Homework 6

Math 126

Due October 29, 2021 by 5pm

Name:

Topics covered: analytic continuation, residue calculus

Instructions:

- This assignment must be typed in LaTeX and submitted on Gradescope by the due date. The Gradescope entry code is V8XWRG
- If you collaborate with other students (which is encouraged!), please mention this near the corresponding problems.
- If you are stuck please ask for help (from me or your classmates). Occasionally problems may require ingredients not discussed in the course.
- You may freely use any fact proved in class. In general, you should provide proof for facts that you use that were not proved in class.

Problem 1. *Use residue theory to compute* $\int_{-\infty}^{\infty} \frac{\cos(x)}{(1+x^2)^2} = \frac{\pi}{e}$. [1]

Solution. □

Problem 2. *Compute* $\int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx$. [2]

Solution. □

Problem 3. *Let f be a holomorphic function defined on $\{0 < |z| < r\}$ for some r. Prove that if $\lim_{z\to 0} zf(z) = 0$, then f can be extended to a holomorphic function defined on $\{|z| < r\}$.* [3]

Solution. □

Problem 4. *Let f be a holomorphic function defined on the right-half plane $H = \{Re(z) > 0\}$, and assume that f satisfies $f(z+1) = zf(z)$ for all $z \in H$. Prove that*

$$\lim_{z\to -n} (z+n)f(z) = \frac{(-1)^n}{n!} f(1)$$

for each integer $n \geq 0$. Conclude that if $f(1) = 0$, then f can be analytically continued to the entire complex plane.

Solution. □

Problem 5. *Prove that the Γ function satisfies $|\Gamma(s)| \leq \Gamma(Re(s))$ for each s in the right-half plane H. Conclude that for any $0 < a < b$, the Γ function is bounded on the vertical strip*

$$\{a \leq Re(z) \leq b\}.$$

Solution. □

Problem 6. *Let H denote the right-half plane. Suppose that $F : H \to \mathbb{C}$ is a holomorphic function such that $F(z+1) = zF(z)$ for all $z \in H$ and F is bounded on the vertical strip $\{1 \leq Re(z) < 2\}$. Prove that $F = a\Gamma$, where $a = F(1)$.* [4] [5]

Solution. □

Problem 7 (Bonus)**.** *Make a meme related to the course.*

Solution. □

[1] Why is this result surprising? How do you explain it?

[2] This problem is tricky. Please don't hesitate to ask for help!

[3] Consider $g(z) = \begin{cases} z^2 f(z) & z \neq 0 \\ 0 & z = 0 \end{cases}$. Prove that g is holomorphic, hence analytic. Write an expression for the power series of g (what is $g'(0)$?), and use this to define an extension of f.

[4] Note that this gives a nice characterization of the Γ function.

[5] First consider $f(z) = F(z) - a\Gamma(z)$. Show that f has an analytic continuation to $\mathbb{C}$. Next consider $s(z) = f(z)f(1-z)$. Prove that s is bounded on $\{0 \leq \mathrm{Re}(z) \leq 1\}$ and that $s(z+1) = -s(z)$. Use this to deduce that s is constant (how?). Conclude.