Homework 4

Math 123

Due February 19, 2021 by 5pm

Name:

Topics covered: matchings, Hall's theorem, maximum matchings, augmenting paths

Instructions:

- This assignment must be submitted on Gradescope by the due date.
- If you collaborate with other students (which is encouraged!), please mention this near the corresponding problems.
- Some problems from this assignment come from West's book, as indicated next to the problem. In some cases, the statements on this assignment differ slightly from the book.
- If you are stuck please ask for help (from me or your classmates). Occasionally problems may require ingredients not discussed in the course.
- You may freely use any fact proved in class. In general, you should provide proof for facts used that were not proved in class.

Problem 1 (West 3.1.8)**.** *Prove or disprove: every tree T has at most one perfect matching.*

Solution. □

Problem 2 (West 3.1.16)**.** *For $k \geq 2$, prove that Q_k (the hypercube graph) has at least $2^{2^{k-2}}$ perfect matchings.*

Solution. □

Problem 3 (West 3.1.9)**.** *Let m be the maximum size of a matching of G. Prove that every maximal matching of G has at least $m/2$ edges.*

Solution. □

Problem 4 (West 3.1.18)**.** *Two people play a game on a graph G, alternatively choosing distinct vertices. Player 1 starts by choosing any vertex. Each subsequent choice must be adjacent to the preceding choice (of the other player). Thus together they follow a path. The last player able to move wins. Prove that the second player has a winning strategy if G has a perfect matching, and otherwise the first player has a winning strategy.*[1]

Solution. □

Problem 5 (West 3.1.24)**.** *Recall that a square matrix is called a permutation matrix P if it has exactly one 1 in each row and each column and the remaining entries are 0, e.g.*

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

(a) Prove that a square matrix with nonnegative integers entries can be expressed as the sum of k permutation matrices if and only if all the row sums and column sums equal k. [2]

(b) Use this to construct your own original 4×4 magic square. Make sure the diagonals also have the same value (you will need to think about how to ensure this). Also make sure your example is not boring.[3]

Solution. □

Problem 6 (West 3.1.26)**.** *A deck with mn cards with m values and n suits consists of one card for each value in each suit. The cards are dealt into an $n \times m$ array. Prove that there is a set of m cards, one in each column, having distinct values.*

Solution. □

[1] Hint: for the second part, the first player should start with a vertex omitted by some maximum matching.

[2] Hint: Here it may help to consider graphs with multiple edges. Use a fact from class about k-regular bipartite graphs.

[3] For example, the magic square where every entry is 1 is very boring.

Problem 7 (Bonus)**.** *This is a continuation of the previous problem. Use that problem to prove that by a sequence of exchanges of cards of the same value, the cards can be rearranged so that each column consists of n cards of distinct suits. (Try to sketch the idea. You do not need to give careful proof.)*

Solution. □