Homework 5

Math 123

Due March 5, 2021 by 5pm

Name:

Topics covered: matchings, König's theorem, vertex covers, Gale–Shapely algorithm

Instructions:

- This assignment must be submitted on Gradescope by the due date.
- If you collaborate with other students (which is encouraged!), please mention this near the corresponding problems.
- Some problems from this assignment come from West's book, as indicated next to the problem. In some cases, the statements on this assignment differ slightly from the book.
- If you are stuck please ask for help (from me or your classmates). Occasionally problems may require ingredients not discussed in the course.
- You may freely use any fact proved in class. In general, you should provide proof for facts used that were not proved in class.

Problem 1 (West 3.1.1)**.** *Find a maximum matching in each graph below. Prove that it is a maximum matching by exhibiting an optimal solution to the dual problem. Explain why this proves that the matching is optimal.*

Solution. □

Problem 2 (West 3.1.29)**.** *Let $G = (V, E)$ be a bipartite graph with maximum vertex degree Δ.*

(a) Use König's theorem to prove that G has a matching of size at least $|E|/\Delta$.

(b) Use (a) to conclude that every subgraph of $K_{n,n}$ with more than $(k-1)n$ edges has a matching of size at least k.

Solution. □

Problem 3. *Use König's theorem to prove the more difficult direction of Hall's theorem: if $G = (X \sqcup Y, E)$ does not have a matching saturating X, then there exists $S \subset X$ so that $|S| > |N(S)|$.* [1]

Solution. □

Problem 4 (West 3.2.3)**.** *Give an example of a stable matching problem of a graph $G = (X \sqcup Y, E)$ with $|X| = |Y| = 2$ in which there is more than one stable matching.*

Solution. □

Problem 5 (West 3.2.4)**.** *Determine the stable matchings resulting from the proposal algorithm run with cats proposing and with giraffes proposing, given the preference lists below.*

Cats $\{u, v, w, x, y, z\}$	Giraffes $\{a, b, c, d, e, f\}$
$u: a > b > d > c > f > e$	$a: z > x > y > u > v > w$
$v: a > b > c > f > e > d$	$b: y > z > w > x > v > u$
$w: c > b > d > a > f > e$	$c: v > x > w > y > u > z$
$x: c > a > d > b > e > f$	$d: w > y > u > x > z > v$
$y: c > d > a > b > f > e$	$e: u > v > x > w > y > z$
$z: d > e > f > c > b > a$	$f: u > w > x > v > z > y$

Solution. □

[1] Hint: Prove the given statement (not its contrapositive). Consider a minimal vertex cover Q. What can you say about the size $Q \cap X$ vs. X? Find a set $S \subset X$ so that $|S| > |N(S)|$. The proof should be relatively short.

Problem 6. *Let $G = (X \sqcup Y, E)$ be a bipartite graph with a preference list. Let M_X be the matching obtained from applying the Gale–Shapely algorithm with vertices in X proposing. Let M be any other stable matching of G. Prove that for each $y \in Y$, if $\{x, y\} \in M_X$ and $\{x', y\} \in M$, then either $x = x'$ or y prefers x' to x. In other words, each $y \in Y$ gets the worst possible matching from the proposal algorithm.*

Solution. □

Problem 7 (Bonus)**.** *Let T_1 be the tiling of the plane by unit squares whose vertices have integer coordinates. Let T_2 be the result of rotating T_1 about the origin by some angle θ. Prove that it is possible to find a bijection between squares of T_1 and squares of T_2 in such a way that the matched squares are within 10 units of each other. The matching will depend on θ.* [2]

Solution. □

[2] Hint: try to make this look like Hall's matching problem.