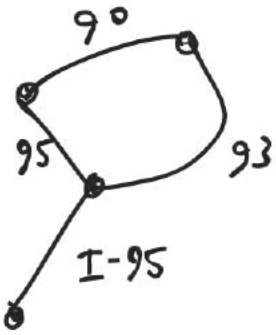
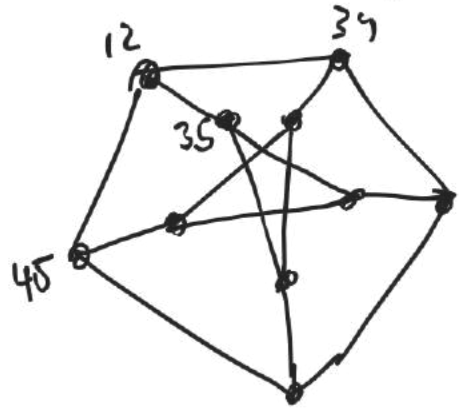
Graphs

Graphs express relationships between things

family tree



routes
PVD $\leftrightarrow$ BOS



2-element subsets of $\{1, \dots, 5\}$
connected by disjointness

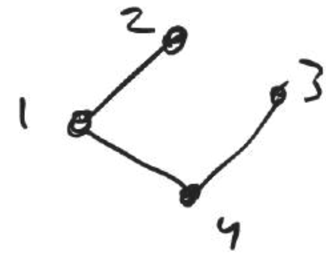
Petersen graph

<u>Formal definition</u> A <u>graph</u> G is pair (V, E) where V is a set and $E \subset \{\text{2-element subsets of } V\}$

$\left(\begin{array}{l} V: \text{vertices} \\ E: \text{edges} \end{array} \right)$

<u>Ex</u> $V = \{1, 2, 3, 4\}$ $E = \{\{1,2\}, \{1,4\}, \{3,4\}\}$

often just draw pictures



Rmks

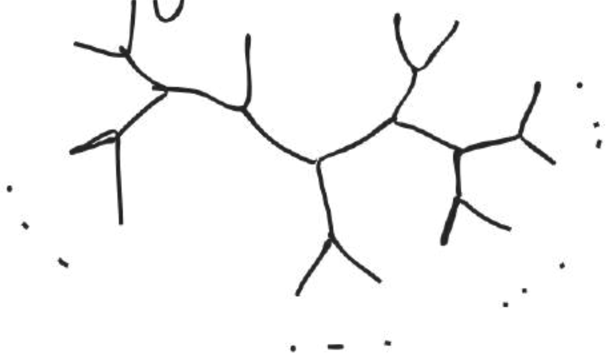
1. Our defn excludes

multiple edges

self loops

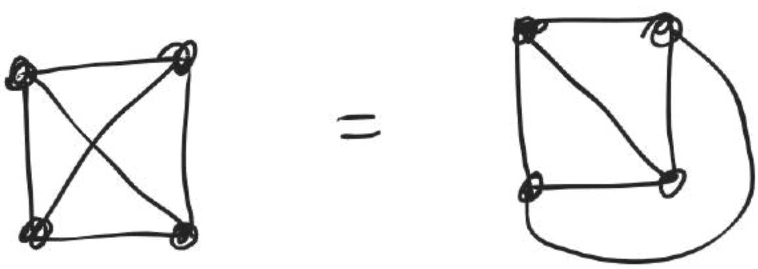
NB: different from West!

2. Only consider G with V finite



infinite 3-regular graph

3. Some graphs can be drawn in plane, some not.



Fact Petersen graph not planar.

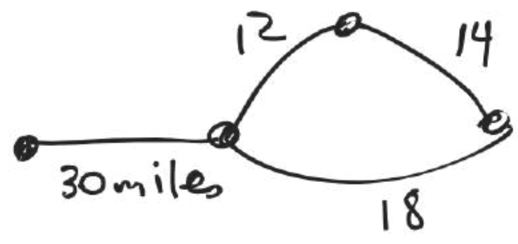
4. A graph can have multiple components

$G =$ 

5. Variations: (not our main focus)

- directed graphs

- weighted graphs

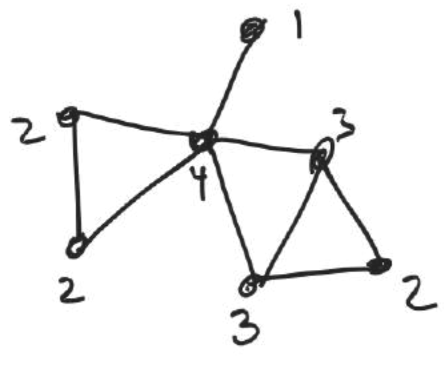


(Next: basic terminology)

$\boxed{\text{Vertex degrees}}$

For $v \in V$, $e \in E$ if $v \in e$ say v, e are incident. Define $\deg(v) = \#$ edges incident to v.

Ex



Q: Does there exist graph with

- 5 vertices each with degree 2
- 5 vertices each w/ degree 3

Lemma For $G = (V, E)$, $\sum_{v \in V} \deg(v) = 2|E|$.

Cor In a graph the number of vertices w/ odd degree is even.

$\Rightarrow \nexists$ 5 vertex 3-regular graph.

Proof of Lemma

Counting each vertex degree counts each edge twice.

More precisely, consider

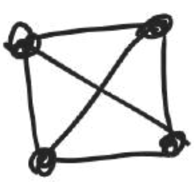
$$\sum_{v \in V} \deg(v) = \sum_{\substack{(v,e) \in V \times E \\ \text{incident pair}}} 1 = \sum_{e \in E} 2 = 2|E|. \qquad \square$$

Ex the complete graph K_n

has vertices $= \{1, \dots, n\}$

edges $=$ all pairs $\{i, j\}$

K_3 

K_4 

in K_n $\deg(v) = n-1$ for each v

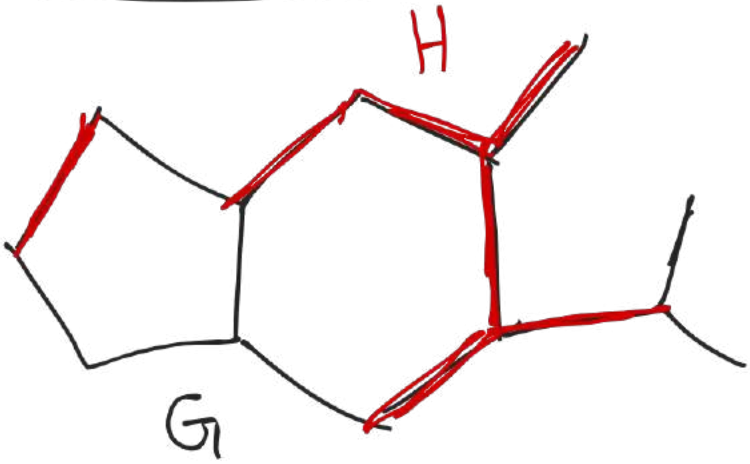
$|E| = \binom{n}{2}$ = # 2-element subsets of $\{1, .., n\}$

Lemma $\Rightarrow$ $n(n-1) = 2\binom{n}{2}$.

Subgraphs

H is a subgraph of G.

$V(H) \subset V(G)$, $E(H) \subset E(G)$



$K = \square$ is not a subgraph of G.

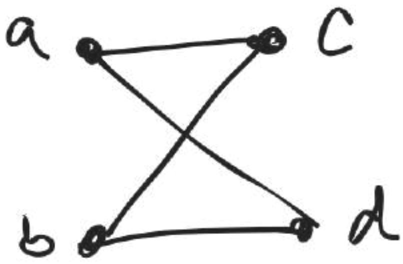
Isomorphism

graphs G_1, G_2 are <u>isomorphic</u> if

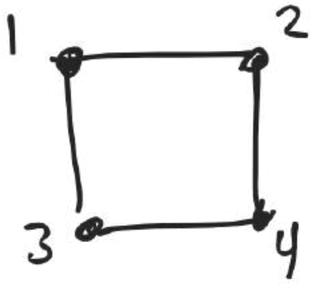
$\exists$ bijection $V(G_1) \xrightarrow{\varphi} V(G_2)$ so that

$\{u, v\} \in E(G_1) \iff \{\varphi u, \varphi v\} \in E(G_2)$

Ex

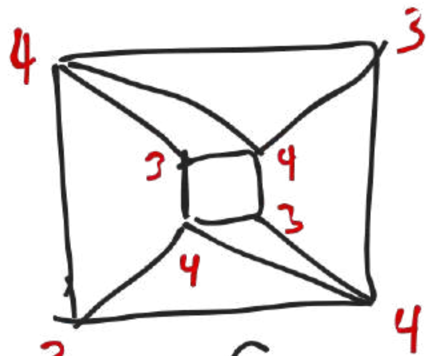


$\cong$

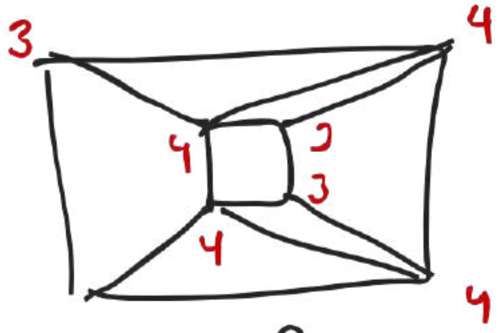


$\varphi: a \mapsto 1,\ c \mapsto 2,\ b \mapsto 4,\ d \mapsto 3$

Ex



G_1



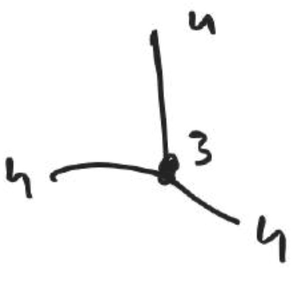
G_2

are G_1, G_2 isomorphic?

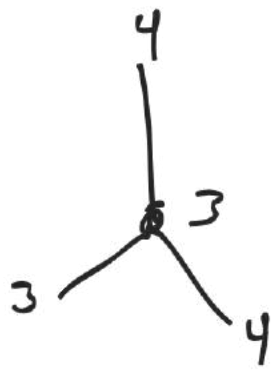
Isomorphism problem hard in general

Look at vertex degrees

in G_1



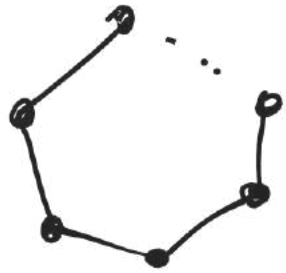
in G_2



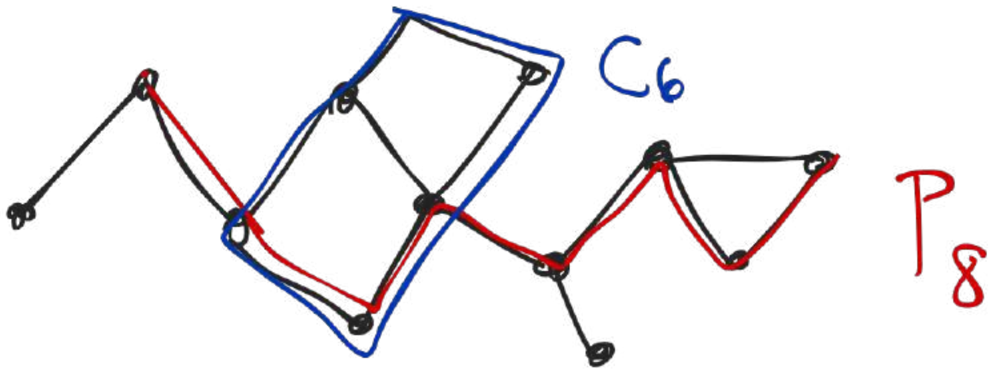
$\Rightarrow G_1 \not\cong G_2$

Some special subgraphs

P_n = 

C_n = 

① A subgraph iso to P_n / C_n is called a <u>path</u> / <u>cycle</u>.



② Given $G=(V,E)$ and $U \subset V$

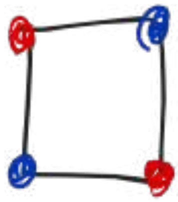
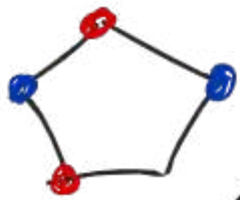
define $G \setminus U$ graph w/ vertices: $V \setminus U$

and edges: $e \in E$ between vertices in $V \setminus U$

eg

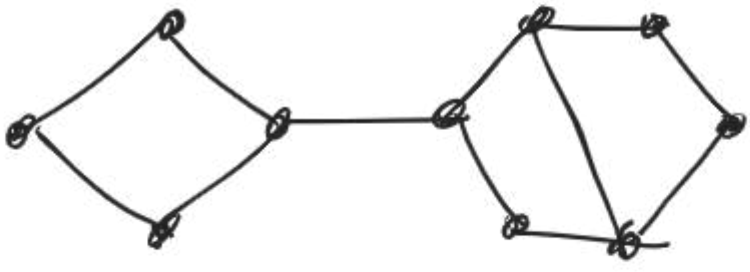
Bipartite graphs

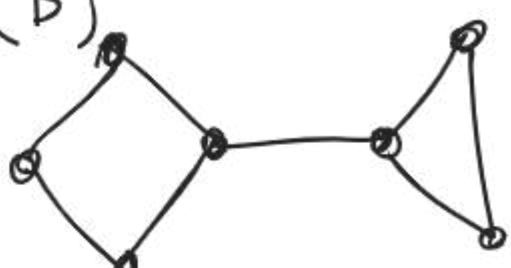
$G=(V,E)$ is <u>bipartite</u> if its possible to color vertices red / blue so there are no monochromatic edges.

ex  C_4 non ex  C_5

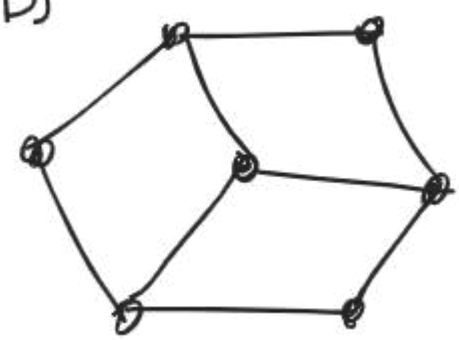
ex
people
jobs


<u>Exercise</u> which of the following are bipartite?

(A) 

(B) 

(C) 

(D) 

- What went wrong in B, C?

B, C have cycles of odd length.

C_{2k+1}

<u>Observation</u> C_{2k+1} is not bipartite and so a graph containing C_{2k+1} as a subgraph is also not bipartite.

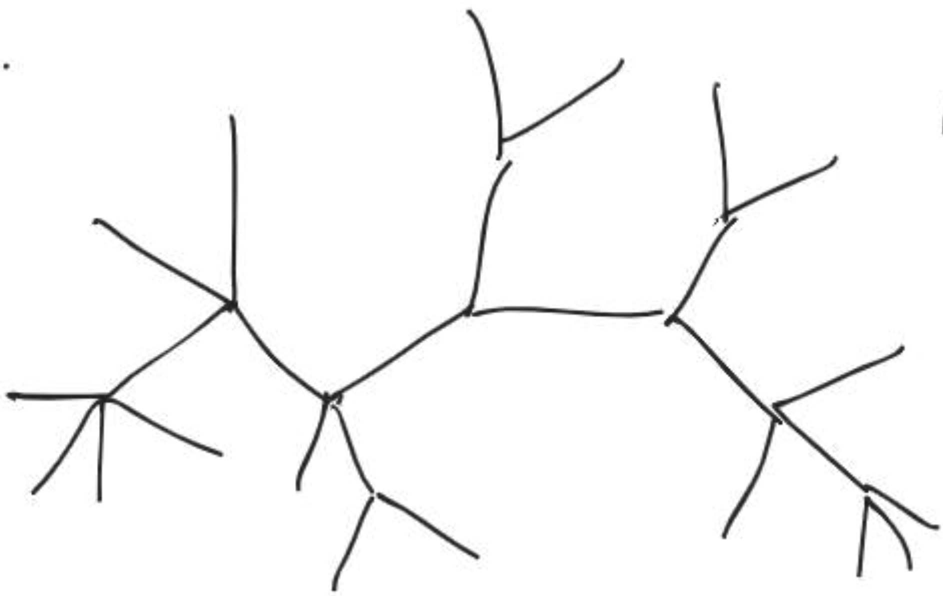
- What if G has has no odd cycle?

<u>Thm</u> G bipartite $\iff$

G does not contain any odd cycle

Eg.

G



is bipartite (no cycles at all!)

A **walk** in $G=(V,E)$ is sequence $v_1,\dots,v_m$ with $v_i \in V$ and

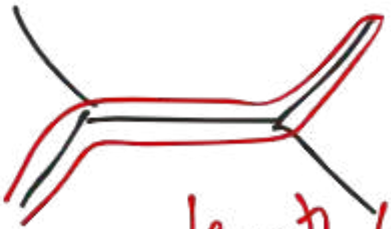
$$\{v_i, v_{i+1}\} \in E \quad i=1,\dots m-1$$

A **closed walk** is a walk with $v_1 = v_m$

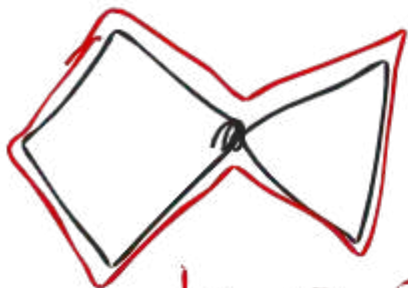
The **length** of a walk is the # of edges $m-1$

Here edges/vertices may repeat.

eg



length 6



length 7

Lemma A closed walk of odd length contains an odd cycle.

Proof. Let w be closed walk, length $2l+1$.

induct on ℓ.

<u>Base case</u> $(\ell = 1)$

v_1, v_2, v_3 distinct since G has no self loops

$\Rightarrow$ w is a cycle C_3.

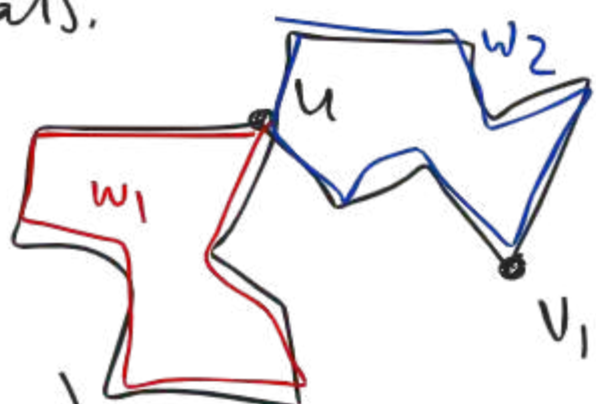
<u>Induction step</u> Fix w length $2\ell+1$.

Assume odd walk of len $< 2\ell+1$ has odd cycle.

<u>Case 1</u> vertices of w don't repeat

$\Rightarrow$ w is odd cycle.

<u>Case 2</u> some vertex repeats.



Extract closed walks w_1, w_2

$\text{len}(w_1) + \text{len}(w_2) = \text{len}(w)$

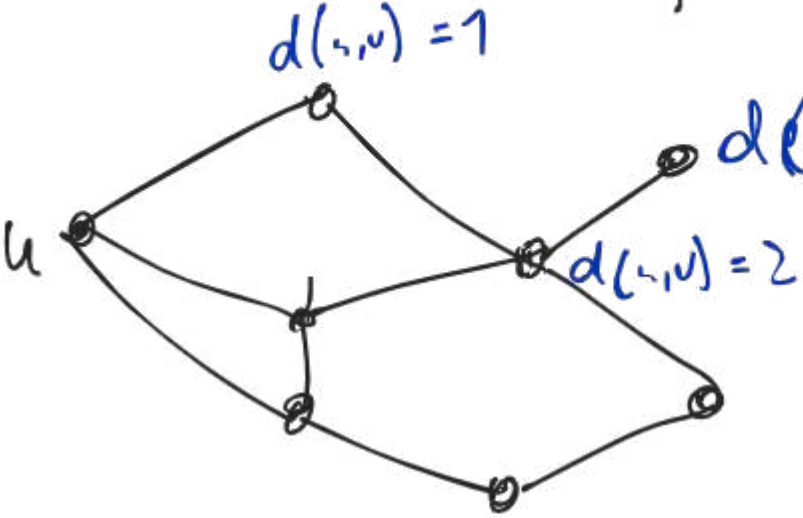
one of $\text{len}(w_i)$ odd. $\Rightarrow$ that w_i has an odd cycle by induction $\square$

Proof ($\Rightarrow$) observed above

($\Leftarrow$) We find a coloring:

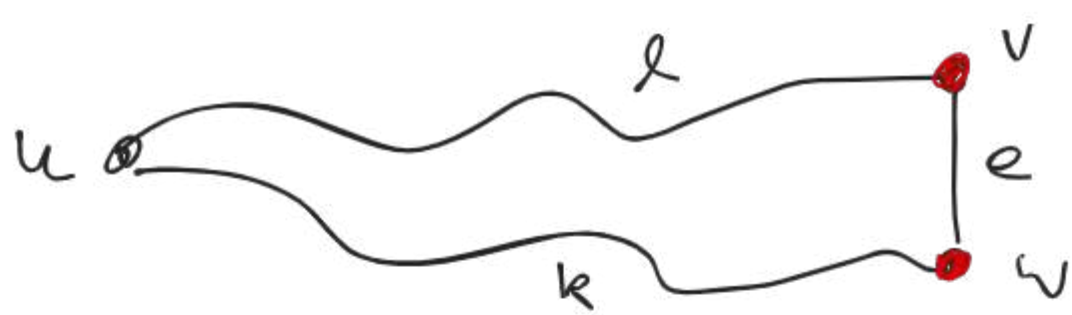
Fix $u \in V$ and for $v \in V$ define

$d(u,v)$ = length of shortest path from u to v.



color v red/blue if $d(u,v)$ even/odd.

If $e \in E$ monochromatic



v, w monochromatic $\Rightarrow$ ℓ, k both even or both odd

$\Rightarrow \ell + k + 1$ odd $\overset{\text{(lemma)}}{\Rightarrow}$ G has odd cycle $\ast$

Then $\nexists$ monochromatic edge $\square$

<u>Rmk</u> It's possible G wasn't connected - apply above to each component.

<u>Rmk</u> (TONCAS)

Given property $\underline{P}$ of graphs (eg bipartite) may ask if given G has $\underline{P}$.

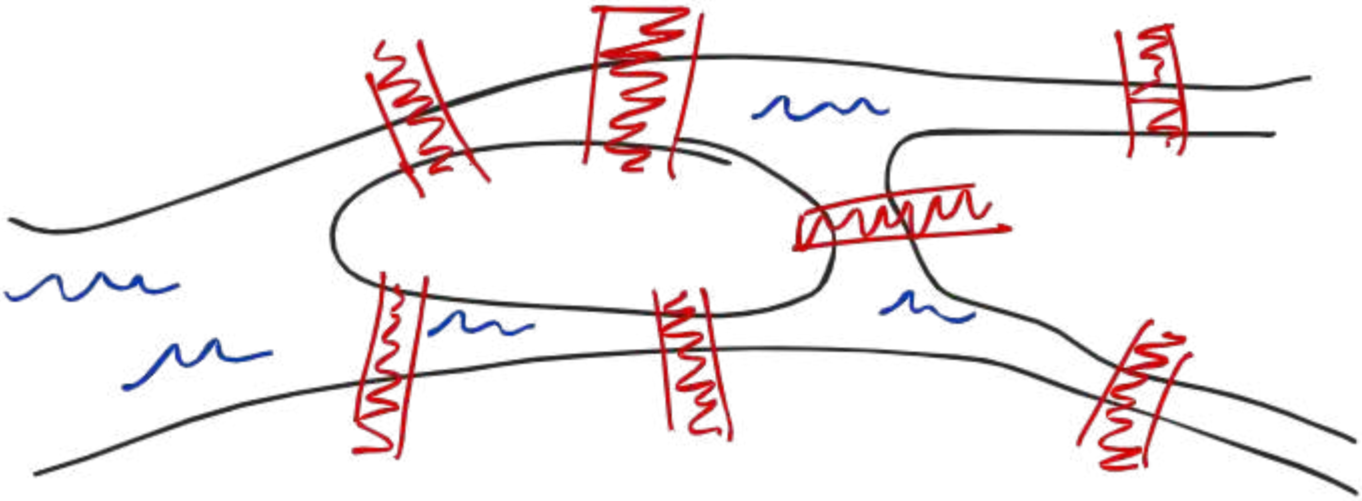
Often there is an "obvious" necessary condition (G bipartite $\Rightarrow$ no odd cycle) and we'll show this is also sufficient

(G has no odd cycle $\Rightarrow$ G bipartite)

TONCAS = "The Obvious Necessary Condition is Also Sufficient"

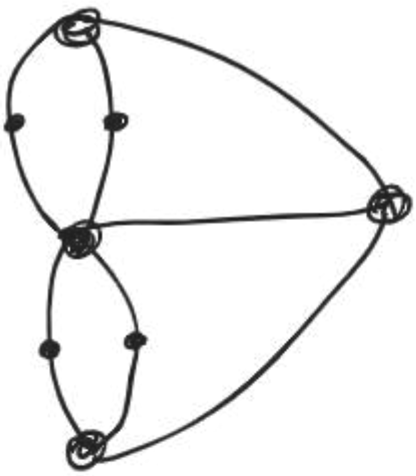
Eulerian Graphs

Bridges of Königsberg



<u>Q</u>: (Euler) Is it possible to cross every bridge exactly once?

Convert to graph theory:

Does $G =$  have a closed walk that contains each edge exactly once?

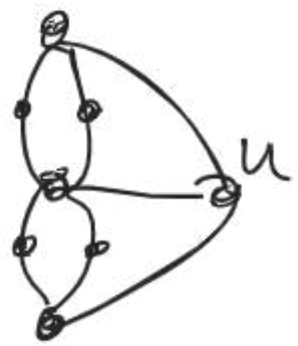
(experiment)

Defn An Euler tour on G is a closed walk that visits each edge exactly once. $>\!\!<$

Assume G connected (one component)

Say G Eulerian if G has Euler tour.

Warm-up



not Eulerian

b/c G has vertex u of odd degree.

In an Euler tour every edge into a u is followed by edge out

Edges don't repeat so need even # incident to each vertex.

<u>TONC</u> G Eulerian $\Rightarrow$ deg(v) even $\forall v \in V$.

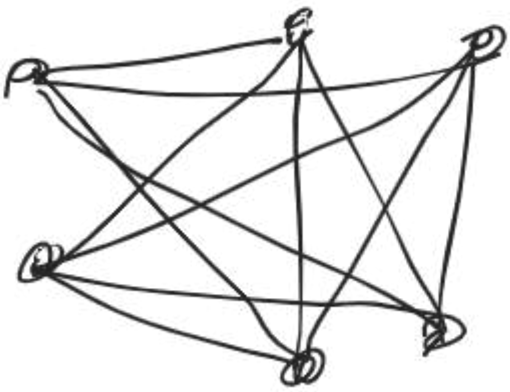
<u>Thm</u> TONCAS, If G connected.

G Eulerian $\iff$ deg(v) even $\forall v \in V$

<u>Ex</u> $P_4 =$  deg = 1

not Eulerian remember Euler tours are closed walks.

<u>Ex</u>

G = 

Eulerian b/c G is 4-regular

in practice not hard to find Euler tour: start & keep going. even vertex degree ensures can't get stuck. (illustrate)

Proof Suppose G has even vertex degrees

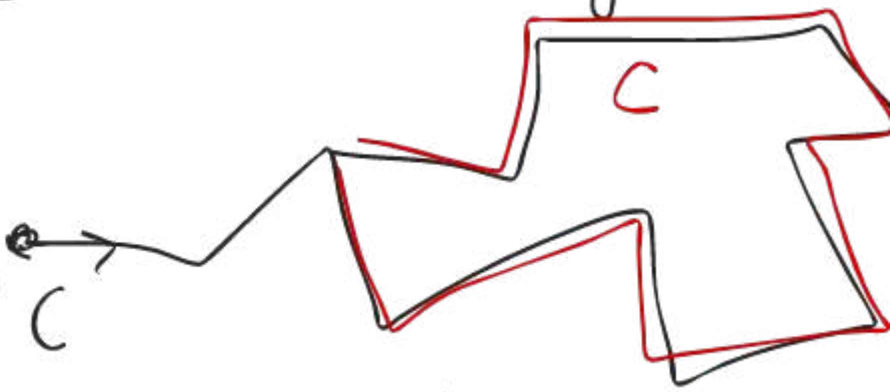
Induct on $|E|$.

Base case $|E| = 0 \Rightarrow G = \bullet$

Then true trivially.

Induction Step Assume then for graphs with $< |E|$ edges.

Since every vertex has even deg G contains a cycle



Let F = edges of C

Consider $G \setminus F = (V, E \setminus F)$

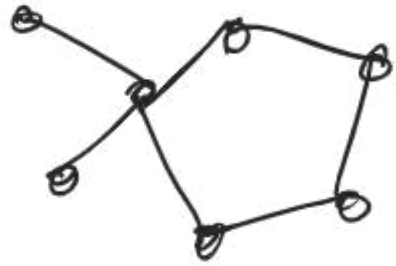
observe $G \setminus F$ has even vertex degrees

IH $\Rightarrow$ G/F has Euler tour. Combine with to get Euler tour $\square$

Connected graphs

A graph is <u>connected</u> if any two vertices are are joined by a path.

disconnected



Connected

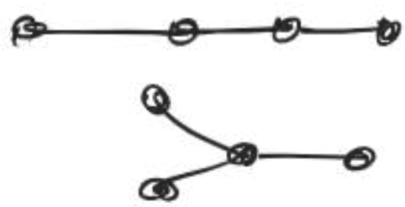
<u>Q</u>: Suppose G connected and has n vertices What is fewest # edges G can have? Is the minimizer unique?

(Experiment)

$n=1$

$n=2$

$n=3$



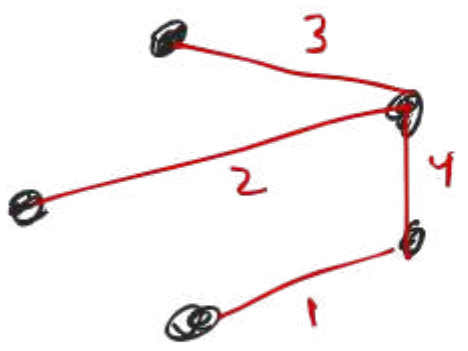
<u>Guess</u> $n-1$ edges.

Indeed start with n disjoint vertices

observe each edge added

decreases # comp by at most 1.

So need $n-1$ edges to get connected graph.



<u>Defn</u> A connected graph with $|V|=n$, $|E|=n-1$

is called a <u>tree</u>.

<u>Facts about trees</u>

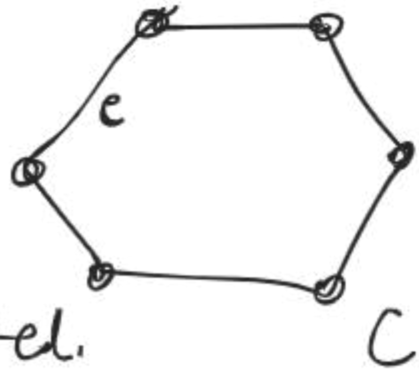
(1) G tree $\Rightarrow$ $G \setminus e$ disconnected for each $e \in E$

(by above)

(2) G tree $\Rightarrow$ G has no cycle.

Prove contrapositive:

Suppose $\exists$ cycle $C \subset G$. and edge e of C.



To show G not tree

suffices to show $G \setminus e$ connected.

Fix $u, v \in V(G)$. WTS $\exists$ path in $G \setminus e$ between u, v

G connected. $\Rightarrow$ $\exists$ path P in G u to v.

Let w_1, w_2 first/last vertices of $P \cap C$

$\exists$ path in $C \setminus e$ w_1 to w_2.

Use to obtain path p' in $G \setminus e$ from u to v.

✓

(3) G tree $\Rightarrow$ $\exists!$ path between any two $u, v \in V$

$\exists$ two path u to v. $\xRightarrow{\text{exercise}}$ G has cycle

$\Rightarrow$ G not tree.

<u>Rmk</u> (extremal problems)

General kind of graph theory prob:

among graphs with property P (connected, n vertices) which graphs minimize property Q (# edges)

Graphs and matrices

$G = (V, E)$

enumerate $V = \{v_1, \dots, v_n\}$ $E = \{e_1, \dots, e_m\}$

<u>incidence matrix</u> $B = (b_{ij})$ where

$$b_{ij} = \begin{cases} 1 & v_i \text{ incident to } e_j \\ 0 & \text{else} \end{cases}$$

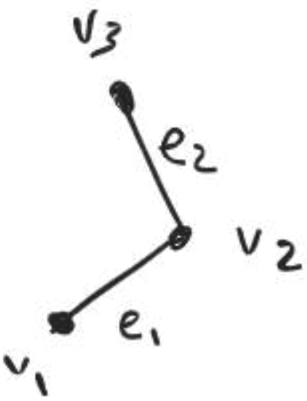
<u>adjacency matrix</u> $A = (a_{ij})$ where

$$a_{ij} = \begin{cases} 1 & \{v_i, v_j\} \in E \\ 0 & \text{else} \end{cases}$$

<u>degree matrix</u> $D = (d_{ij})$ where

$$d_{ij} = \begin{cases} \deg(v_i) & i = j \\ 0 & i \neq j \end{cases}$$

Ex



$$B: \quad \begin{array}{c|cc} & e_1 & e_2 \\ \hline v_1 & & \\ v_2 & & \\ v_3 & & \end{array} \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$A: \quad \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \end{array} \begin{array}{c} v_1 \quad v_2 \quad v_3 \\ \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \end{array} \qquad D: \quad \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \end{array} \begin{array}{c} v_1 \quad v_2 \quad v_3 \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array}$$

What info about G can we extract from B, A, D?

Example (powers of A)

$$(A^2)_{ij} = \sum_{r=1}^{n} a_{ir} a_{rj} = \text{\# walks length 2 from } v_i \text{ to } v_j.$$

$$a_{ir} a_{rj} = 1 \iff$$

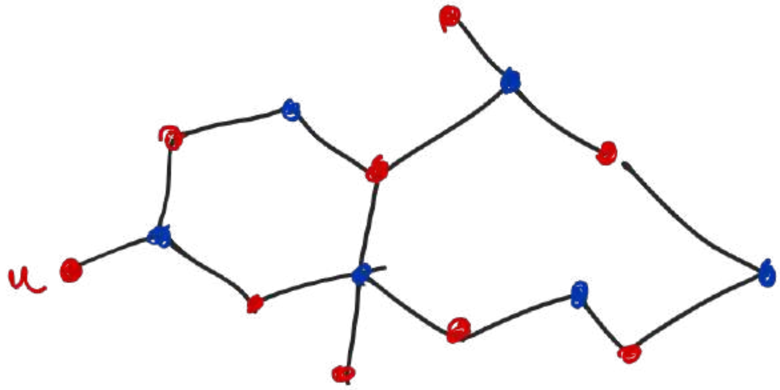
note $(A^2)_{ii} = \deg(v_i)$

Similarly $(A^d)_{ij}$ = # walks length d from v_i to v_j

Last time:

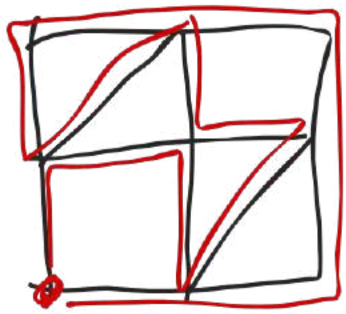
- G bipartite $\iff$ G has no odd cycle

eg

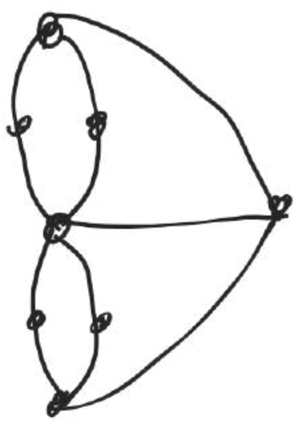


- G is <u>Eulerian</u> if $\exists$ closed walk on G that visits every edge exactly once (call such a walk an Euler tour)

eg

G =  is Eulerian

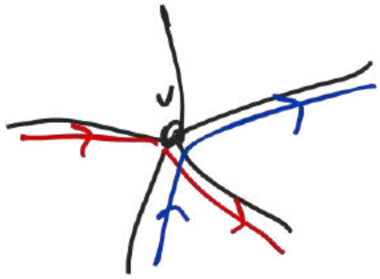
<u>Ex</u> Königsberg graph



is <u>NOT</u> Eulerian because it has vertices of odd degree.

Observation: if $G=(V,E)$ is Eulerian then
(TONC) $\deg(v)$ even for each $v \in V$.

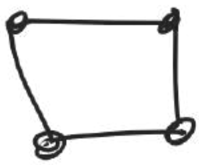
Indeed, a closed walk gives a pairing of edges incident to v.



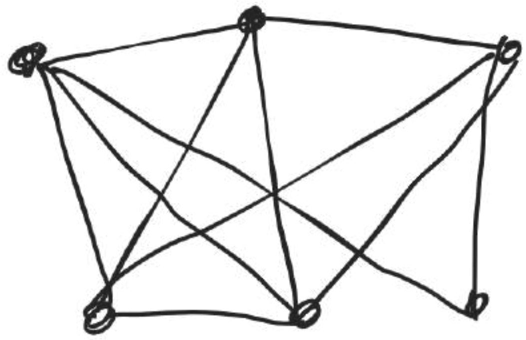
Theorem TONCAS: Assume G connected

G Eulerian $\Longleftrightarrow$ every vertex has even degree.

connected means any two vertices joined by path. A disconnected graph can't be Eulerian



Ex



Eulerian by Thm

In practice, not hard to find an Euler tour as proof will show.

Proof of Thm WTS $(\Leftarrow)$

use induct on $|E|$ number of edges

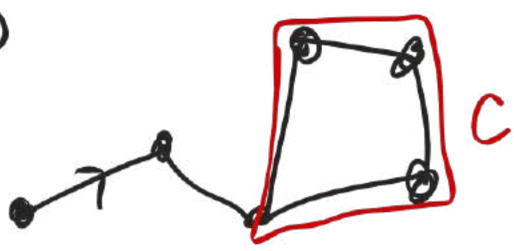
Base case $|E| = 0 \Rightarrow G = \cdot$

This graph is Eulerian (vacuously)

Induction step Fix $G = (V, E)$ with $\deg(v)$ even $\forall v \in V$.

Assume thm true for graphs with $< |E|$ edges.

Observe: even degrees $\Rightarrow$ G has a cycle C



Let $F = \{ \text{edges of } C. \} \subset E$

Consider subgraph $H := (V, E \setminus F)$

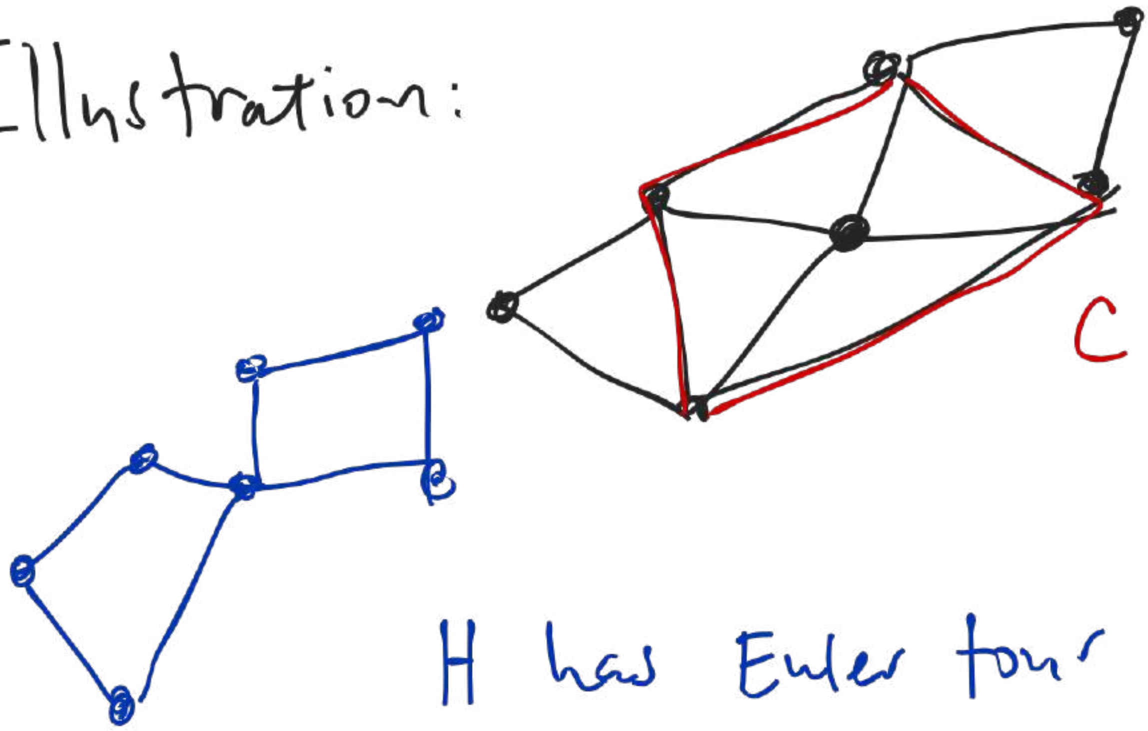
Each $v \in V$ has even degree in H

$\Rightarrow$ H has an Euler tour w

Combine w and C to get

Euler tour of G.

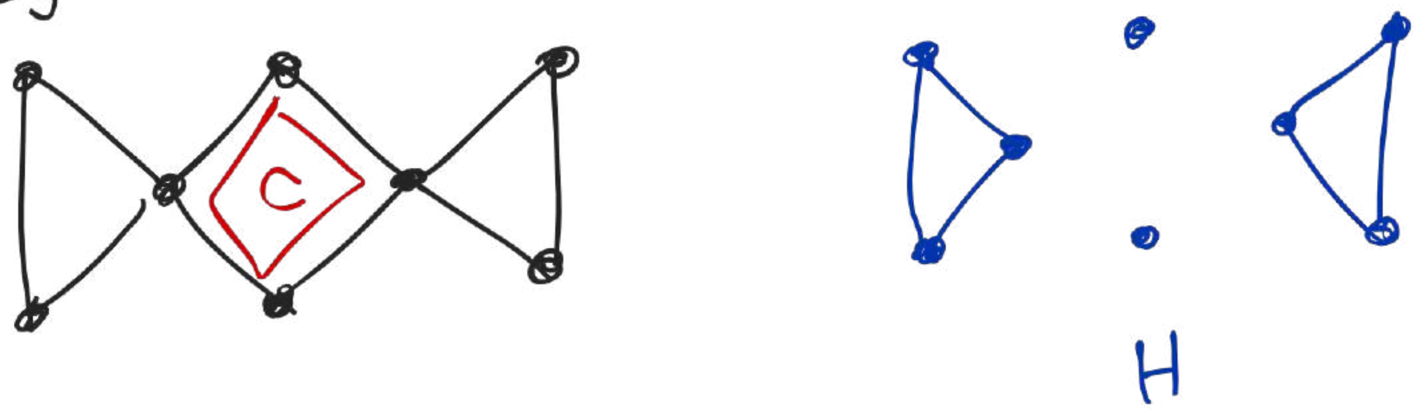
Illustration:

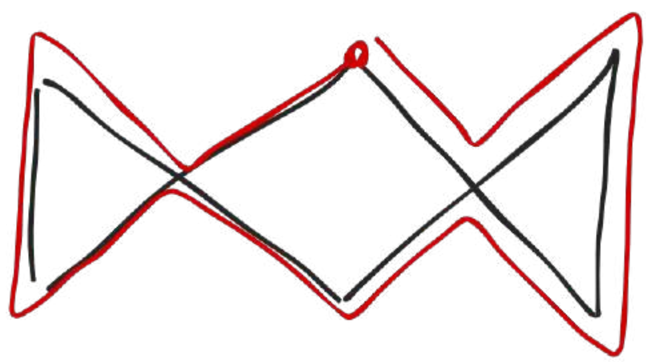


H has Euler tour

Potential issue: H may not be connected.

Eg

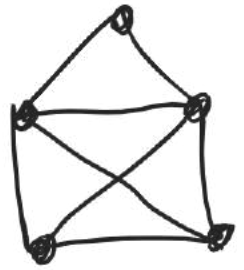




To combine H, C into a Euler tour

start walking along C, and take detour at each vertex part of a nontrivial component of H, following the Euler tour given by the induction step. ◻

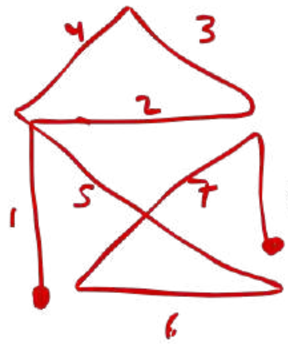
Example



Not Eulerian

deg 3

But here's a walk visiting every edge once



Why doesn't this contradict the theorem?

(Not a closed walk)

$\boxed{\text{Connected graphs}}$

G connected if any two edges are joined by a path

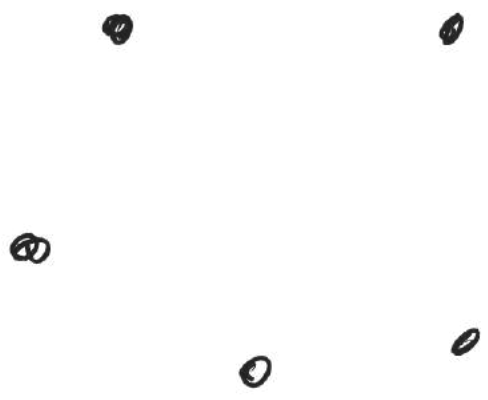
Any graph is union of connected graphs "connected components"

<u>Question</u> What's the fewest number of edges in a connected graph with n vertices?

<u>Ans</u>: $n-1$

eg P_n is connected w/ n vertices, $n-1$ edges.

This is the best possible



Start w/ n points each edge decreases # components by at most 1.

After k edges

there are $\geq n-k$ components

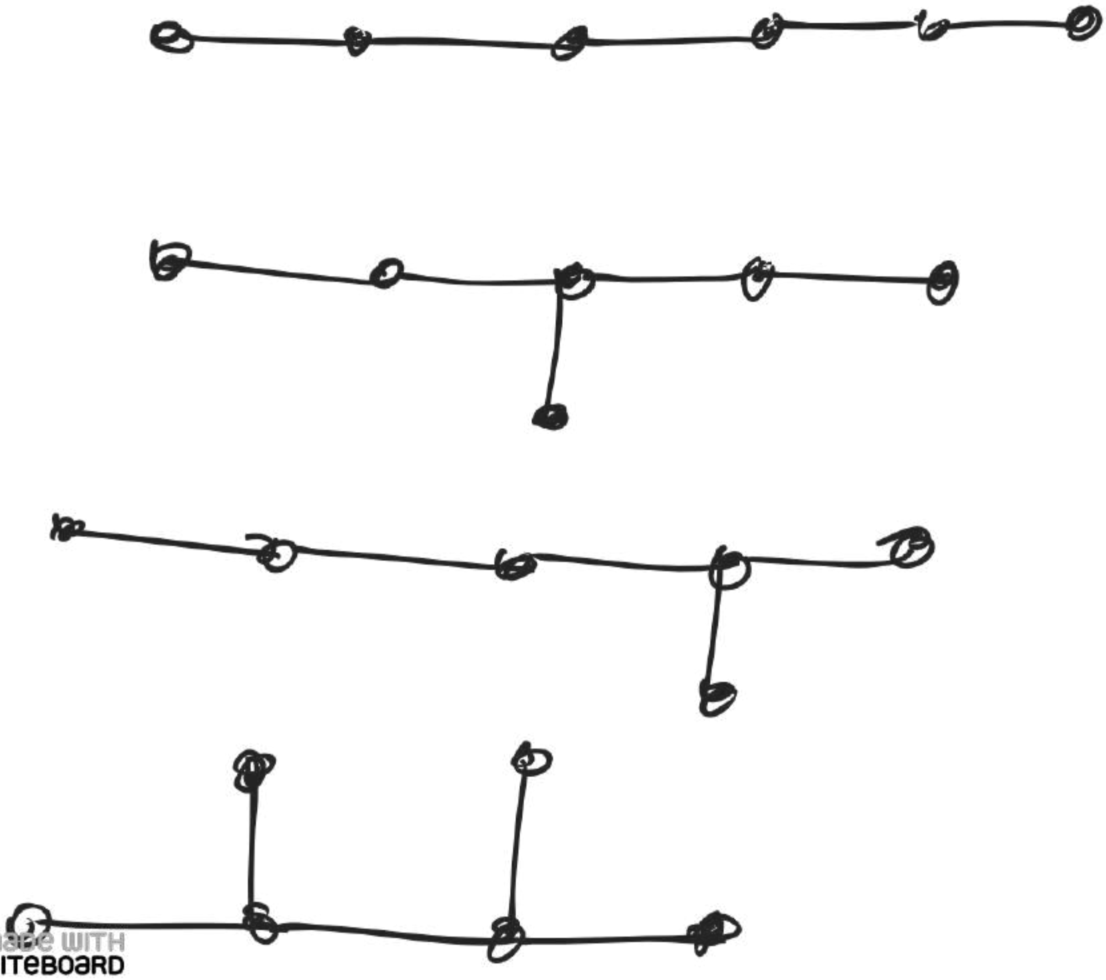
so need at least $n-1$ edges

to get 1 component.

Examples w/ n vertices, $n-1$ edges

(minimizers) These are called

trees.



Rmk Above question is simple example of extremal problem.

Such problems ask: among graphs with property P (conn. n vertices) which graphs minimize property Q (#edges)?

Next time

Thm (Characterization of trees)

Fix connected G. TFAE

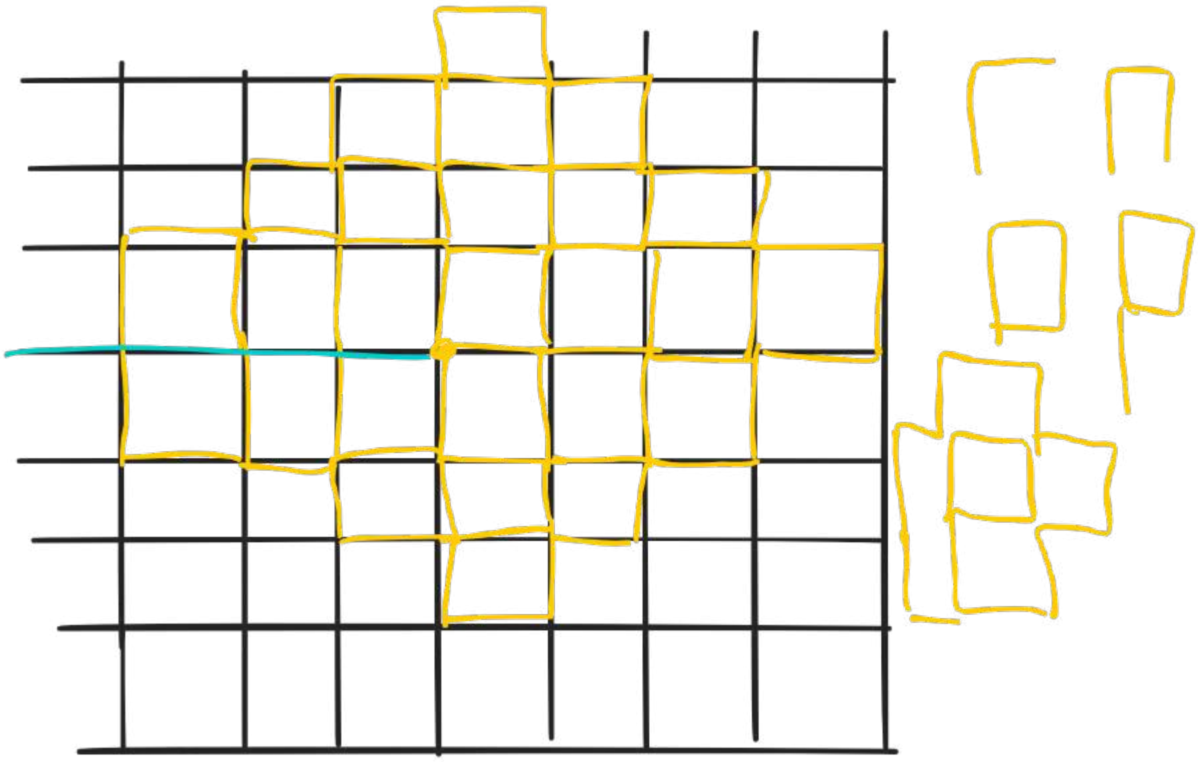
1) G has n vert, n-1 edges
2) removing any edge disconn. G
3) G contains no cycle
4) Between $u,v \in V$ $\exists!$ path.

Euler tours on infinite graphs

Last time: G Eulerian $\iff$ vertex deg are even

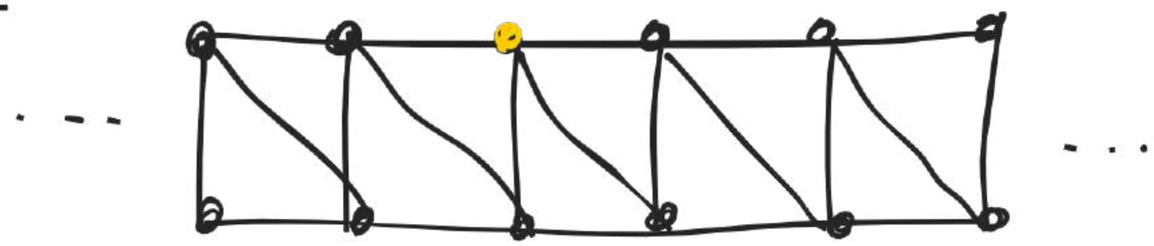
What about infinite graphs? $(|V| = \infty)$

eg infinite grid



An ∞ graph is Eulerian if $\exists$ walk $(\ldots w_{-2}, w_{-1}, w_0, w_1, w_2, \ldots)$ that visits every edge once.

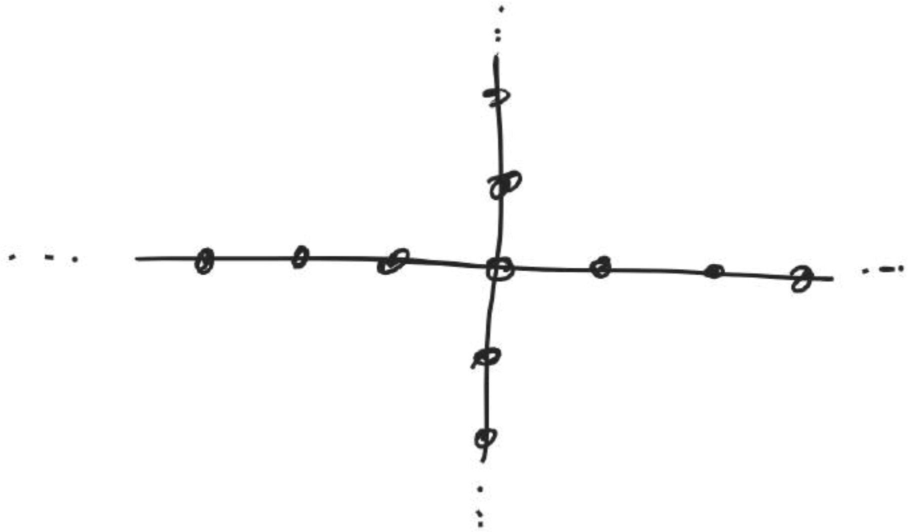
Ex

$G =$ 

is Eulerian

Example The infinite grid is Eulerian!

Ex Having even vertex deg is not enough to be Eulerian (TONC but not TONCAS)



not Eulerian

Possible final project: characterize infinite Eulerian graphs (Erdos - Grünewald - Weiszfeld)

Trees

A tree is a connected graph with n vertices and $n-1$ edges.

Thm (characterization of trees)

Let G be a connected graph w/ n vertices

TFAE

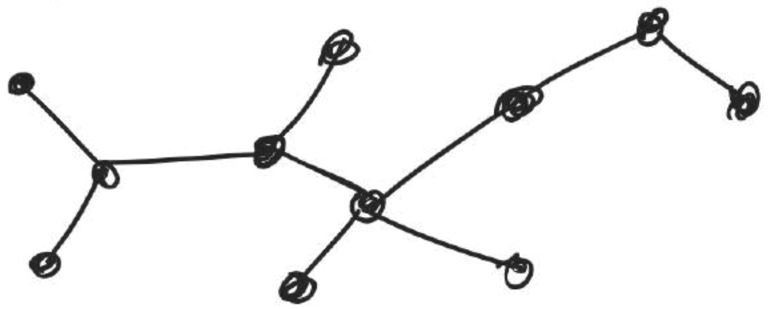
(i) $G \setminus e$ disconnected for each edge e

(ii) G has $n-1$ edges

(iii) G does not contain a cycle

(iv) Any two vertices are joined by unique path.

Ex



Can take any of these as definition of tree.

Will be easier to prove TFAE for connected graphs $G=(V,E)$

(1) $\exists$ edge e s.t. $G \setminus e$ connected

(2) $|E| > |V| - 1$

(3) G has a cycle

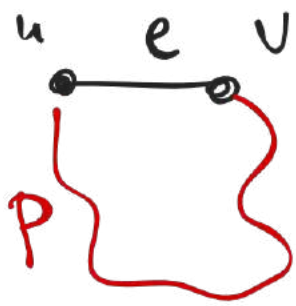
(4) $\exists$ $u, v \in V(G)$ that are joined by two different paths

easy implications (do quickly write for exercise)

(1) implies (2), (3), (4)

- (1) $\Rightarrow$ (2) by last time min # edges for conn. graph w/ n vert. is $n-1$ (gives contrapositive)

- (1) $\Rightarrow$ (3) $G \setminus e$ connected



$C = P \cup e$ cycle

• $(1) \Rightarrow (4)$ $G \setminus e$ connected

e, P (above) are paths joining u, v.

To finish proof, show

$$\begin{tikzcd}
(2) \arrow[r, Leftrightarrow] & (1) \arrow[r, Rightarrow] \arrow[d, Leftrightarrow] & (4) \arrow[dl, Rightarrow] \\
 & (3) &
\end{tikzcd}$$

$(3) \Rightarrow (1)$
$(4) \Rightarrow (3)$
$(2) \Rightarrow (1)$

$(3) \Rightarrow (1)$: last time

if $C \subset G$ cycle with

$e \in E(C)$, then $G \setminus e$ connected.

$(4) \Rightarrow (3)$: Assume paths not unique.

Choose vertices u, v s.t

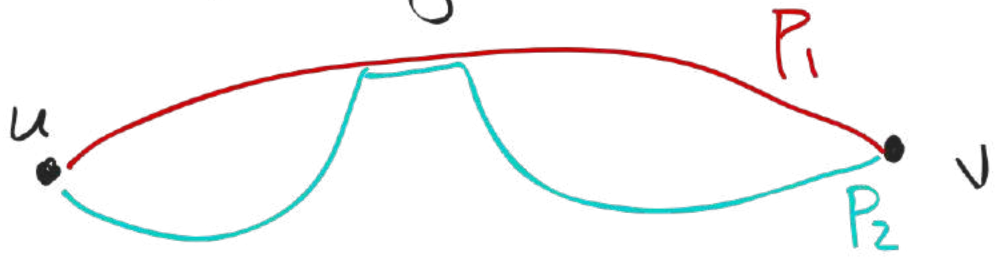
(i) $\exists$ two paths $P_1, P_2 \subset G$ between u, v

(ii) $d(u,v)$ minimal among all

examples satisfying (i)

Claim $P_1 \cup P_2$ form a cycle.

if not



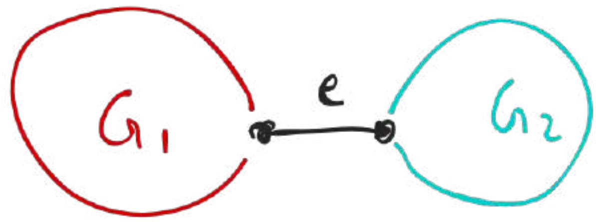
this would contradict (ii).

(2) $\Rightarrow$ (1): Contrapositive

$G \setminus e$ disconnected $\forall e \Rightarrow |E| = |V| - 1$

Proof by induction on $|E|$

- Base case: $|E| = 0 \Rightarrow G = \bullet$ $\quad |V| = 1$, $|E| = 0$ ✓
- Induction step: $|E(G)| = m$



$$m - 1 = |E(G_1)| + |E(G_2)|$$
$$= |V(G_1)| - 1 + |V(G_2)| - 1 \quad \text{(induction)}$$
$$= |V(G)| - 2$$

$$\Rightarrow |E(G)| = m = |V(G)| - 1 \qquad \square$$

Counting trees

Fix n and let $V=\{1,\dots,n\}$.

<u>Q</u> Among graphs $G=(V,E)$ how many are trees?

<u>Rmk</u> total # graphs is $2^{n(n-1)/2}$ (choose edges)

Here we are <u>not</u> counting up to iso.

<u>Exercise</u> (1) Answer Q for $2 \le n \le 5$

(2) Make a conjecture about the general case.

$\boxed{n=2}$

1 graph

$\boxed{n=3}$

3 graphs

$\boxed{n=4}$

$\frac{4!}{2} = 12$ graphs

4 graphs

$\boxed{n=5}$

$\frac{5!}{2} = 60$

$5 \cdot 4 \cdot 3 = 60$

5

n	trees
2	$1 = 2^0$
3	$3 = 3^1$
4	$16 = 4^2$
5	$25 = 5^2$

Guess n^{n-2}

<u>Thm</u> (Cayley) Among graphs $G=(V,E)$ with $V=\{1,\dots,n\}$ There are n^{n-2} trees.

Strategy: To count fingers on left hand:

- $\exists$ bijection Left/Right
- count fingers on right hand.

Consider sequences $(a_1,\dots,a_{n-2})$ with

$a_i \in \{1,\dots,n\}$

Observe that there are n^{n-2} of these.

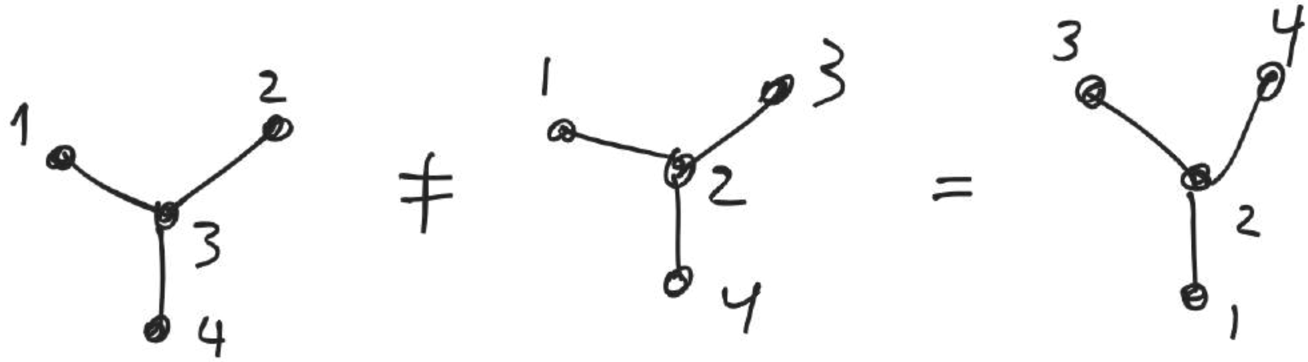
To prove Cayley's Thm we find bijection

Trees with $V(T)=\{1,\dots,n\}$ $\overset{1\text{-}1}{\longleftrightarrow}$ Sequences $(a_1,\dots,a_{n-2})$ as above

"Prüfer code"

Trees and Prüfer codes

T tree with vertices $1, \dots, n$



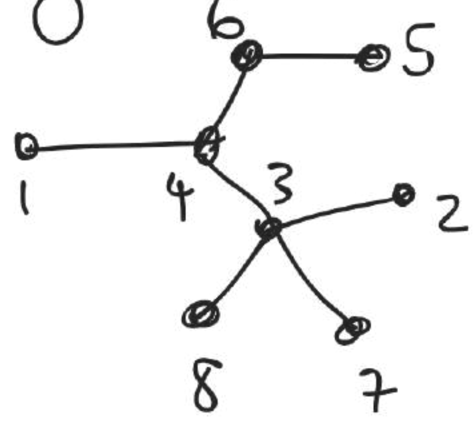
The Prüfer code $P(T) = (a_1, \dots, a_{n-1})$

obtained inductively by

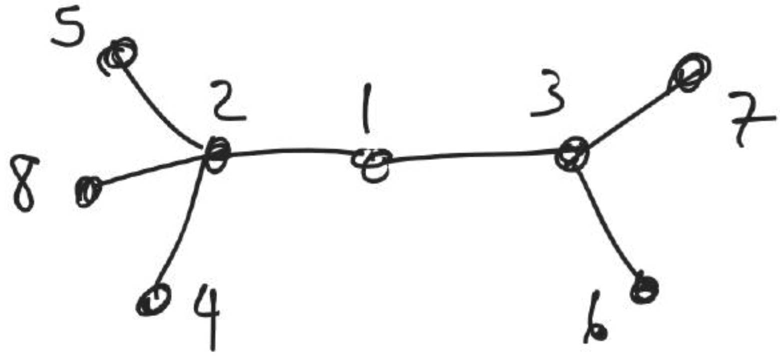
deleting smallest valence 1 vertex

and adding its neighbor to the sequence

Ex $T =$



$P(T) = (4, 3, 6, 4, 3, 3)$

Ex $T =$ 

$P(T) = (2, 2, 3, 3, 1, 2)$

Observations

- degree-1 vertices of T (leaves) don't appear in $P(T)$
- $v \in T$ appears $\deg(v) - 1$ times in $P(T)$:

each v has valence 1 at some pt in algorithm to get to this pt, $\deg(v) - 1$ of its nbrs have been deleted.

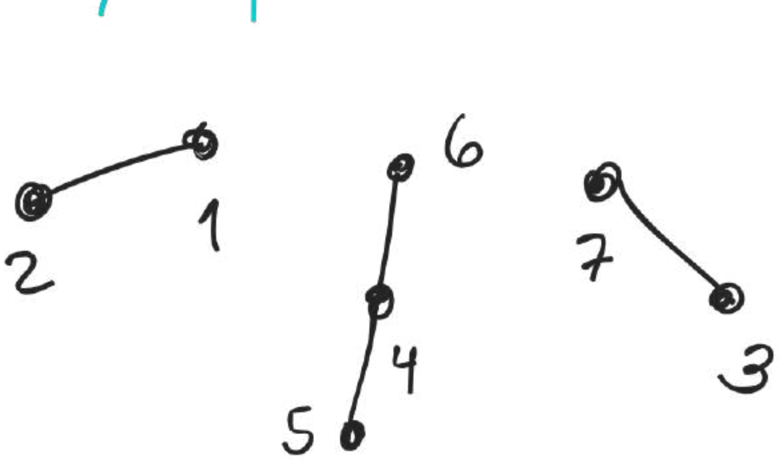
Working backwards:

given $P = (1,7,4,6,1)$ find T with $P(T) = P$.

- T will have 7 vertices
- 2, 3, 5 don't appear in P so they're leaves of T
- the smallest was deleted first

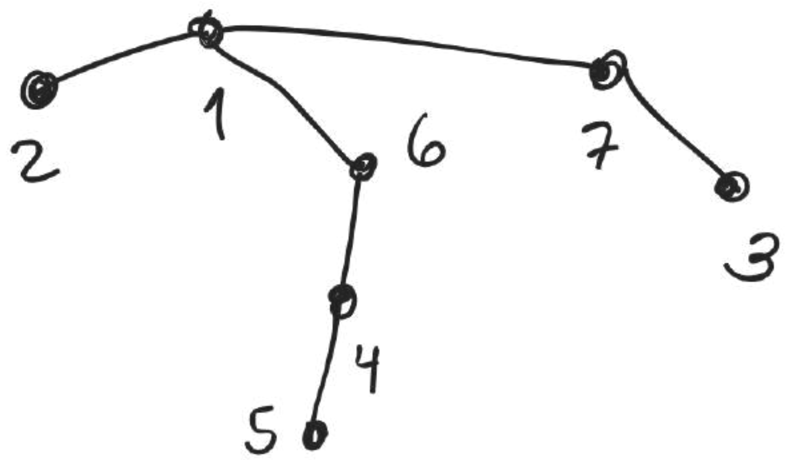
- continue: have graph w/ verts $1, \cancel{3}, \cancel{4}, \cancel{5}, 6, 7$ and code $(\cancel{7}, \cancel{4}, \cancel{6}, 1)$

continue ...



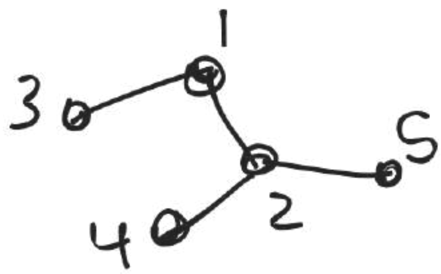
- Final step: want graph with

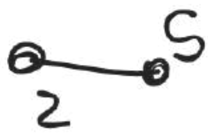
Vertices 1, 6, 7 and code (1)



Exercise Give the code for graphs

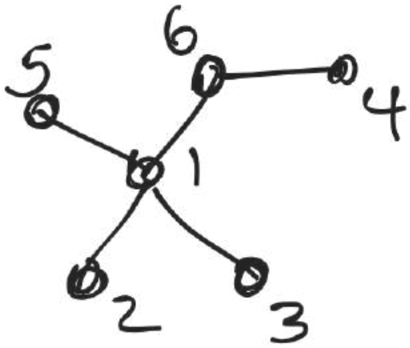
(1) $T =$





(1, 2, 2)

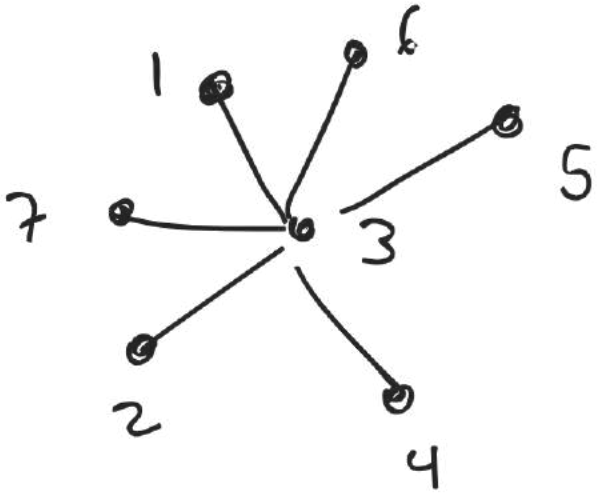
(2) $T =$



(1, 1, 6, 1)

<u>Exercise</u> draw trees with code

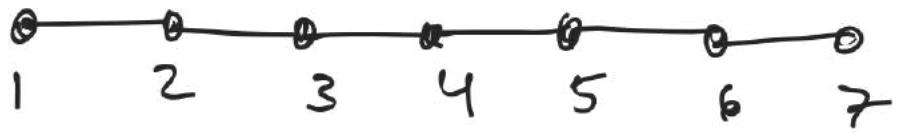
- $(3,3,3,3,3)$



- $(2,3,4,5,6)$

graph w/ vert and code

$\cancel{1}\,\cancel{2}\,\cancel{3}\,\cancel{4}\,5\,6\,\underline{7}$ $\qquad$ $\cancel{2}\,\cancel{3}\,\cancel{4}\,\cancel{5}\,6$



<u>Thm</u> $T \longmapsto P(T)$ gives a bijection

$$\left\{\begin{array}{l}\text{trees with}\\ \text{vertices } \{1,..,n\}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{sequences}\\ (a_1,..,a_{n-2}) \quad a_i \in \{1,..,n\}\end{array}\right\}$$

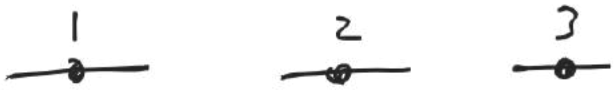
<u>Cor</u> LHS set has n^{n-2} elements

__Proof of Thm__ By induction on n

__Base case__ ($n=2$) There is only one tree 

There is only one sequence $()$.

(alternatively do $n=3$, have 3 graphs



and 3 seq $(1), (2), (3)$)

__Induction Step__ Focus on surjective. Inj. will follow

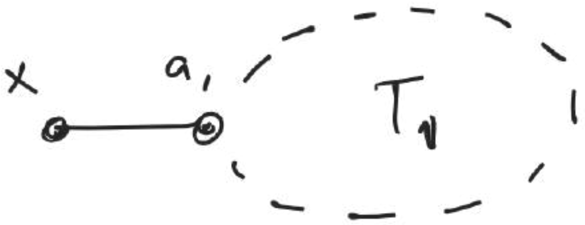
given $P = (a_1, \dots, a_{n-2})$ let x smallest

index not in P.

__Key observation__ if $P(T) = P$ then ① x is a leaf, ② $x - a_1$ is an edge, and ③ deleting x leaves tree with vertices $\{1, \dots, n\} \setminus \{x\}$ and code $(a_2, \dots, a_{n-2})$

By induction $\exists!$ tree T_1 with

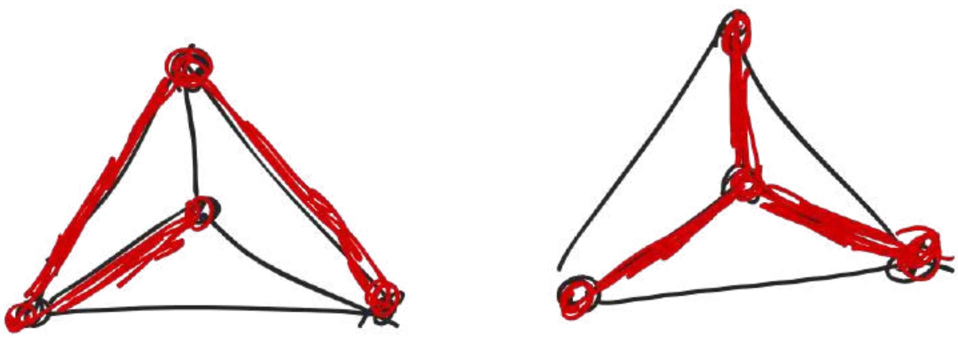
$$P(T_1) = (a_2, \dots, a_{n-2})$$

Then $T =$  is tree with $P(T) = P$. Uniqueness of T_1 and key observation $\Rightarrow$ T unique tree with $P(T) = P$ $\square$

Spanning Trees

__Defn__ Given a graph G, a __spanning tree__ is a subgraph $T \subset G$ that is a tree and contains every vertex of G.

__Ex.__



__Motivation__ (extremal problem) Smallest connected subgraphs containing all vertices

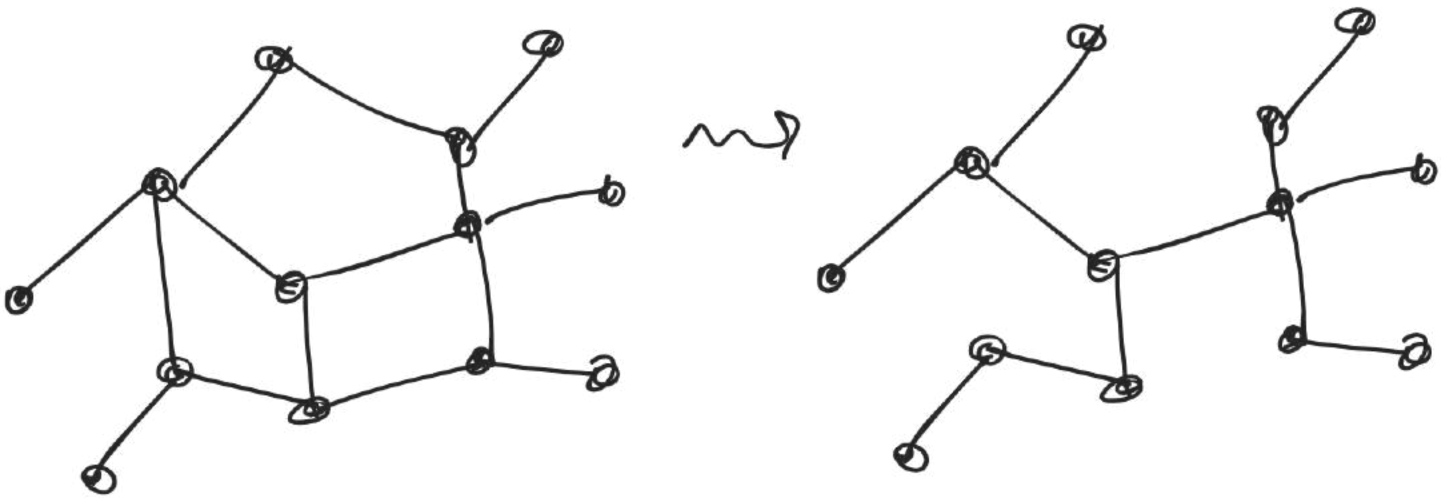
- networks. fewest roads to keep open during construction.

so ppl can still travel to diff parts of town.

<u>Prop</u> Every graph has a spanning tree.

<u>Proof</u> Recall that we proved:
if G has a cycle C, then $G \setminus e$ connected
for each edge e of C.

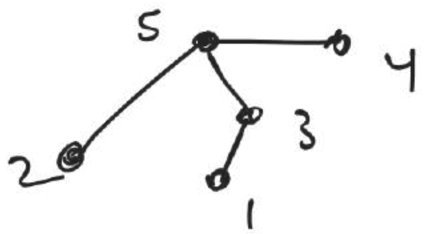
Inductively find cycle of G and
remove an edge.
Eventually arrive at tree (no cycles)
w/ same vertex set $\square$

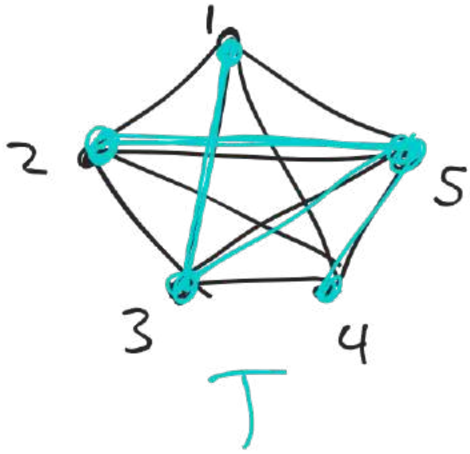


Two problems

(1) Given graph G, how many spanning trees.

Eg $G = K_n$ complete.



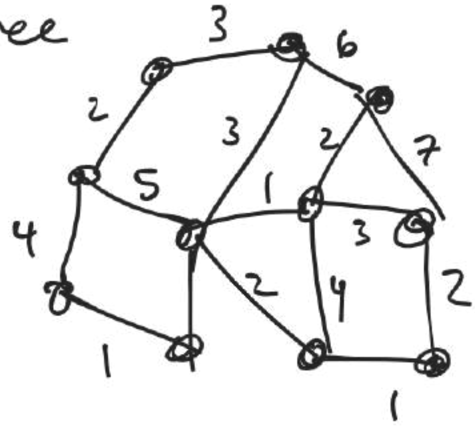


spanning trees is n^{n-2} by Cayley

generalization?

(2) Given weighted graph find minimal spanning tree

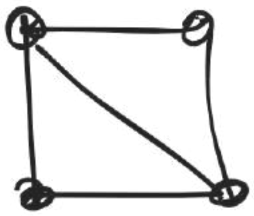
ie tree with smallest total edge weight.



Problem Given graph G how many spanning trees.

Examples

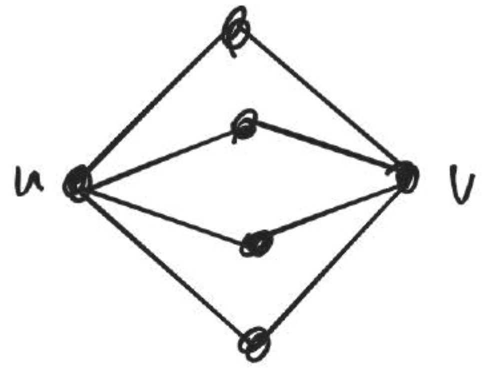
(1) G =



has spanning trees

(8)

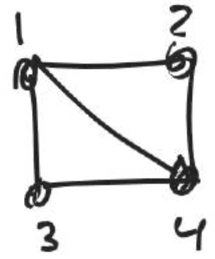
(2) G =



Has $4 \cdot 2^3$ span. trees:

a spanning tree contains a unique path u, v (4 choices) and there are 2^3 choices for rest of tree.

Matrix tree theorem (Kirchhoff)

$G =$ 

$$M := \begin{pmatrix} 3 & & & 0 \\ & 2 & & \\ & & 2 & \\ 0 & & & 3 \end{pmatrix} - \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \end{array} = \begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & 0 & -1 \\ -1 & 0 & 2 & -1 \\ -1 & -1 & -1 & 3 \end{pmatrix}$$

degree matrix adjacency matrix

M_{11}

$$\det(M_{11}) = 2(6-1) - 1(2) = 10 - 2 = 8$$

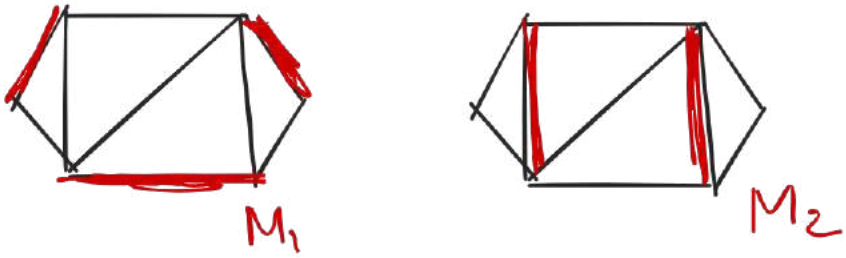
Theorem $\det(M_{11}) = \#$ spanning trees.

Matchings

<u>Defn</u> A <u>matching</u> in $G=(V,E)$ is a subset $M \subseteq E$ so that no two edges of M share a vertex.

<u>Examples</u>

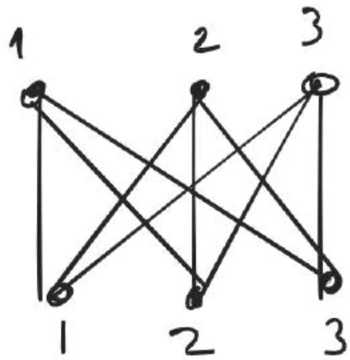
(1)



For $U \subseteq V$ say M <u>saturates</u> U if every $u \in U$ incident to some $e \in M$.

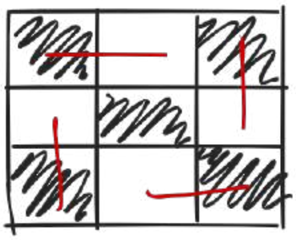
A matching that saturates V is called <u>perfect</u>.

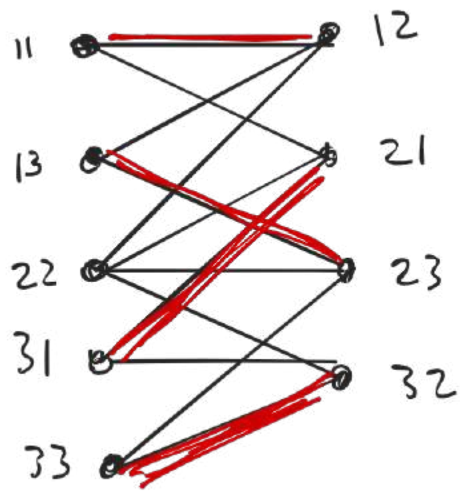
(2) $K_{n,n}$



perfect matchings of $K_{n,n}$ $\longleftrightarrow$ bijections $\{1,\dots,n\} \to \{1,\dots,n\}$

There are $n!$ of these.

(3) 

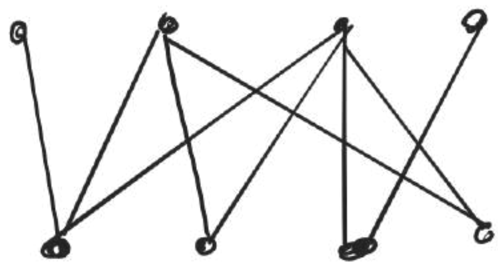


A tiling of board by tiles   gives a matching of associated bipartite graph.

(HW2): A bipartite graph $V = A \cup B$ where $|A| \neq |B|$ doesn't have a perfect matching.

(4) jobs

applicants



<u>Q</u>: Let $G = (V, E)$ be a bipartite graph $V = X \sqcup Y$. Does there exist a matching $M \subset E$ that saturates X?

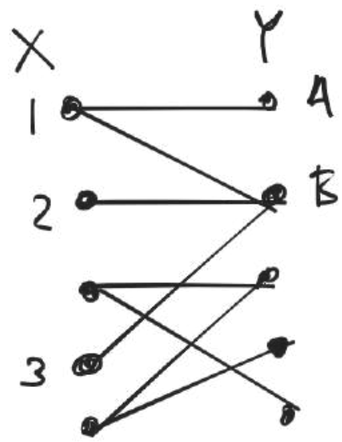
<u>Want</u>: TONC and TONCAS.

Exercise Determine if G has matching that saturates X

No. 4 applicants competing for 3 jobs

Yes

Yes



No. 3 applicants (1,2,3) competing for 2 jobs (A,B)

For $S \subset X$ define $N(S) = \{ y \in Y \mid \exists \text{ edge from } y \text{ to some } s \in S \}$

neighbors of S.

TONC If $G = (X \sqcup Y, E)$ has a matching that saturates X, then $|N(S)| \geq |S|$ for each $S \subset X$.

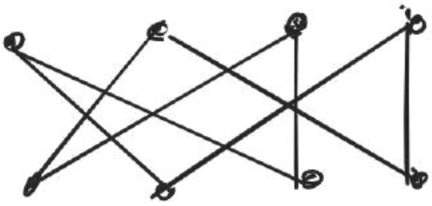
Thm (Hall's matching) TONCAS.

For $G = (X \cup Y, E)$ bipartite

G has matching saturating X $\iff$ $|N(S)| \geq |S|$ for every $S \subset X$.

<u>Cor</u> Fix $k \geq 1$. A k-regular bipartite graph G has a perfect matching

<u>Example</u> graph above is 2-regular



<u>Proof of Cor</u> Write $G = (X \cup Y, E)$.

<u>Step 1</u> Show $|X| = |Y|$.

$$k|Y| = \sum_{u \in Y} \deg(u) = |E| = \sum_{v \in X} \deg(v) = k|X|$$

<u>Step 2</u> Since $|X| = |Y|$, to show there is perfect matching suffices to show $\exists$ matching that saturates X.

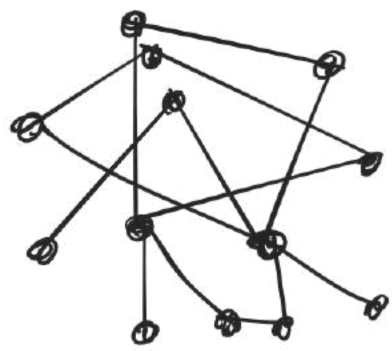
Apply Hall: Fix $S \subset X$.

$$\underbrace{\#\text{ edges leaving } S}_{k|S|} \le \underbrace{\#\text{ edges entering } N(S)}_{k|N(S)|}.$$

$$\Rightarrow |S| \le |N(S)|. \quad ✓ \qquad \square$$

Prove Hall next time

$\boxed{\text{Maximum matchings}}$



Let G be any graph.

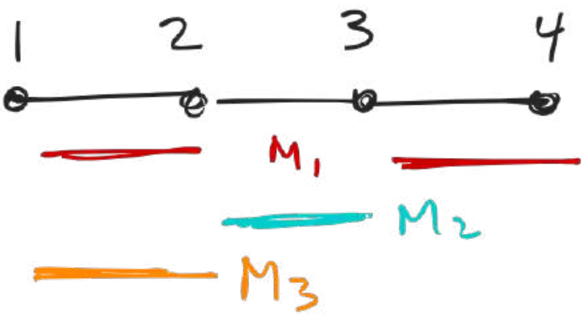
Q: What's the max size of a matching?

Defn Say M is a maximal matching if $\nexists$ matching M' with $M \subsetneq M'$.

Say M is a maximum matching if $\nexists$ matching M' with $|M| < |M'|$.

(Maximum $\Rightarrow$ Maximal)

Example $G = P_4$

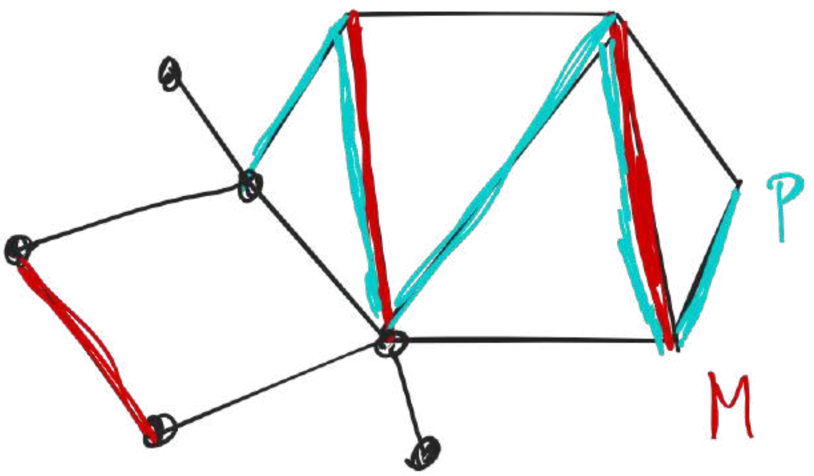


$M_1 = \{12, 34\}$ maximum

$M_2 = \{23\}$ maximal not maximum.

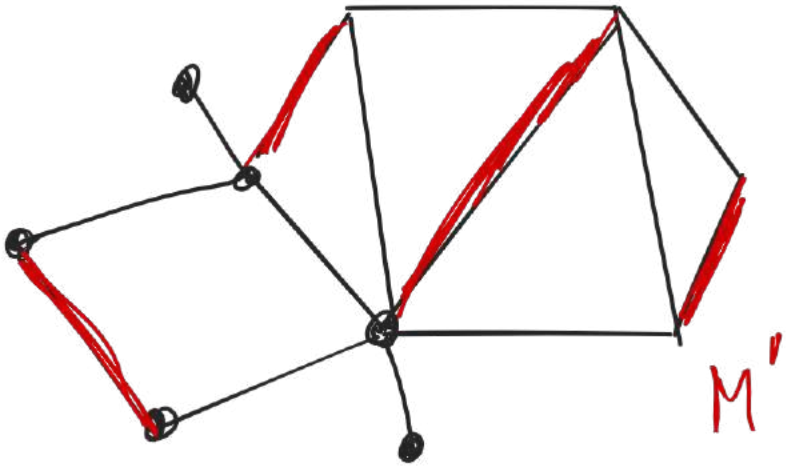
$M_3 = \{12\}$ not maximal

Ex (how to show a matching is not maximum)



Consider path P whose edges alternate between $E \setminus M$ and M.

Here the endpoints aren't incident to any edge of M so we can use P to get a larger matching



Call P an M-augmenting path.

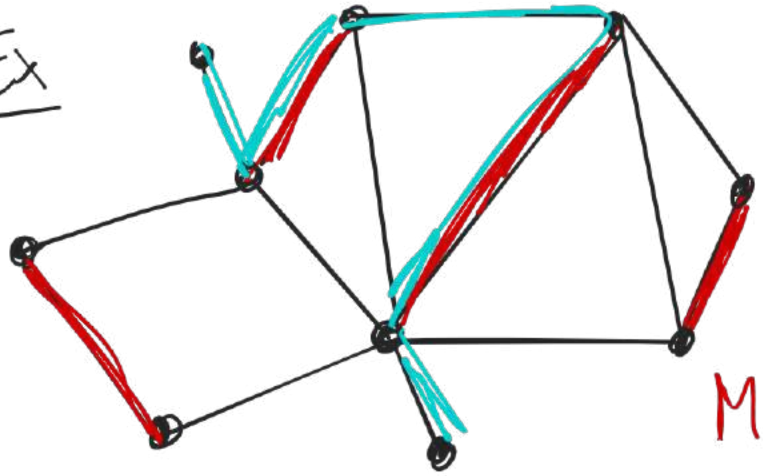
TONC If M is a matching of G and G has an M-augmenting path, then M is not a maximum matching.

Thm (maximum matchings) TONCAS

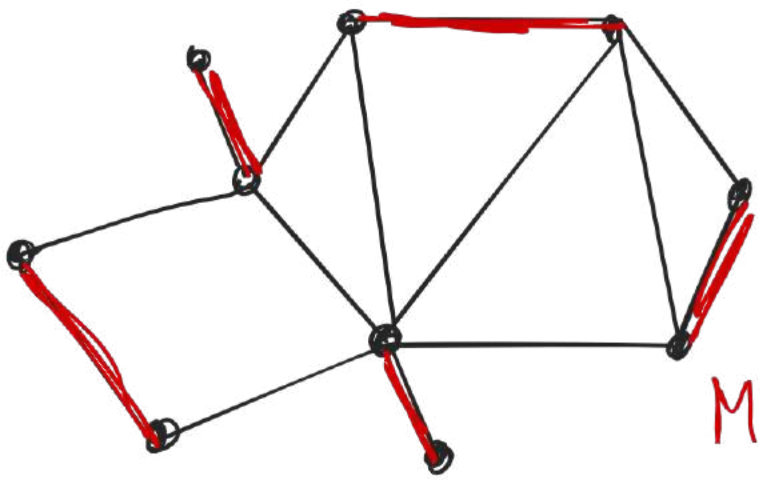
M is maximum matching of G $\iff$ G has no M augmenting paths.

Rmk Thus to find a maximum matching we could start w/ any matching M and look for M-augmenting paths to replace M with larger matching, until $\nexists$ M-aug path.

Ex



M'



M''

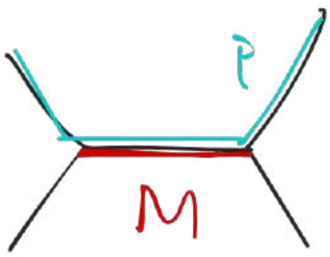
M'' has no M''-aug paths.

(indeed every vertex is M''-saturated)

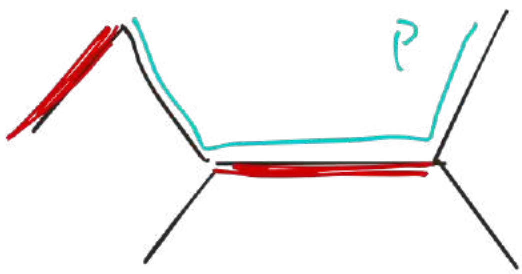
Maximum matchings

Recall $G=(V,E)$

- $M \subset E$ is matching if no $e, e' \in M$ have common vertex
- a matching M is maximum if it has most edges of any matching
- for matching M, an M-augmenting path $P \subset G$ has edges alternating between M and $E \setminus M$ and endpts of P not incident to any edge of M.



P is M-aug



P not M-aug.

- TONC:

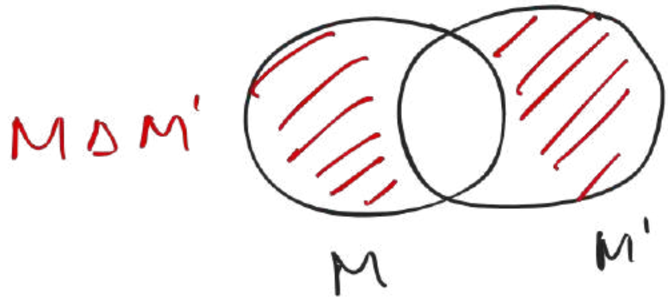
  if M maximum, then $\nexists$ M-aug path.

Thm (maximum matchings) TONCAS

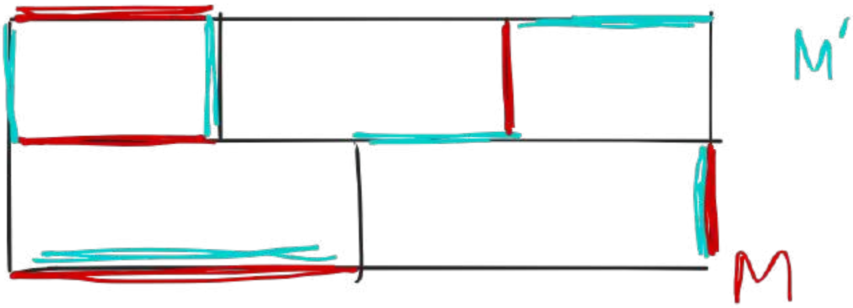
if M not maximum, then $\exists$ M-aug path

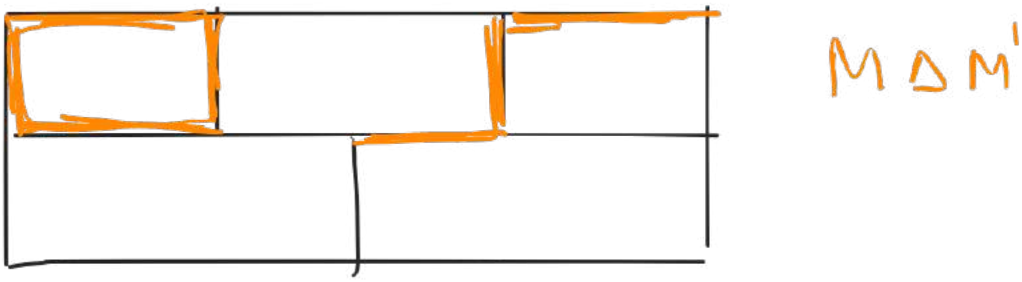
Toward proof: if M not maximum $\exists$ M' with more edges. consider symmetric diff

$$M \Delta M' := (M \setminus M') \cup (M' \setminus M)$$



Example



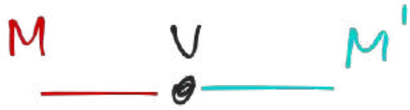


Lemma M, M' matchings of G.

Then $M \Delta M'$ is union of paths and even cycles

Proof (sketch)

- Vertices of $M \Delta M'$ have degree 1 or 2



- graphs with vertex degrees ≤ 2 are unions of P_n's and C_m's. (compare to HW2 #3)
- cycles are even length b/c ... They alternate btwn M and M'. $\square$

Proof of maximum matchings Thm

<u>WTS</u> if M is not maximum then G has M-augmenting path.

Let M' be a matching with more edges than M.

By lemma $M \Delta M'$ union of paths and even cycles. Since $|M'| > |M|$, $M \Delta M'$ must have a component $\bullet \overset{M'}{\text{---}} \circ \overset{M}{\text{---}} \circ \cdots \circ \overset{M}{\text{---}} \circ \overset{M'}{\text{---}} \bullet$

Endpts are not incident to M by construction

So this is an M-augmenting path. $\square$

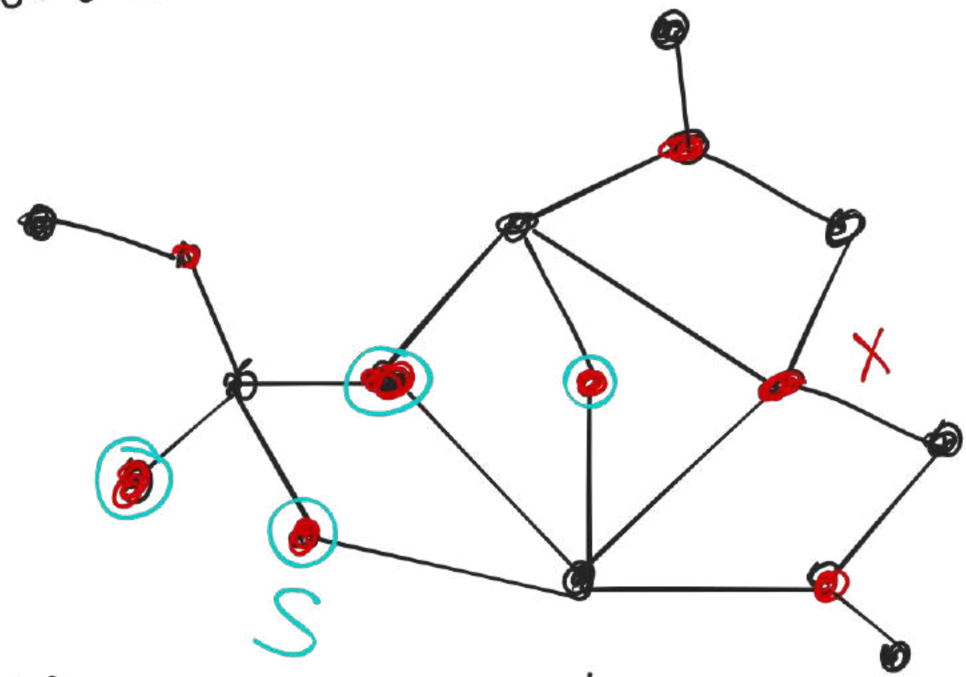
Hall's Theorem

$G = (X \sqcup Y, E)$ bipartite

$\exists$ matching M saturating X $\iff$ $|S| \leq |N(S)|$ $\forall\, S \subset X$.

$(\Rightarrow)$ is "obvious"

Example



Bipartite

$|X| = |Y|$

$|S| = 4 \quad |N(S)| = 3 \quad \Rightarrow \nexists$ matching saturating X.

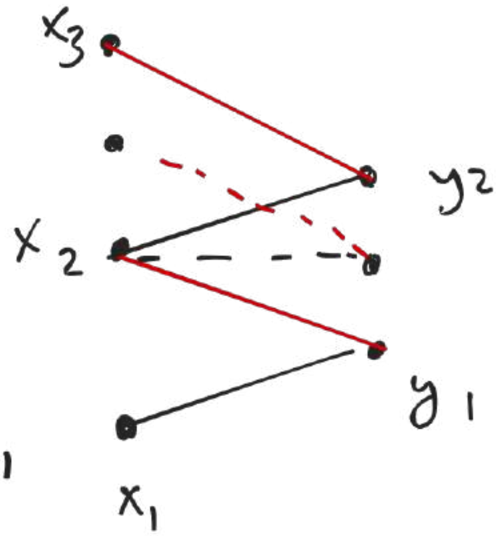
Pre-proof Assume $|S| \leq |N(S)|$ $\forall\, S \subset X$.

Let M be a maximum matching and suppose for a contradiction that M doesn't saturate X.

What goes wrong?

Choose $x_1 \in X$ not saturated by M

$|N(\{x\})| \geq |\{x\}| = 1 \Rightarrow \exists y_1$



M max $\Rightarrow$ y_1 saturated

$\Rightarrow \exists \ x_2$

$|N(\{x_1, x_2\})| \geq 2 \Rightarrow \exists y_2 \neq y_1$

M max $\Rightarrow$ y_2 saturated $\Rightarrow \exists \ x_3$

G is finite so eventually this process ends. If it ends at y_k then we found M-aug path contradicting M max

If it ends at x_k would like to conclude $|N(S)| < |S|$ for some $S \subset X$ . . .

<u>Proof</u> Consider M-alternating paths starting at x_1. Let

$$X' = \{ x \in X \text{ end point of } M \text{ alt. path starting at } x_1 \}$$

$$Y' = \{ \text{penultimate vertices of these paths} \}$$

Observe M gives matching X' to Y'

$$\Rightarrow |X'| = |Y'|$$

Consider $S = X' \cup \{x_1\}$

By assumption $|N(S)| \geq |S|$

$\Rightarrow \exists\, y \in N(S) \setminus Y'$.

<u>Key</u> $y \notin Y' \Rightarrow y$ not saturated by M.

<u>Case 1</u> $s = x_1$. $\Rightarrow$

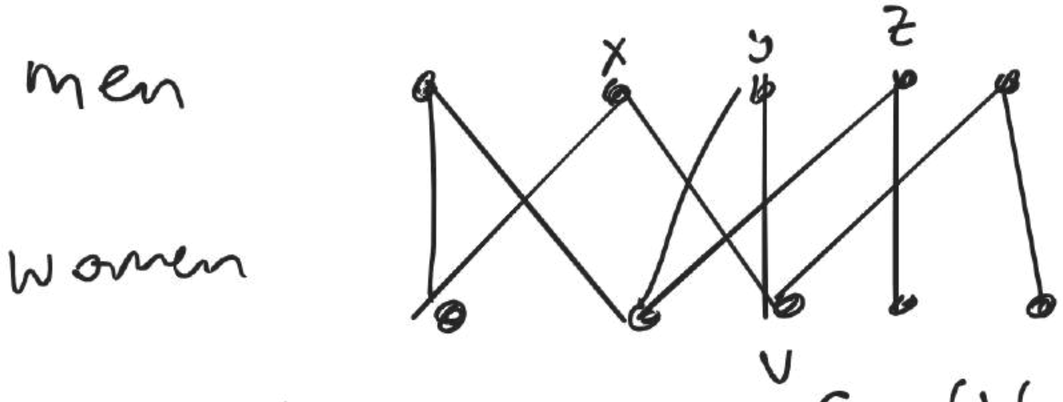
sy is M alt. path. $\times$.

Case 2 $s \in X'$

Take alt path P from x_1 to s. Then $P \cup \{sy\}$ is M-augmenting path. $*$

From this conclude x_1 doesn't exist ie M saturates X. $\square$

Stable Marriage Problem



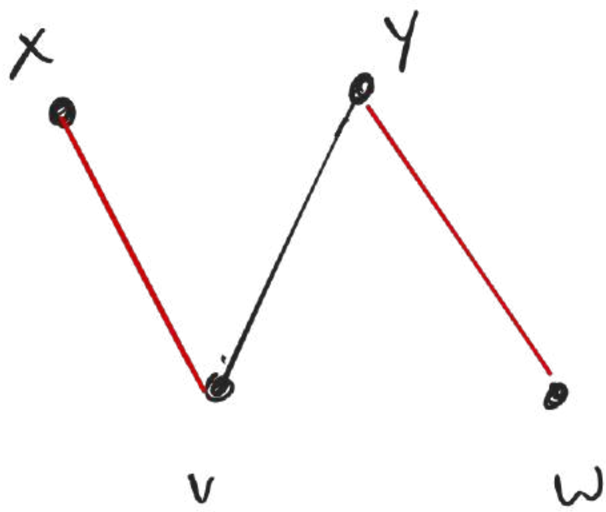
acquaintance graph. $G = (V, E)$

Goal Find matching incorporating preferences

For each $v \in V$, given an ordering $<_v$ on $N(v)$.

$x <_v z <_v y$.

A matching M is <u>stable</u> if no unmatched pair is motivated to change their matching eg.

don't have → 

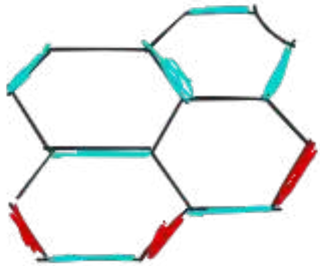
with

$x <_v y$ and

$w <_y v$.

Konig's Theorem on maximum matchings

<ins>Q</ins>: Given $G=(V,E)$ what is size of maximum matching?



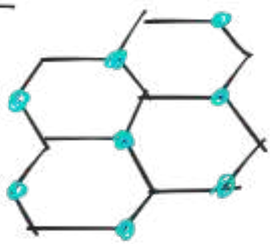
<ins>Recall</ins> M maximum $\Leftrightarrow$ G has no M aug-path

Can use this to find maximum M but testing for M-aug path can be tedious.

Better way:

<ins>Defn</ins> A <ins>vertex cover</ins> of $G=(V,E)$ is subset $Q \subset V$ s.t. every $e \in E$ has at least one endpt in Q.

<ins>Ex</ins>

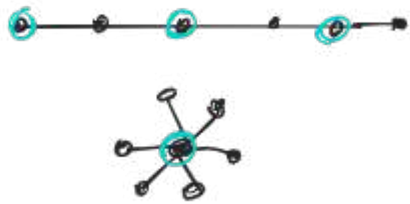


Q

Note $Q=V$ is always a vertex cover. We're interested in small covers

Ex Social network

efficient way to spread a message.



Connection to matchings

Lemma M max matching of G.

Then (i) every vertex cover $Q \subset V$ has $\geq |M|$ vertices

(ii) $\exists$ vertex cover w/ $2|M|$ vertices

Proof

(i) For each $e \in M$ ≥ 1 endpt in Q

so $|Q| \geq |M|$. (no $e, e' \in M$ share endpt)

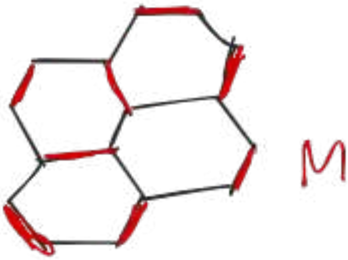
(ii) Take $Q =$ endpts of edges of M. (so $|Q| = 2|M|$)

Q is vertex cover:

For $e \in E$ either one of both endpts saturated by M, since M maximum. $\square$

Eg

$\Rightarrow$ G does not have a matching w/ 8 edges.



$\Rightarrow$ M is maximum (easy!)

Thm (König) G bipartite.

Max size of matching = Min size of vertex cover.

Rmks ① Given lemma, main part is to show G has vertex cover w/ $|M|$ vertices.

② It's important that G bipartite!

eg 

max size matching = 2

min size of vertex cover = 3

③ König's Thm is example of <u>min/max relation</u>.

Consider problems of max matching

& min vertex cover

to be <u>dual problems</u>

This differs from TONCAS.

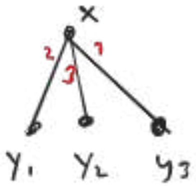
Proving min/max relation can save work.

More examples later.

Stable Matchings

setup: $G = (X \cup Y, E)$ bipartite

For $v \in V$ given ordering $<_v$ on $N(v)$



$y_2 <_x y_1 <_x y_3$

A matching $M \subset E$ is <u>stable</u> if for each edge $\{x,y\} \notin M$ $\exists$ either

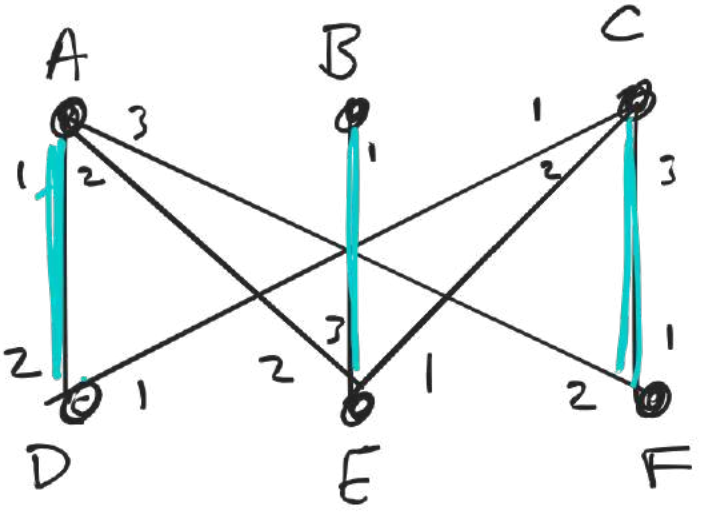
$\{x,y'\} \in M$ so $y <_x y'$

or $\{x',y\} \in M$ so $x <_y x'$

ie either x or y is matched to someone/thing they prefer more.

Examples (Add from prev lecture)

Example

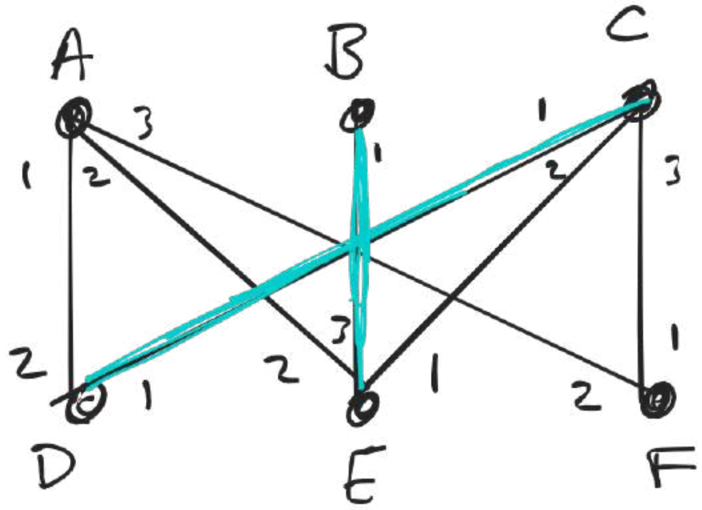


unstable

eg

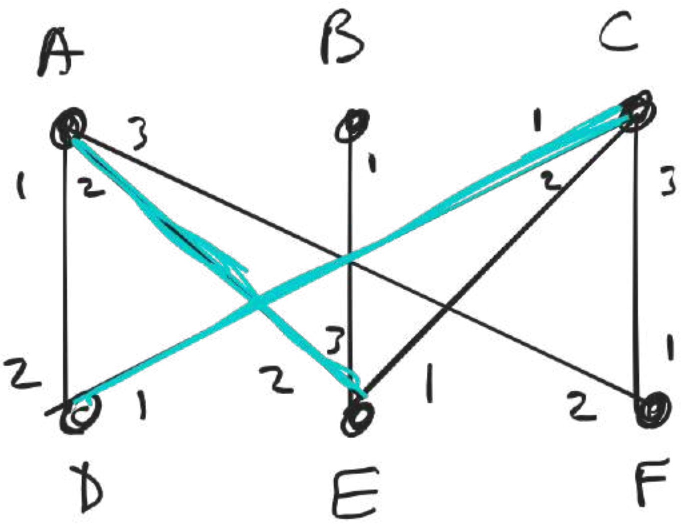
- D prefers C to A

  C prefers D to F



also unstable

- A prefers E to nothing
- E prefers A to B.



Stable

eg

- A prefers D but D prefers C
- E prefers C but C prefers D.
- B prefers E but E prefers A.

Gale-Shapely proposal algorithm

Thm (Gale-Shapely)

Stable matchings always exist!

Idea

Build stable matching inductively.

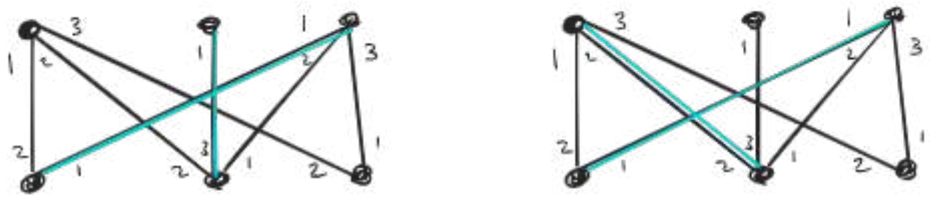
- 1st round: each $x \in X$ proposes to top choice. $y \in Y$ matches w/ best offer. Remaining $x \in X$ are unmatched.
- subsequent rounds: each unmatched $x \in X$ proposes to top choice they have not yet proposed to. Each $y \in Y$ compares

any new offer to previous and matches w/ top choice.

Remaining x remain/become unmatched.

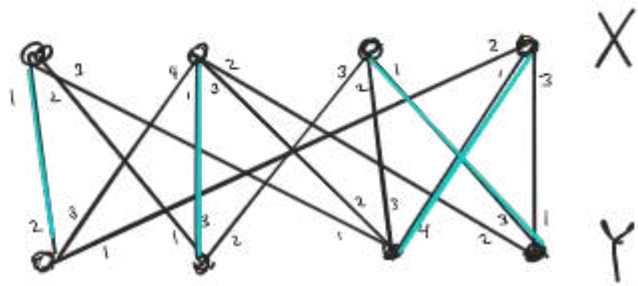
- Repeat until each $x \in X$ either matched or has proposed to full list.

Example

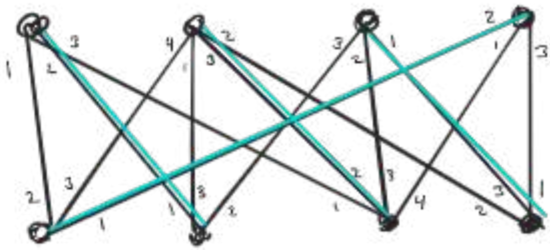


Stable

Example (use algorithm, check it's stable)



X proposes



Y proposes

Features

- algorithm stops:
  at most $|E|$ proposals made,
  none made twice

- Matching at end stable

- proposers end up w/ best possible match (among all stable matchings) and proposees get worst possible match

Rank algorithm used for med school residencies.

König's Thm

Recall $G=(V,E)$. A vertex cover is $Q\subset V$ s.t. every edge has ≥ 1 endpt in Q.

Ex $G=K_n$

Every two vertices connected by edge $\Rightarrow$ a vertex cover must have $\geq n-1$ vertices

Thm (König) G bipartite.

min size of vertex cover $=$ max size of matching.

Last time: $\geq$ (each edge of M has ≥ 1 endpt in Q)

Ex $G=K_n$ max size of matching is $\lfloor \frac{n}{2} \rfloor$

(generally max size is $\leq n/2$)

So in general diff. btwn
min vertex cut size & max matching size
can be arb. large.

Proof $G = (X \cup Y, E)$

Fix $M \subset E$ maximum matching.

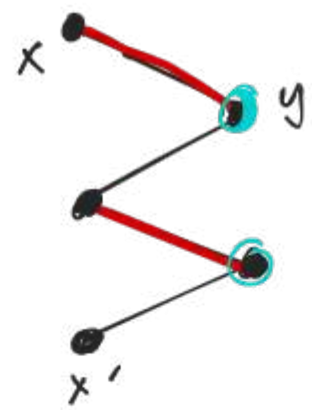
WTF: vertex cover Q w/ $|M|$ vertices

idea try to use one vertex from each edge of M

Fix $e = \{x, y\} \in M$. How to decide:

$x \in Q$ or $y \in Q$?

Case 1: $\exists$ alternating path from unsaturated $x' \in X$ ending in edge e.



Then choose $y \in Q$.

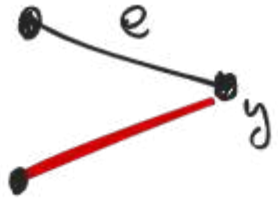
Case 2 Otherwise $x \in Q$.

Check Q is a vertex cover.

Fix edge $e = \{x, y\} \in E$. If $e \in M$, done

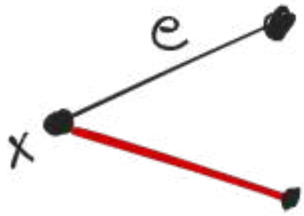
If $e \notin M$, one endpt is saturated.

<u>case</u>



by defn, $y \in Q$. ✓

<u>case</u>

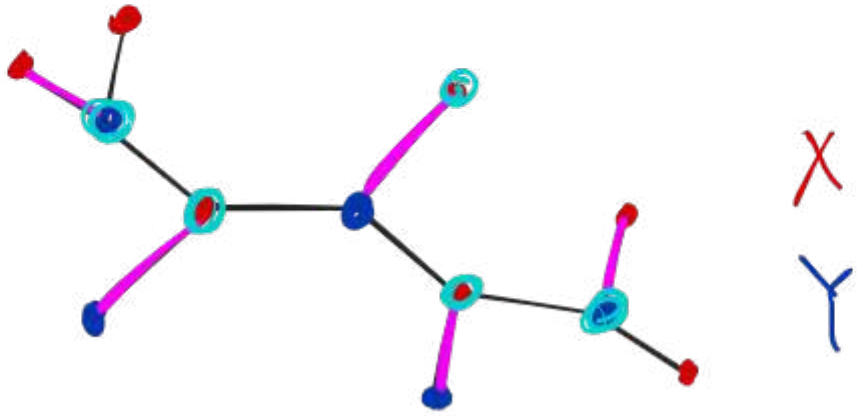


$x \in Q$

(M maximum so $\nexists$ M-aug path)

$\square$

<u>Ex</u>



<u>Application:</u>

<u>Hall's Thm</u> $G = (X \cup Y, E)$ has matching saturating $X \iff |S| \le |N(S)| \quad \forall S \subset X$.

Proof of $(\Leftarrow)$ using König

By contrapositive. Let M be a maximum matching. Suppose it doesn't saturate X.

Let Q be a min vertex cover.

$$|Q| = |M| < |X|.$$

$$|Q| = |Q \cap X| + |Q \cap Y|$$

Observe all edges from $x \in X \setminus Q$ land in $Q \cap Y$ since Q vertex cover.

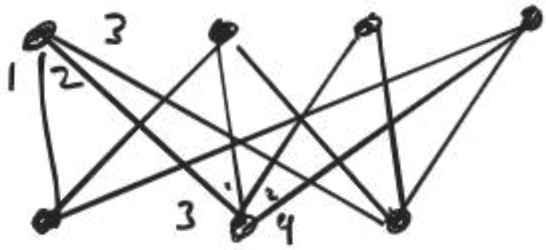
Ie for $S = X \setminus Q$, $N(S) \subset Q \cap Y$.

$$|S| = |X \setminus Q| = |X| - |X \cap Q| > |Q \cap Y| \geq |N(S)|$$

$\square$

Gale-Shapely Theorem

Given bipartite $G = (X \cup Y, E)$ with preferences



A matching $M \subset E$ is <u>stable</u> if never have $\{x, y\} \in E$ st

- x prefers y to x's match and y prefers x to y's match.
- x unmatched and y prefers x to y's match. Same switching x, y.
- x, y both unmatched.

<u>Thm</u> (Gale-Shapely) For any bipartite G w/ preferences, $\exists$ stable matching.

<u>Proof</u> by construction

Short summary of proposal algorithm

vertices in X propose to top choice that they haven't proposed to.

vertices in Y accept their best offer.

Stop when all $x \in X$ matched or have no more proposals to make.

Call this matching M_X

Claim M_X is stable.

Key property if $\{x, y\} \in M_X$ then x has been rejected by each $y' \in Y$ that x prefers. Also y has not been proposed to by any $x' \in X$ that y prefers.

Proof of claim Fix $\{x_0, y_0\} \in E \setminus M_X$

There are several cases

- x_0, y_0 both matched

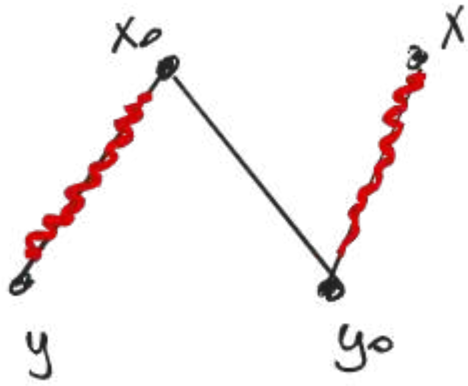
  if x_0 prefers y_0 to y

  (o.w. we're done)

  then x_0 was rejected

  by y_0 which means

  y_0 rejected x_0, so y_0 prefers x to x_0.



- other cases similar. $\square$

Further properties / comments

(1) Can't cheat algorithm by lying.

eg if $x \in X$ puts #1 choice at bottom of list that pairing becomes less likely. as it gives all x's other choices a chance to accept first.

(2) Algorithm optimal for proposers

let M any other stable matching.

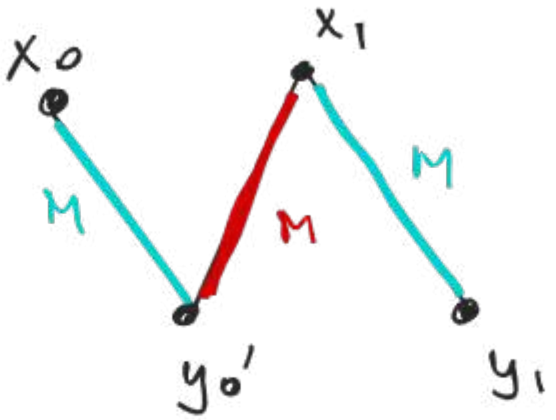
For each $x \in X$ if x paired w/ y in M_X and $y' \in M$ then x prefers y to y'. (or $y = y'$)

Proof Sketch

By contradiction. Suppose $\exists$ M, $x_0 \in X$, $\{x_0, y_0\} \in M_X$ $\{x_0, y_0'\} \in M$, and x_0 prefers y_0' to y_0

There may be many x's like this

Choose x_0 that is rejected by corresponding y_0' earliest in proposal algorithm.



x_0 rejected by y_0'

$\Rightarrow$ y_0' already accepted proposal from x_1 that y_0' prefers.

$\{x_0, y_0'\} \in M$ and y_0' prefers x_1 and M stable $\Rightarrow$ $\{x_1, y_1\} \in M$ with x_1 preferring y_1 to y_0'.

$\{x_1, y_1\} \notin M$ $\Rightarrow$ x_1 rejected by y_1

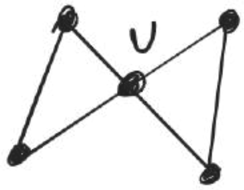
This happened before x_1 proposed to y_0' and hence before y_0' rejected x_0 $\times$.

(3) This algorithm used for med school residency matching. Initially w/ schools proposing. Later switched to students proposing. (algorithm ethics /bias)

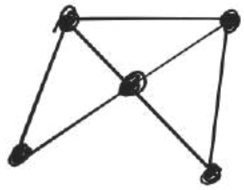
I. Graph Connectivity

"Some graphs are more connected than others"

Ex



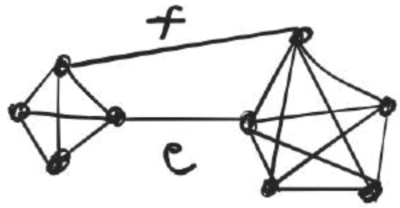
vs



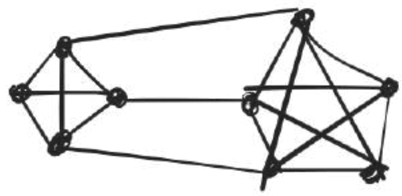
$G \setminus v$ disconnected

$G \setminus v$ connected $\forall v \in V$

Ex



vs

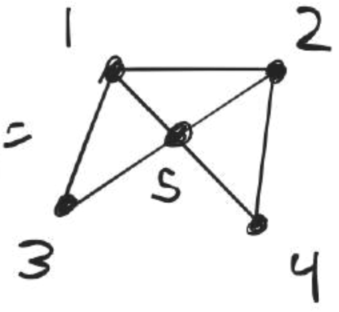


$G \setminus \{e, f\}$ disconnected

$G \setminus \{e, f\}$ connected $\forall e, f \in E$

Defn A <u>vertex cut</u> of $G = (V, E)$ is subset $S \subset V$ s.t. $G \setminus S$ disconnected

The <u>vertex connectivity</u> $\kappa(G)$ is the smallest size of a vertex cut.

Ex. $G =$ 

$S = \{1, 5\}$ is vertex cut

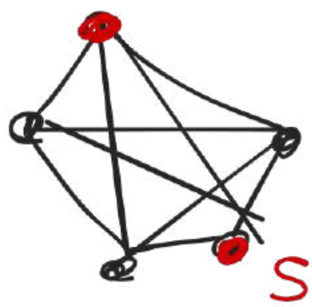
$S = \{3, 4\}$ not vertex cut

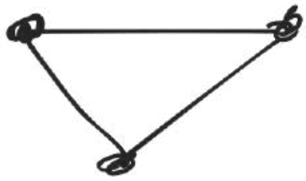
Since $G \setminus \{v\}$ connected $\forall v$

$\kappa(G) = 2$.

Ex $G = K_n$ For any $S \subset V$

$K_n \setminus S \cong K_{n-|S|}$





So $\kappa(K_n) = \ldots$ (what's min of empty set?)

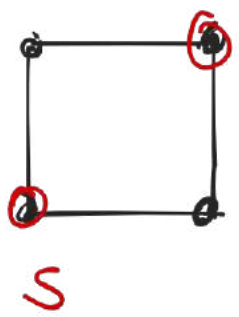
Leave $\kappa(K_n)$ undefined for now.

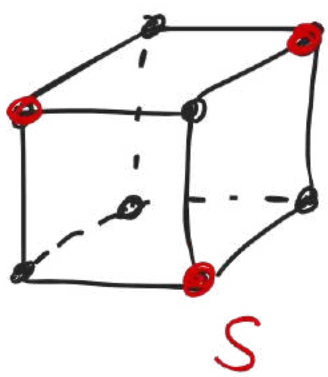
Lemma Fix $G=(V,E)$. If $G\neq K_n$, then G has a vertex cut.

Pf. $G \neq K_n \Rightarrow \exists\, u,v \in V$ s.t. $\{u,v\} \notin E$. Take $S = V \setminus \{u,v\}$.

Then $G\setminus S = \underset{u}{\bullet} \quad \underset{v}{\bullet}$ $\square$

Ex $\kappa(Q_n) = n$

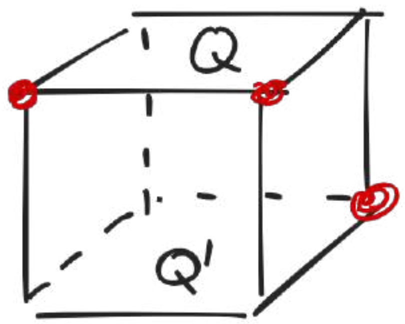




Q_n has vertex cut obtained by removing all neighbors of $(0,\dots,0)$. There are n. Thus $\kappa(Q_n) \leq n$

Harder: any vertex cut has at least n vertices

Prove this by induction on n.



$Q \cong Q_{n-1} \cong Q'$

(cf. HW4 # 2)

Induction step Fix vertex cover S of Q_n.

Case 1 $Q \cap S$, $Q' \cap S$ both connected

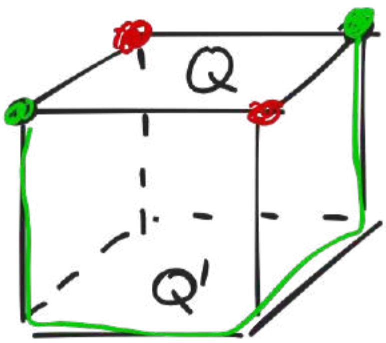
Then S must contain one vertex from each edge connecting Q, Q'

$\Rightarrow |S| \geq 2^k$

Case 2 (wlog) $Q \cap S$ disconnected.

Then $|Q \cap S| \geq n-1$.

If $|Q' \cap S| = 0$ then $Q_n \setminus S$ connected



So $|Q' \cap S| \geq 1$

$\Rightarrow |Q| \geq k$. $\square$

Remarks

① Want effective way to compute $\kappa(G)$.

Upper bounds are "easy": to show $\kappa(G) \leq m$ need only to find vertex cut of size m.

Lower bounds are harder.

(Similar to finding size of max matching.)

② Similarly can define edge cuts, edge connectivity.

Ex submarine cable map

K-connectedness

Say G is m-connected if $\kappa(G) \geq m$ i.e. $G \setminus S$ connected for any $S \subset V$ with $|S| < m$.

Eg Every graph is 0-connected.

$\underbrace{\text{1-connected}}_{S=\phi \text{ is not a vertex cut.}} \iff \underbrace{\text{Connected}}_{\text{every pair of vertices connected by path}}$

Thm (Whitney) Fix $G = (V, E)$ w/ $|V| \geq 3$

TFAE

1. 2-connected
2. $\forall\ u, v \in V\ \exists\ 2$ disjoint (u,v)-paths (*)
3. G has an ear (+) decomposition (++)

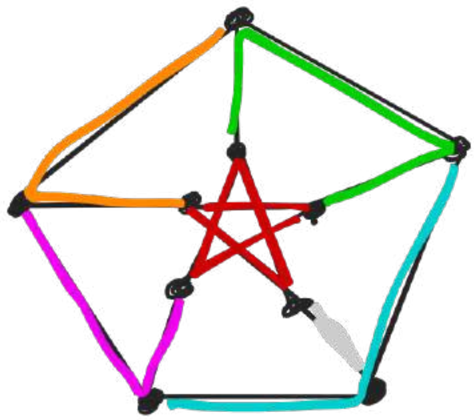
(*) paths u to v sharing no interior vertices

(+) paths whose internal vertices have $\deg = 2$

(++) obtained from C_n by adding ears.

eg

eg Petersen graph



Easier ③ $\Rightarrow$ ② $\Rightarrow$ ①. (exercise)

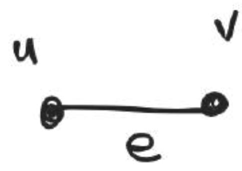
Proof of ① $\Rightarrow$ ②

Assume G 2-connected.

Want: disjoint (u,v)-paths for each pair $u, v \in V$.

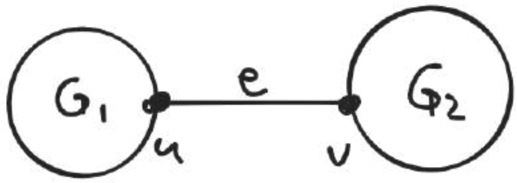
Use induction on $d(u,v)$

base case $d(u,v) = 1$.



• Observe that $G \setminus e$ is connected:

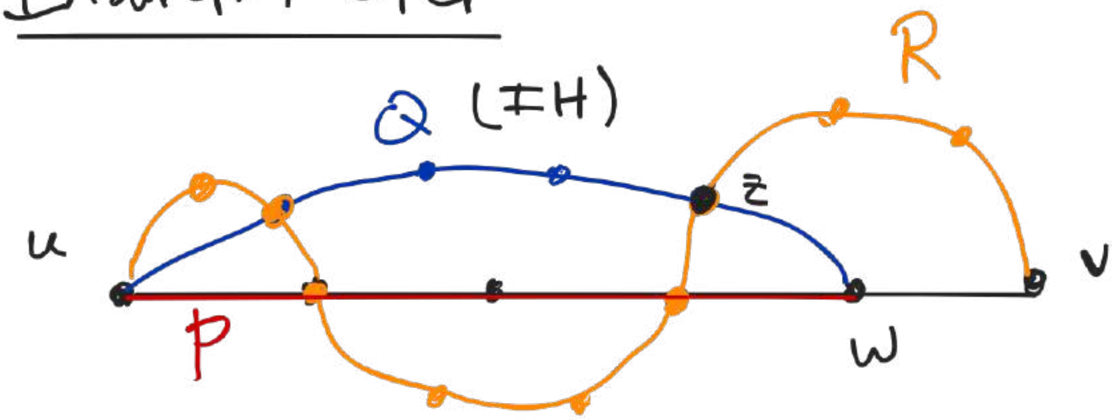
o.w.



wlog G_1 has ≥ 2 vertices $\Rightarrow$ $G \setminus u$ disconnected $\times$.

• $G \setminus e$ connected $\Rightarrow$ $\exists$ disjoint (u,v) paths.

Induction Step



$G \setminus w$ connected $\Rightarrow \exists$ path R from v to u missing w.

if R disjoint from P or Q, done

else $z :=$ 1st contact of R w/ $P \cup Q$. wlog $z \in Q$.

define $P_1 =$ follow R to z, then follow Q

$P_2 =$ v to w to P. $\square$

$\boxed{\text{Menger's Thm}}$

$G = (V, E)$

$$\kappa(G) = \min \{ |S| : S \subset V,\ G \setminus S \text{ disconnected} \}$$

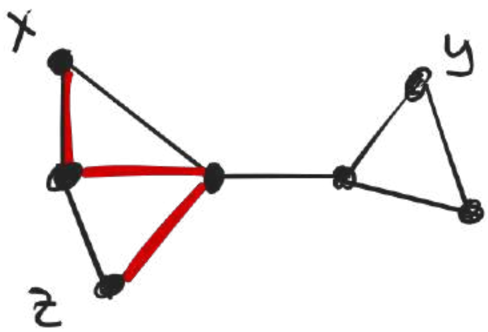
(vertex) connectivity.

For $x, y \in V$ define

$$\kappa(x,y) = \min \{ |S| : \underbrace{S \subset V \setminus \{x,y\},\ x,y \text{ in diff comp. of } G \setminus S}_{(x,y)\text{-vertex cover}} \}$$

Then $\kappa(G) = \min_{x,y} \kappa(x,y)$.

eg



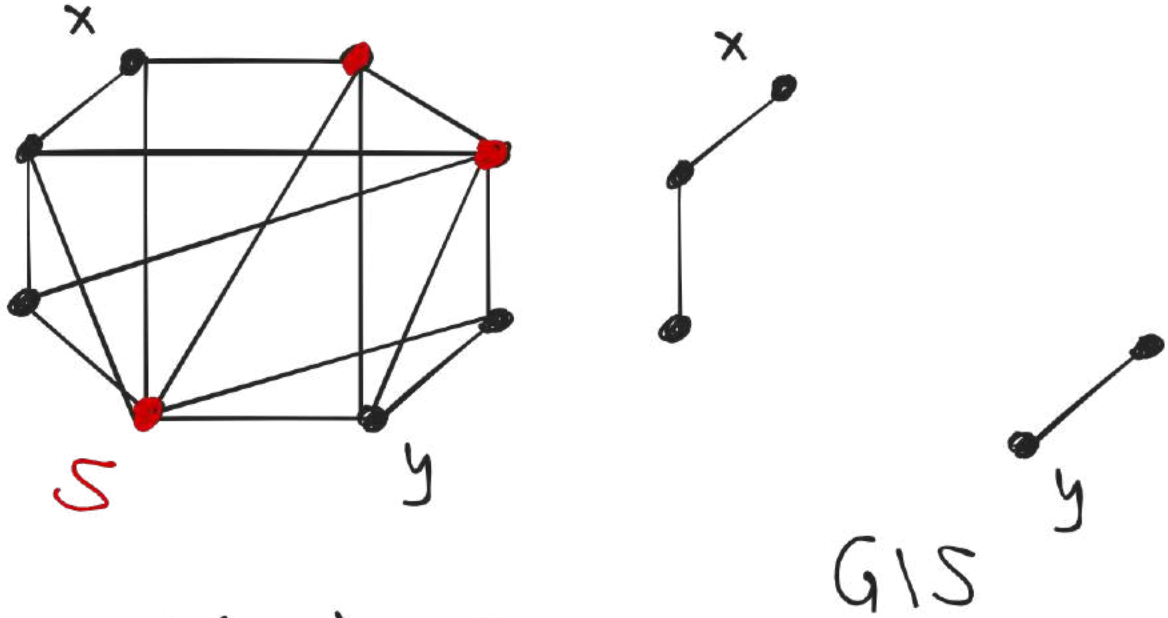
(not every path is part of maximum family)

$$\kappa(x,y) = \kappa(z,y) = 1, \quad \kappa(x,z) = 2.$$

<u>Rmk</u> $\{x,y\} \in E \Rightarrow \kappa(x,y) = \min(\emptyset)$.

As with $\kappa(G)$ not obvious how to efficiently compute $\kappa(x,y)$.

Ex.

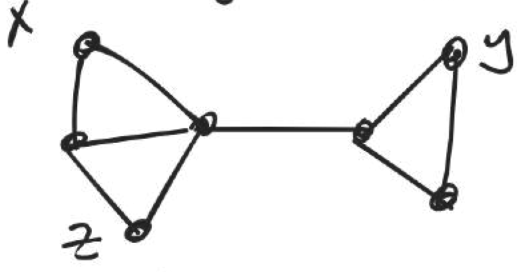


So $\kappa(x,y) \leq 3$.

But how do we show $\kappa(x,y) \geq 3$ without casework?

Define $\lambda(x,y)$ = max # of pairwise disjoint (x,y)-paths.

In example



$\lambda(x,y) = 1 = \lambda(z,y)$, $\lambda(x,z) = 2$.

Thm (Menger) Fix $G = (V, E)$

$x, y \in V$ s.t. $\{x, y\} \notin E$. Then

$\kappa(x, y) = \lambda(x, y)$.

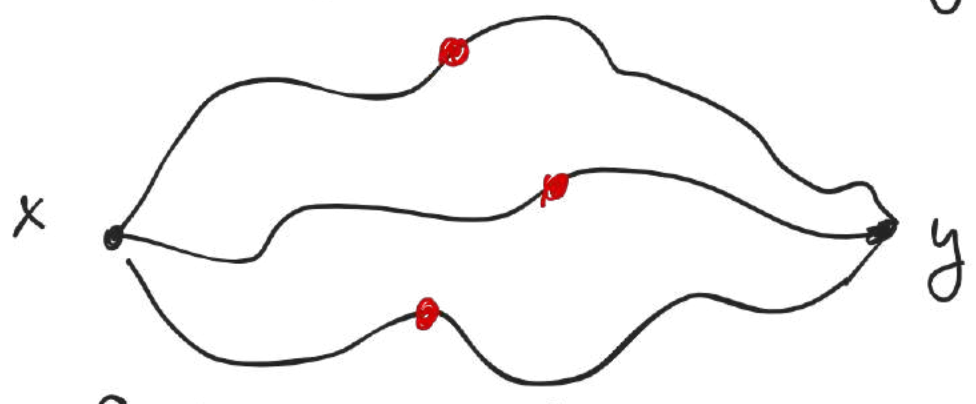
Rmks

(1) This is like König's Thm

max matching $\longleftrightarrow$ min vertex cover

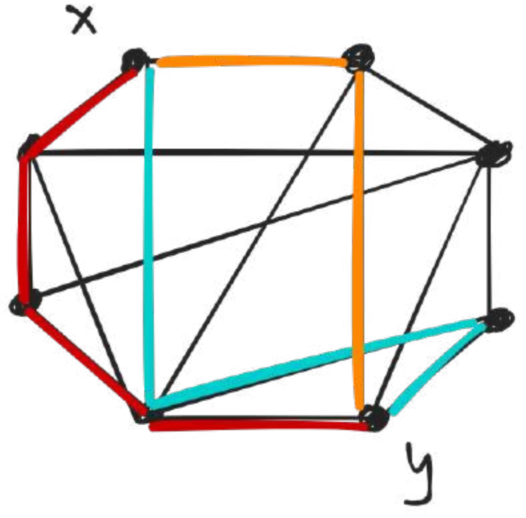
Here max disjoint (x,y)-paths $\longleftrightarrow$ min (x,y) vertex cut.

(2) $\kappa(x,y) \geq \lambda(x,y)$ easy:



if S is an (x,y)-vertex cut then S must meet each path from x to y.

(3) Example above. $\kappa(x,y) \le 3$.



$\Rightarrow \lambda(x,y) \geq 3$

$\Rightarrow \kappa(x,y) = 3$
(Menger)

(4) $\kappa(G) = \min_{x,y} \kappa(x,y) = \min_{x,y} \lambda(x,y)$

gives way to compute κ.

eg $\kappa(Q_3) = 3$ ((last time)

$\kappa(Q_3) \le 3$ easy. For $\kappa(Q_3) \geq 3$
show between any two pts $\exists$
3 disjoint paths.