Homework 8

Math 126

Due November 12, 2021 by 5pm

Name:

Topics covered: Singularities, Schwarz lemma, argument principle, final projects

Instructions:

- This assignment must be typed in LaTeX and submitted on Gradescope by the due date. The Gradescope entry code is V8XWRG
- If you collaborate with other students (which is encouraged!), please mention this near the corresponding problems.
- If you are stuck please ask for help (from me or your classmates). Occasionally problems may require ingredients not discussed in the course.
- You may freely use any fact proved in class. In general, you should provide proof for facts that you use that were not proved in class.

Problem 1. *Find the isolated singularities of* $\frac{z}{\sin(z)}$*, and determine whether they are removable, poles, or essential (with proof). Determine the order of any pole.*[1]

Solution. □

Problem 2. *True or false: if* $f(z)$ *has an isolated non-removable singularity at* z_0*, then* $e^{f(z)}$ *has an essential singularity at* z_0*.*

Solution. □

Problem 3. *Assume* f *is a holomorphic function on* $\mathbb{C} \setminus \{0\}$*. Prove there exists* c *so that* $f(z) - \frac{c}{z}$ *has an antiderivative on* $\mathbb{C} \setminus \{0\}$*.*

Solution. □

Problem 4. *Let* $\mathbb{D} = \{|z| < 1\}$ *denote the (open) unit disk.*

(a) *Fix* $a \in \mathbb{C}$*. Mimic the proof of the Schwarz lemma to show that if* $B = \{|z - a| < d\}$ *and* $h : B \to \mathbb{D}$ *is holomorphic satisfying* $h(a) = 0$*, then* $|h'(a)| \leq \frac{1}{d}$*.*

(b) *Let* $f : \mathbb{D} \to \mathbb{D}$ *be a biholomorphism. Let* d *be the distance from* $f(0)$ *to* $\partial\mathbb{D}$*. Show that* $d \leq |f'(0)|$*.* [2]

Solution. □

Problem 5. *Find the number of zeros of* $p(z) = z^9 + 2z^5 - 2z^4 + z + 3$ *in the right-half plane.*[3] *Deduce that* p *has a positive real root.*[4]

Solution. □

Problem 6. *Submit a final project outline. See below for an example and more information. (This should be more detailed than the paragraph you submitted last week!)* [5]

Solution. □

[1] Recall that the order of a pole w of f is the unique number $n > 0$ so that $f(z) = \frac{1}{(z-w)^n} g(z)$, where $g(z)$ is holomorphic and $g(w) \neq 0$.

[2] Hint: Observe that f^{-1} gives a function on the ball of radius d around $f(0)$ taking values in $\mathbb{D}$. Apply part (a) and the chain rule.

[3] Do this similar to class. If you like, you can use Mathematica to plot the image of a relevant closed curve under p.

[4] Hint: roots of a polynomial with real coefficients come in complex conjugate pairs (why?).

[5] I'm assigning this as a homework problem because I am expecting you to put real thought into this!

In preparing your outline, please keep in mind the following.

1. Organization. Your project should have a clear focus and should be balanced, well-organized, and fit into a 15 minute presentation. You should give at most one proof (and possibly none). The rest of your presentation should be ideas, examples, motivations, applications.
2. Breadth of research. Please include at least two sources. Your outline topics/subtopics should connect well to the theme of the project.
3. Depth of information. Your outline should contain enough detail/specifics (e.g. what examples will you use, what statements will you prove, for your application how background info will you need to explain) and illustrate a thorough exploration of the related material.

Below is a sample outline.

Talk title. The Casorati–Weierstrass theorem

Sources: Gamelin VI.2 and Ahlfors's "Complex analysis" Chapter 4.3

Slogan: essential singularities of a holomorphic function are very weird!

1. Statement: If w is an essential singularity of f, then f is maps densely near w.
   (a) Context/recall: an isolated singularity is either removable, a pole, or essential. For the first two, we understand well the behavior of f near w.
   (b) Example: $f(z) = e^{1/z}$. $w = 0$ is an essential singularity. The behavior is unpredictable as we approach 0 along radial paths. Mathematica demonstration of image of f along circles of shrinking radius.
2. Proof of Casorati–Weierstrass
   (a) By contrapositive. If f does not map densely, then we can find $u \in \mathbb{C}$ so that $g(z) := \frac{1}{f(z)-u}$ is a bounded holomorphic function near w. By Removable Singularity Theorem, g is holomorphic. If we write $g(z) = (z-w)^n h(z)$, where h is holomorphic and $h(w) \neq 0$, then rearranging, we find that $f(z) = \frac{1}{(z-w)^n} \frac{1}{h(z)} + u$ either has a removable singularity or a pole at w. This proves the theorem.
3. Further directions.
   (a) Picard theorem: in fact, near an essential singularity w, f takes on all possible complex values, with at most one exception. (!) Return to $f(z) = e^{1/z}$ example.