Homework 3

Math 123

Due February 12, 2021 by 5pm

Name:

Topics covered: trees, Prüfer codes, spanning trees, counting graphs

Instructions:

- This assignment must be submitted on Gradescope by the due date.
- If you collaborate with other students (which is encouraged!), please mention this near the corresponding problems.
- Some problems from this assignment come from West's book, as indicated next to the problem. In some cases, the statements on this assignment differ slightly from the book.
- If you are stuck please ask for help (from me or your classmates). Occasionally problems may require ingredients not discussed in the course.
- You may freely use any fact proved in class. In general, you should provide proof for facts used that were not proved in class.

Problem 1 (West 1.2.12). *Prove or disprove: if G is Eulerian with edges e, f that share a vertex, then G has an Euler tour where e, f appear consecutively.*

Solution. □

Problem 2 (West 2.2.1). *Determine which trees have Prüfer codes that*

(a) contain only one value;

(b) contain exactly two values;

(c) have distinct values.

You should explain your answer, but you don't need to give careful proof.

Solution. □

Problem 3 (West 2.1.72). *Prove that if $T_1, \ldots, T_k$ are pairwise-intersecting subtrees of a tree T, then T has a vertex that belongs to all of $T_1, \ldots, T_k$.* [1] [2]

Solution. □

Problem 4 (West 1.2.17). *Let G_n be the graph whose vertices are orderings of the elements of $\{1, \ldots, n\}$ with $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ adjacent if they differ by switching a pair of adjacent entries.*[3]

(a) The graph G_3 is isomorphic to a familiar graph. Which one is it?

(b) Show that G_n is connected. [4]

Solution. □

Problem 5 (West 2.2.7). *Use Cayley's formula to prove that the graph obtained from K_n by deleting an edge has $(n-2)n^{n-3}$ spanning trees.*

Solution. □

Problem 6 (West 1.3.32). *Prove that the number of even graphs (i.e. graphs where every vertex has even degree) with vertex set $\{1, \ldots, n\}$ is $2^{\binom{n-1}{2}}$.* [5]

Solution. □

Problem 7 (Bonus). *Consider the graph with vertices and edges pictured below. Show it is impossible to draw a line (without picking up your pen) that crosses each edge exactly once.*

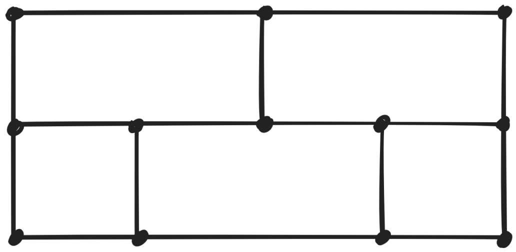
[1] Remark: This is a graph-theoretic analog of Helly's theorem.

[2] Hint: use induction on k.

[3] Note: Here the sequence $(a_1, \ldots, a_n)$ does not have repetition. Each element of $\{1, \ldots, n\}$ appears exactly once.

[4] Tangential remark: here you are proving that a certain collection of permutations generate the symmetric group.

[5] Hint: establish a bijection to the set of all graphs with vertex set $\{1, \ldots, n-1\}$.



Solution. □