Homework 9

Math 123

Due April 2, 2021 by 5pm

Name:

Topics covered: planar graphs, Kuratowski's theorem, convex embeddings, Tutte's theorem

Instructions:

- This assignment must be submitted on Gradescope by the due date.
- If you collaborate with other students (which is encouraged!), please mention this near the corresponding problems.
- Some problems from this assignment come from West's book, as indicated next to the problem. In some cases, the statements on this assignment differ slightly from the book.
- If you are stuck please ask for help (from me or your classmates). Occasionally problems may require ingredients not discussed in the course.
- You may freely use any fact proved in class. In general, you should provide proof for facts used that were not proved in class.

Problem 1. *Prove that $K_{3,3}$ is not planar by an argument similar to the one we gave in class for K_5.* [1]

Solution. □

Problem 2 (West 6.1.20)**.** *Prove that a planar graph G is bipartite if and only if every region of $\mathbb{R}^2 \setminus G$ has an even number of sides. (Note: the boundary of a region does not necessarily correspond to a cycle in the graph.)*[2]

Solution. □

Problem 3 (West 6.1.29)**.** *Prove that the complement of a planar graph with at least 11 vertices is nonplanar.*[3] *Give an example of a planar graph with 8 vertices whose complement is also planar.*[4]

Solution. □

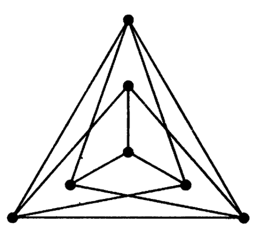
Problem 4 (West 6.1.31)**.** *Let G_n be the graph with vertices $v_1, \ldots, v_n$ and an edge between v_i and v_j whenever $|i - j| \leq 3$. Prove that G_n is a maximal planar graph.*[5]

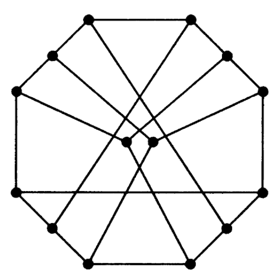
Solution. □

Problem 5. *Let $G \subset \mathbb{R}^2$ be a plane graph, and let $C \subset G$ be a cycle. Assume that G is 3-connected. Prove that C is the boundary of a region of $\mathbb{R}^2 \setminus G$ if and only if G has exactly one C-fragment.*[6]

Solution. □

Problem 6 (West 6.2.4)**.** *For each graph below, if the graph is planar, give a convex embedding, and if the graph is non-planar give two proofs: one using Kuratowski's theorem and one using the conflict graph.*





Solution. □

[1] Hint: What is the smallest length of a closed walk/cycle in $K_{3,3}$?

[2] Hint: for one direction, use induction on the number of regions of $\mathbb{R}^2 \setminus G$.

[3] Hint: your solution should be very short.

[4] Hint: in fact you can find one that is self-complementary. This should simplify your search.

[5] Hint: you can do this by induction. Note that there are two things to show: maximal and planar.

[6] Remark: to better understand this problem, it might help to construct an example of a graph G with a cycle $C \subset G$ and a planar embedding $G \subset \mathbb{R}^2$ so that C is not the boundary of a region.

Problem 7 (Bonus)**.** *Give an intelligent[7] solution to the magic square puzzle posted by Cole in Piazza post @93.*

Solution. □

[7] i.e. using math, not by brute force