Homework 1

Math 123

Due January 29, 2021 by 5pm

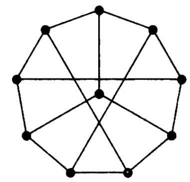
Name:

Topics covered: graph/subgraph, cycle, path, vertex degrees, isomorphism

Instructions:

- This assignment must be submitted on Gradescope by the due date. Gradescope Entry Code:WYB7BK.
- If you collaborate with other students (which is encouraged!), please mention this near the corresponding problems.
- Some problems from this assignment come from West's book, as indicated next to the problem. In some cases, the statements on this assignment differ slightly from the book.
- If you are stuck please ask for help (from me or your classmates). Occasionally problems may require ingredients not discussed in the course.

Problem 1 (West 1.1.24). *Prove that the graph below is isomorphic to the Petersen graph.*[1]



Solution. □

Problem 2 (West 1.3.20). *How many cycles of length n are there in the complete graph K_n?*

Solution. □

Problem 3 (West 1.3.24). *Define the hypercube graph Q_k as the graph with a vertex for each tuple $(a_1, \dots, a_k)$ with coordinates $a_i \in \{0,1\}$ and with an edge between $(a_1, \dots, a_k)$ and $(b_1, \dots, b_k)$ if they differ in exactly one coordinate.*[2]

(a) Prove that two 4-cycles in Q_k are either disjoint, intersect in a single vertex, or intersect in a single edge.

(b) Let $K_{2,3}$ be the complete bipartite graph with 2 red and 3 blue vertices. Prove that $K_{2,3}$ is not a subgraph of any hypercube Q_k.

Solution. □

Problem 4 (West 1.1.31). *Let $G = (V, E)$ be a graph. The complement of G is the graph $\bar{G} = (V, \bar{E})$, where $\{u, v\} \in \bar{E}$ if and only if $\{u, v\} \notin E$.*

(a) Determine the complement of the graphs P_3 and P_4. (Recall that P_n is the path with n vertices.)

(b) We say that G is self-complementary if G is isomorphic $\bar{G}$. Prove that if G is self-complementary with n vertices, then either n is divisible by 4 or $n-1$ is divisible by 4.

(c) Prove that if either n or $n-1$ is divisible by 4, then there exists a self-complementary graph with n vertices. You do not need to write a careful proof that these graphs are self-complementary, but you should convince yourself of this.[3]

Solution. □

Problem 5. *True or false: if G is isomorphic to H, then the complements $\bar{G}$ and $\bar{H}$ are also isomorphic.*[4]

[1] Hint: label the graph.

[2] Suggestion: Draw Q_k for $k = 2$ and $k = 3$. This should help explain why the graph is called the hypercube.

[3] Hint: when $n = 4k$, start with k copies of P_4. It may help to start by carefully considering the case $n = 8$. When $n = 4k + 1$, start by adding one vertex to the graph constructed for $n = 4k$. Again, start with the case $n = 5$.

[4] True/false questions require either a proof (if the statement is true) or a counterexample (if the statement is false).

Solution. □

Problem 6 (West 1.1.25). *For this problem, let G denote the Petersen graph.*

(a) *Prove that G has no cycles of length 3 or 4.* [5]

(b) *Prove that G has no cycle of length 7.* [6]

Solution. □

[5] Hint: use the definition of Petersen graph given in class.

[6] Hint: argue by contradiction. Here are some useful things to consider. Every vertex of G has degree 3, so each vertex on a 7-cycle C has one edge not on C. Can such an edge connect two different vertices on C? How many vertices of G are not part of C? How many total edges are in G? Use part (a) several times.