Homework 7

Math 123

Due March 19, 2021 by 5pm

Name:

Topics covered: Graph coloring, Mycielski construction, Brooks' theorem

Instructions:

- This assignment must be submitted on Gradescope by the due date.
- If you collaborate with other students (which is encouraged!), please mention this near the corresponding problems.
- Some problems from this assignment come from West's book, as indicated next to the problem. In some cases, the statements on this assignment differ slightly from the book.
- If you are stuck please ask for help (from me or your classmates). Occasionally problems may require ingredients not discussed in the course.
- You may freely use any fact proved in class. In general, you should provide proof for facts used that were not proved in class.

Problem 1.

(a) *Identify the graph on the course webpage and explain what is special about it.*

(b) *Let $G = K_{n_1,\ldots,n_r}$ be a complete r-partite graph with $n = n_1 + \cdots + n_r$ vertices. Show that if $n_i - n_j \geq 2$ for some i, j, then there exists an r-partite graph with n vertices and more edges than G.*

Solution. □

Problem 2 (West 5.1.19)**.** *Find the error in the following incorrect proof of Brooks' Theorem: If $G \neq C_{2k+1}$ and $G \neq K_n$, then $\chi(G) \leq \Delta(G)$. A complete solution requires not only pointing out the error, but also explaining why it's wrong (with an example).*

> *We induct on the number of vertices n. The statement holds for $n = 1$, trivially. For the induction step, suppose that G is not complete or an odd cycle. Since $\kappa(G) \leq \delta(G)$ (where κ is the vertex connectivity and δ is the minimal vertex degree), the graph G has a vertex cut S of size at most $\Delta(G)$. Let $G_1, \ldots, G_m$ be the components of $G \setminus S$, and let $H_i \subset G$ be the subgraph generated by $V(G_i) \cup S$. By the induction hypothesis, each H_i is $\Delta(G)$-colorable. Permute the names of the colors used on these subgraphs to agree on S. This yields a coloring of G with $\Delta(G)$ colors.*

Note that this argument is similar to the argument we used to show a critical graph doesn't have a cut vertex.

Solution. □

Problem 3 (West 5.1.22)**.** *Given a set of lines in the plane with no three meeting at a point, form a graph G whose vertices are the intersections of the lines, with two vertices adjacent if they appear consecutively on one of the lines. Prove that $\chi(G) \leq 3$.* [1] [2]

Solution. □

Problem 4 (West 5.1.33)**.** *Prove that every graph G has a vertex ordering relative to which the greedy coloring uses $\chi(G)$ colors.*

Solution. □

Problem 5 (West 5.2.9)**.** *For a graph G, let G' be the graph obtained by the Mycielski construction. Show that if G is critical, then G' is also critical.* [3] [4]

Solution. □

[1] If you are unsure of exactly how this graph works, see the problem statement in West for an example.

[2] Hint: use a greedy coloring with an appropriate vertex ordering.

[3] Hint: once you figure this out for the Grötzsch graph, you can probably quickly detail the general case.

[4] Show $\chi(G' \setminus e) < \chi(G')$ by cases. The trickiest case is $e = \{v_i, u_j\}$. One way to approach this is to start by coloring H' with $H \subset G \setminus e'$, where $e' = \{v_i, v_j\}$...

Problem 6 (West 5.1.38)**.** *Prove that $\chi(G) = \omega(G)$ when the complement $\bar{G}$ is bipartite.* [5] [6]

Solution. $\square$

Problem 7 (Bonus)**.** *Let $G = (V, E)$ be the unit distance graph in the plane: $V = \mathbb{R}^2$, and two points are adjacent if their Euclidean distance is 1. Prove that $4 \leq \chi(G) \leq 7$.*

Solution. $\square$

[5] Recall that $\omega(G)$ is the clique number.
[6] Hint: look to apply König's theorem (!)