Homework 8

Math 123

Due March 26, 2021 by 5pm

Name:

Topics covered: Ramsey theory, pigeonhole principle

Instructions:

- This assignment must be submitted on Gradescope by the due date.
- If you collaborate with other students (which is encouraged!), please mention this near the corresponding problems.
- Some problems from this assignment come from West's book, as indicated next to the problem. In some cases, the statements on this assignment differ slightly from the book.
- If you are stuck please ask for help (from me or your classmates). Occasionally problems may require ingredients not discussed in the course.
- You may freely use any fact proved in class. In general, you should provide proof for facts used that were not proved in class.

Problem 1 (West 8.3.15)**.** *Prove that $R(k,\ell) \le \binom{k+\ell-2}{k-1}$ and deduce that $R(k,k) \le 2^{2k}$.* [1]

Solution. □

Problem 2 (West 8.3.3)**.** *Use the pigeonhole principle to prove the following statements.*

(a) *Every set of n integers $\{a_1, \dots, a_n\}$ contains a nonempty subset whose sum is divisible by n.* [2][3]

(b) *Given $x \in \mathbb{R}$, prove that at least one of $\{x, 2x, \dots, (n-1)x\}$ differs by at most $1/n$ from an integer.* [4]

Solution. □

Problem 3. *Let S be a set of five points in the plane, no three collinear. Prove that S contains a vertex set of a convex quadrilateral.* [5] [6]

Solution. □

Problem 4 (West 8.3.1)**.** *Each of two concentric discs has 20 radial sections of equal size. For each disc, 10 sections are painted red and 10 blue, in some arrangement. Prove that the two discs can be aligned so that at least 10 sections on the inner disc match colors with the corresponding sections on the outer disc.* [7] [8]

Solution. □

Problem 5 (West 8.3.13)**.** *Fix $r \ge 2$ and let $R(k, \dots, k)$ (with r copies of k) denote the minimal n so that any r-coloring of the edges of K_n contains at a monochromatic K_k (i.e. we are generalizing the Ramsey numbers to more colors). Prove that every r coloring of $1, \dots, R(3, \dots, 3)$ (with r copies of 3) contains a monochromatic x, y, z so that $x + y = z$.* [9] [10]

Solution. □

[1] Use induction for the first part. You will probably want to use some well-known facts about binomial coefficients.

[2] Hint: consider the partial sums $S_i = a_1 + \cdots + a_i$.

[3] Remark: The set $\{1, n+1, 2n+1, \dots, (n-1)n+1\}$ shows you cannot improve this result to a set of $n-1$ integers.

[4] Hint: consider the fractional parts of these integers: every real number r can be written uniquely in the form $r = \lfloor r \rfloor + \{r\}$, where $\lfloor r \rfloor$ is an integer and $\{r\} \in [0, 1)$ (this is the fractional part).

[5] A set $C \subset \mathbb{R}^2$ is *convex* if for any $x, y \in C$, the line segment between x and y is contained in C. For any finite subset $X \subset \mathbb{R}^2$, the *convex hull* of X is a smallest convex set C containing X. This is always a polygon.

[6] Hint: consider cases for the shape of the convex hull of S.

[7] Hint: There is a very short solution using the pigeonhole principle.

[8] Sizable hint: fixing a section of the outer disk, how many positions of the inner disk give a matching in this one "coordinate"?

[9] Hint: set $n = R(3, \dots, 3)$. Given a coloring of $1, \dots, n$, construct an edge r-coloring of K_n so that a monochromatic triangle in K_n corresponds to monochromatic x, y, z with $x + y = z$.

[10] Sizable hint: labelling the vertices of K_n by $v_1, \dots, v_n$, choose the coloring of the edge $v_i v_j$ using the color of $|j - i|$ as a guide.

Problem 6. *A theorem similar to Negami's theorem says that for any embedding of K_6 in $\mathbb{R}^3$, there exists a pair of triangles that form a Hopf link.*[11] *Verify this result for the following set of 6 points in $\mathbb{R}^3$.*

$$A = (0, 3, 2), \quad B = (-2, -6, 0), \quad C = (6, 3, 2), \quad D = (-6, -10, 7), \quad E = (-9, -6, 9), \quad F = (3, -1, 9)$$

To help solve this, use this Geo-gebra notebook:
`https://www.geogebra.org/calculator/bat5mzx6`[12] *Include a screenshot in your solution.*

Solution. □

Problem 7 (This is required). *Submit a project outline. See the course webpage for more information and a sample. Submit separately on Gradescope.*

Problem 8 (Bonus). *Give a set of eight points S in the plane, no three on a line, with the property that S does not contain the vertex set of any convex 5-gon. Be sure to give an explanation.*[13]

Solution. □

[11] Google this.
[12] Fill the notebook with the data you're given. Then you can click the circles on the left to show/hide line segments.
[13] In fact, this is the best you can do: any collection of 9 points in the plane (no three collinear) contains the vertices of a convex pentagon.