Homework 2

Math 123

Due February 5, 2021 by 5pm

Name:

Topics covered: bipartite graphs, Euler tours, vertex-degree sum formula, trees

Instructions:

- This assignment must be submitted on Gradescope by the due date.
- If you collaborate with other students (which is encouraged!), please mention this near the corresponding problems.
- Some problems from this assignment come from West's book, as indicated next to the problem. In some cases, the statements on this assignment differ slightly from the book.
- If you are stuck please ask for help (from me or your classmates). Occasionally problems may require ingredients not discussed in the course.
- You may freely use any fact proved in class. In general, you should provide proof for facts used that were not proved in class.

Problem 1 (West 1.2.8)**.** *The complete bipartite graph $K_{n,m}$ is the graph with $n + m$ vertices $v_1, \ldots, v_n$ and $u_1, \ldots, u_m$ and edges $\{v_i, u_j\}$ for each $1 \leq i \leq n$ and $1 \leq j \leq m$. Determine the values n, m so that $K_{n,m}$ is Eulerian.*

Solution. □

Problem 2 (West 1.2.10)**.** *Prove or disprove:*

(a) Every Eulerian bipartite graph has an even number of edges.

(b) Every Eulerian graph with an even number of vertices has an even number of edges.

Solution. □

Problem 3.

(a) Classify trees with exactly two vertices of degree 1. [1]

(b) What can you say about the shape of trees with either 3 or 4 vertices of degree 1? (Give a qualitative statement – you do not need to provide a formal argument.)

Solution. □

Problem 4 (West 1.1.17)**.** *Determine the number of graphs with 7-vertices, each of degree 4 (up to isomorphism).* [2]

Solution. □

Problem 5 (West 1.1.14)**.**

(a) Prove that removing opposite corner squares from an 8×8 checkerboard leaves a sub-board that cannot be partitioned into 1×2 and 2×1 rectangles. [3]

(b) Using the same argument, make a general statement about all bipartite graphs.[4]

Solution. □

Problem 6 (West 1.3.13)**.** *Suppose there are two mountain trails, each starting at sea level and ending at the same elevation. Suppose hikers A,B start hiking these two different trails at the same time. The Mountain Climber Problem asks if it is possible for A and B to hike to the top of their individual trails in a way so that they have the same elevation at every time.*[5] *We model the trails by functions $f, g : [0, 1] \to [0, 1]$ with $f(0) = g(0) = 0$ and $f(1) = g(1) = 1$. In this problem you solve the Mountain Climber Problem in the case when f and g are piecewise linear continuous functions.*[6]

[1] At some point, you should use the vertex-degree sum formula from Lecture 1.

[2] Hint: consider the complement. Your solution should not be long. Use the previous problem.

[3] Hint: your solution should be very short.

[4] This problem is not particularly deep. The point here is to see how to translate to bipartite graphs. If you get stuck look up "matching" in a (bipartite) graph.

[5] It is important to note that the hikers are allowed to backtrack.

[6] A function $f : [0, 1] \to \mathbb{R}$ is piecewise linear if it's possible to express $[0, 1]$ as a union of finitely many intervals, so that f is linear ($x \mapsto ax + b$) on each.

(a) Consider

$$Z = \{(x, y) \in [0,1] \times [0,1] : f(x) = g(y)\}$$

Assuming f, g are piecewise linear, determine the local picture near (x, y) in Z, considering cases based on the local pictures of f and g near x and y, respectively.

(b) Observe that Z can be given the structure of a graph G. Show that G has exactly two vertices of odd degree. Deduce that there is a path in G from $(0,0)$ to $(1,1)$.

Solution. □