{v_iv_j\in E} (x_i - x_j)^2 \iff x_i = x_j \text{ whenever } v_iv_j \in E$$

so $L(f) = 0 \iff f$ constant on components of G.

and $\dim \operatorname{Ker}(L) = \#$ components $\square$

<u>Cor</u> Eigenvalues of L are ≥ 0.

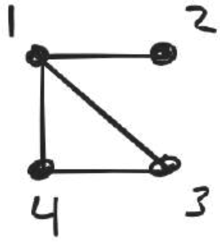
<u>Pf</u> $Lx = \lambda x$

$$\Rightarrow \lambda |x|^2 = x^t L x = \sum_{v_iv_j\in E} (x_i - x_j)^2$$

$$\Rightarrow \lambda = \frac{1}{|x|^2} \sum (x_i - x_j)^2 \geq 0. \qquad \square$$

Matrix-Tree Theorem

Fix G



$$\begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}$$

$L = D - A$

$$L_{11} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

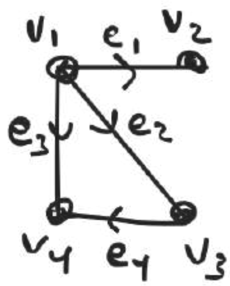
remove 1^{st} row & 1^{st} column

Thm $\det(L_{11}) = \#$ spanning trees of G.

eg above $L_{11} = 1(2\cdot 2 - (-1)\cdot(-1)) = 3$

So Laplacian knows about spanning trees!

Define incidence matrix B of G



$$E$$

$$V \quad \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} e_1 & e_2 & e_3 & e_4 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & -1 \end{bmatrix} \end{array}$$

$e_k = \{v_i, v_j\}$ $i < j$

$B_{ik} = 1$ $B_{jk} = -1$

0 otherwise

Lemma $L = BB^t$

Rmk $V = \{v_1, \dots, v_n\}$ $E = \{e_1, \dots, e_N\}$

B is $n \times N$ matrix

B^t " $N \times n$ "

BB^t " $n \times n$ "

Proof Compute

$$(BB^t)_{ij} = \sum_{r=1}^{N} B_{ir} (B^t)_{rj}$$

$$= \sum_{r=1}^{N} B_{ir} B_{jr}$$

$$B_{ir}B_{jr} = \begin{cases} 1 & i=j \text{ and } e_r \text{ incident to } v_i \\ -1 & e_r = \{v_i, v_j\} \\ 0 & \text{else} \end{cases}$$

So if $i \neq j$ $(BB^t)_{ij} = \begin{cases} 1 & \text{if } v_i v_j \in E \\ 0 & \text{else} \end{cases}$

and $(BB^t)_{ii} = \deg(v_i)$ $\square$

Cauchy-Binet Thm

$$X = \begin{pmatrix} 1 & 2 & 5 \\ -2 & 4 & 3 \end{pmatrix} \qquad Y = \begin{pmatrix} 3 & 4 \\ 2 & 1 \\ -1 & 2 \end{pmatrix}$$

$$XY = \begin{pmatrix} 2 & 16 \\ -1 & 2 \end{pmatrix}$$

$$\det(XY) = \sum \det(X_S)\cdot \det(Y_S)$$

$(2)(2)-(-1)(16)=20$ ranging over 2x2 minors

$$\underset{8}{\det\begin{pmatrix} 1 & 2 \\ -2 & 4 \end{pmatrix}}\ \underset{-5}{\det\begin{pmatrix} 3 & 4 \\ 2 & 1 \end{pmatrix}} + \underset{13}{\det\begin{pmatrix} 1 & 5 \\ -2 & 3 \end{pmatrix}}\ \underset{10}{\det\begin{pmatrix} 3 & 4 \\ -1 & 2 \end{pmatrix}}$$

$$+ \underset{-14}{\det\begin{pmatrix} 2 & 5 \\ 4 & 3 \end{pmatrix}}\ \underset{5}{\det\begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}}$$

$$= -40 + 130 - 70 = 20$$

Proof Sketch ① observe that row ops on X and column ops on Y change quantities

$\det(XY)$ and $\sum \det(X_S)\det(Y_S)$

in the same way. (eg mult row 1 of X by 2)

(2) Then suffices to prove for X, Y in RREF. This case is easy $\square$

Graph Laplacian

- $G = (V, E)$ $\quad V = \{v_1, \dots, v_n\}$ $\quad E = \{e_1, \dots, e_N\}$
- Laplacian

degree

adjacency

$$BB^t = L = D - A$$

incidence matrix $n \times N$.

- Eigenvalues of L

$$0 \le \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$$

carry info about G. eg

$$\lambda_1 = \lambda_2 = \cdots = \lambda_k = 0 \iff G \text{ has } \ge k \text{ components}$$

<u>Prop</u> λ eigenvalue of $L \Rightarrow \lambda \geq 0$.

<u>Proof</u> Fix $x \neq 0$ w/ $Lx = \lambda x$.

$$\lambda(x^t x) = x^t L x \underset{\text{last time}}{=} \sum_{v_i v_j \in E} (x_i - x_j)^2$$

$$\Rightarrow \lambda = \frac{1}{|x|^2} \sum (x_i - x_j)^2 \geq 0. \qquad \square$$

<u>Thm</u> Assume G connected and d-regular

Then $\lambda_n \leq 2d$ with equality $\iff$

G is bipartite

<u>Ex</u> K_4 is 3 regular,

Thm says $\lambda_4 \leq 6$.

Last time showed $\lambda_4 = 4$.

(But Interesting direction is
$\lambda_n = 2d \implies G$ bipartite)

<u>Toward Proof</u>

<u>Fact</u> $\lambda_n = \max_{\substack{x \in \mathbb{R}^n \\ |x|=1}} x^t L x$ (realized by eigenvector)

<u>Rmk</u> $x^t L x = Lx \cdot x$ dot product

$= |Lx| \cdot |x| \cdot \cos\theta$

angle btwn x, Lx.

$\cos\theta$ max when $\theta = 0$

ie Lx parallel to x, ie $Lx = \lambda x$

in this case $x^t L x = \lambda$ $\left(\begin{array}{c}\text{when} \\ |x| = 1\end{array}\right)$

(This is not a proof — no a priori reason that $|Lx| \cos\theta$ maximized when $\cos\theta$ maximized.)

Fact can be proved w/ MVC (Lagrange multipliers)

Proof of Thm

$$x^t L x = \sum_{v_i v_j \in E} (x_i - x_j)^2$$

$$= \sum_{v_i v_j \in E} 2(x_i^2 + x_j^2) - (x_i + x_j)^2$$

$$\underset{\text{d-reg}}{=} 2d \underbrace{\sum x_i^2}_{=1 \text{ if } |x|=1} \quad \sum_{v_i v_j \in E} (x_i + x_j)^2$$

Then

$$\lambda_n = \max_{|x|=1} x^t L x = \max_{|x|=1} 2d - \sum_{v_iv_j \in E} (x_i + x_j)^2$$

$$= 2d - \min \underbrace{\sum_{v_iv_j \in E} (x_i + x_j)^2}_{\geq 0.}$$

$$\Longrightarrow \quad \lambda_n \leq 2d$$

equality case

- Suppose G bipartite. $V = X \sqcup Y$.

  Consider x with $x_i = 1$ if $v_i \in X$

  $x_i = -1$ if $v_i \in Y$.

$$(Lx)_i = \underbrace{\deg(v_i)}_{d} \underset{1 \text{ or } -1}{x_i} - \sum_{v_iv_j \in E} \underset{-1 \text{ or } 1.}{x_j}$$

$$= \begin{cases} 2d & \text{if } x_i = 1 \\ -2d & \text{if } x_i = -1 \end{cases}$$

$$\Longrightarrow Lx = (2d) \cdot x$$

- conversely suppose $\exists\, x$ w/ $Lx = (2d)x$

By computation above $\sum_{v_iv_j \in E} (x_i + x_j)^2 = 0$

$\Longrightarrow$ $x_i = -x_j$ whenever $v_iv_j \in E$

G connected $\Rightarrow$ if $x_i = 0$ for some i then $x = 0$

Define partition $V = X \sqcup Y$

$X = \{v_i$ w/ $x_i > 0\}$
$Y = \{v_i$ w/ $x_i < 0\}$ bipartition $\square$

<u>Other λ_i</u> are interesting but more complicated

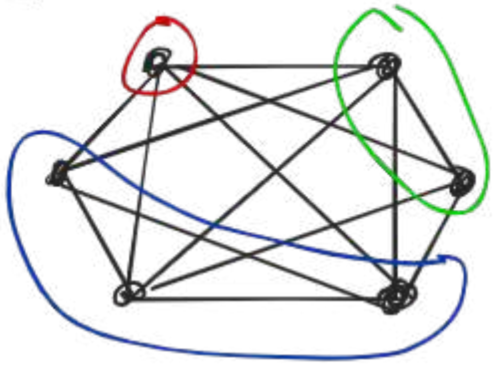
- smallest nonzero eigenvalue (λ_2 if G connected) related to

<u>Cheeger's Constant</u> measure of bottlenecks

$$h(G) = \min_{A \subset V} \frac{\#\text{ edges btwn } A \text{ \& } A^c}{\min\{|A|, |V \setminus A|\}}$$

(for G regular)

Ex



$\frac{5}{1}$ $\frac{8}{2}$ $\frac{9}{3}$

$h(G) = 3$

Ex

$\frac{1}{4}$

$h(G) \le \frac{1}{4}$

Thm (Cheeger's inequality) G connected, d-regular

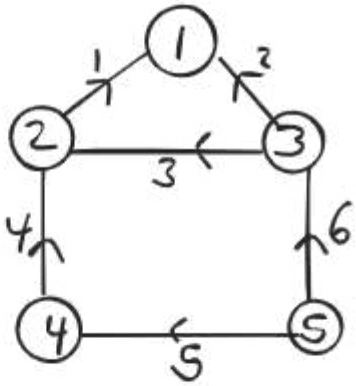
$$\frac{h(G)^2}{2d} \le \lambda_2 \le 2h(G)$$

Matrix-Tree Theorem Fix G, $L = D - A$

L_{11} obtained from L by removing row 1, col 1

$\det(L_{11}) = \#$ spanning trees of G

$n = |V| \quad N = |E|$



Incidence matrix B $\quad n \times N$

$$
\begin{array}{c|cccccc}
 & 1 & 2 & 3 & 4 & 5 & 6 \\
\hline
1 & 1 & 1 & 0 & 0 & 0 & 0 \\
2 & -1 & 0 & 1 & 1 & 0 & 0 \\
3 & 0 & -1 & -1 & 0 & 0 & 1 \\
4 & 0 & 0 & 0 & -1 & 1 & 0 \\
5 & 0 & 0 & 0 & 0 & -1 & -1
\end{array}
$$

$B_1 \quad (n-1) \times N$

Check $L_{11} = B_1 B_1^t$

Cauchy-Binet $\quad \det(L_{11}) = \sum_{S \subset \{1,\dots,N\},\ |S| = n-1} \det(B_{1,S}) \det(B^t_{1,S})$

$$= \sum \left[\det(B_{1,S})\right]^2$$

Given $S \subset \{1, \dots, N\} = E$

$|S| = n-1$, let G_S be the subgraph of G with these edges.

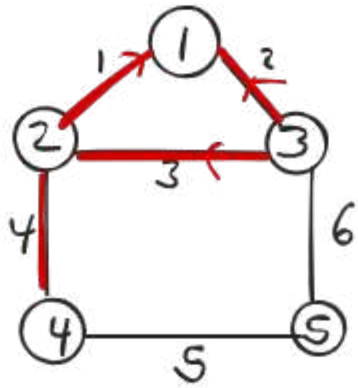
<u>Exercise</u> ① Compute $\det(B_{1,S})^2$ and draw G_S for $S = \{1,2,3,4\}$

and $S = \{1,3,4,5\}$

② Make a conjecture

- $S = \{1,2,3,4\}$

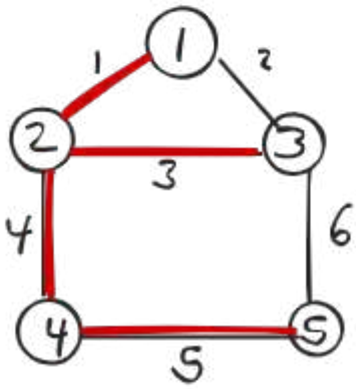
$$\begin{bmatrix} -1 & 0 & 1 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 \end{bmatrix}$$



$B_{1,S}$

- $S = \{1,3,4,5\}$

$$\begin{bmatrix} -1 & 0 & 1 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 \end{bmatrix}$$



Proof of Thm Fix $S \subset \{1, \dots, N\}$

Claim 1 If G_S not spanning tree
then $\det(B_{1,S}) = 0$

Claim 2 If G_S spanning tree
then $\det(B_{1,S}) = \pm 1$.

Claim 1 + Claim 2 $\Rightarrow$ Thm

Proof of Claim 1 (sketch)

G_S not spanning tree $\Rightarrow$ contains cycle C. Edges in C give columns of $B_{1,S}$ that are linearly dependent.

$\Rightarrow \det(B_{1,S}) = 0$

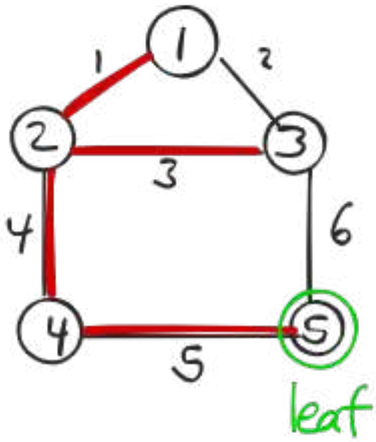
Proof of Claim 2 (sketch)

Assume G_S spanning tree

Choose a leaf v of G_S

- $S = \{1, 3, 4, 5\}$

$$\begin{bmatrix} -1 & 0 & 1 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 \end{bmatrix}$$



corresponding row has one nonzero entry in $B_{1,S}$.

$\Longrightarrow \det(B_{1,S}) = \pm \det(B'_{1,S})$

$B'_{1,S} \longleftrightarrow$ spanning tree of $G \setminus v$

inductively conclude $\det(B'_{1,S}) = \pm 1$.

$\square$