# Homework 6

Math 123

Due March 12, 2021 by 5pm

## Name:

Topics covered: vertex cuts, connectivity, Menger's theorem, network flows

Instructions:

- This assignment must be submitted on Gradescope by the due date.
- If you collaborate with other students (which is encouraged!), please mention this near the corresponding problems.
- Some problems from this assignment come from West's book, as indicated next to the problem. In some cases, the statements on this assignment differ slightly from the book.
- If you are stuck please ask for help (from me or your classmates). Occasionally problems may require ingredients not discussed in the course.
- You may freely use any fact proved in class. In general, you should provide proof for facts used that were not proved in class.

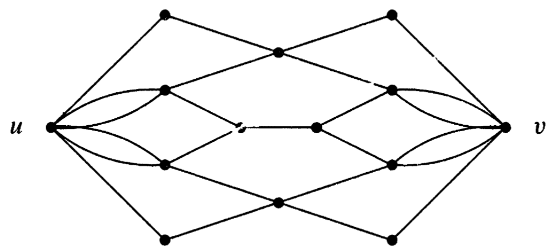
**Problem 1** (West 4.1.2). *Let $G$ be a graph.*

*(a) Give a counterexample to the following statement: If $e$ is a cut-edge of $G$, then at least one vertex of $e$ is a cut-vertex of $G$.*

*(b) Add a hypothesis to correct the above statement.*

*Solution.* □

**Problem 2** (West 4.2.1). *Compute (with proof) $\kappa(u, v)$ for the graph below.*




*Solution.* □

**Problem 3** (West 4.1.10). *Find (with proof) the smallest 3-regular graph having connectivity 1.* [1]

*Solution.* □

**Problem 4** (West 4.2.20). *Fix $k \geq 2$. Prove that the hypercube $Q_k$ is $k$-connected by constructing $k$ pairwise internally disjoint $x, y$-paths for each pair of vertices $x, y$.*

*Solution.* □

**Problem 5** (West 4.2.23). *Use Menger's theorem to prove König's theorem: if $G = (X \sqcup Y, E)$ is bipartite the maximum size of a matching of $G$ is equal to the minimum size of a vertex cover of $G$.* [2]

*Solution.* □

**Problem 6** (West 4.3.10). *Use network flows to prove König's theorem.*[3] [4] [5]

*Solution.* □

**Problem 7** (Required!). *Sign up for a project on the Google doc on the course webpage. If you will be unable to present during lectures in the last two weeks of the course (e.g. for time zone reasons), please email me about this, so we can figure out an alternative.*

[1] Hint: consider a 1-element vertex cut $S$. What does $G \setminus S$ look like?

[2] Hint: consider graph $G'$ obtained by adding vertices $a, b$ to $G$ and connecting $a$ to every vertex of $X$ and $b$ to every vertex of $Y$.

[3] Hint: given the hint from the previous problem, you should be able to solve this problem without a hint.

[4] Be sure to give details.

[5] I guess you could do this by combining our proof of Menger from class with the solution for the problem above, but there is a more direct route.