Homework 4

Math 242

Due April 17, 2020 by 5pm

Topics covered: cohomology, Künneth theorem, Poincaré duality, final project

Instructions:

- This assignment must be submitted on Canvas by the due date.
- If you collaborate with other students, please mention this near the corresponding problems.
- Some problems from this assignment come from Hatcher or Bredon, as indicated next to the problem. Note that the statements on this assignment might differ slightly from the books.
- If you are stuck please ask for help (from me or your classmates). Occasionally problems may require ingredients not discussed in the course.

Problem 1.

(a) Look up the Yoneda lemma and give a brief explanation of what it says (this can be more philosophical and less precise). [1]

(b) Let B be an abelian group. Show that the functor $\operatorname{Hom}(B,-)$ is right adjoint to $(-)\otimes B$, i.e. for abelian groups A, C, there is a natural (in A, B, C) isomorphism

$$\tau : \operatorname{Hom}(A\otimes B, C) \xrightarrow{\cong} \operatorname{Hom}(A, \operatorname{Hom}(B, C)).$$

(c) Use the fact that $\operatorname{Hom}(-,X)$ is a left-exact (contravariant functor), the adjunction, and the Yoneda lemma to deduce that $(-)\otimes M$ is right exact (covariant functor). [2]

Solution. □

Problem 2. *Compute $H_k(\mathbb{R}P^n;\mathbb{Z}/p\mathbb{Z})$ and $H^k(\mathbb{R}P^n;\mathbb{Z}/p\mathbb{Z})$ for any prime p. Please show your work and don't just write down the answer.*

Solution. □

Problem 3.

(a) Prove that $\mathrm{SO}(4)$ is homeomorphic to $(S^3\times S^3)/\pm 1$. [3]

(b) Show that $(S^3\times S^3)/\pm 1$ is a product.

(c) Compute $H_k(\mathrm{SO}(4))$ for each $k\geq 0$.

Solution. □

Problem 4. *Prove that Euler characteristic is multiplicative $\chi(X\times Y) = \chi(X)\times\chi(Y)$.*

Solution. □

Problem 5. *Let M be a simply connected closed 3-manifold. In this problem you prove that M is homotopy equivalent to S^3.* [4]

(a) Prove that M has the same homology as S^3, and compute $\pi_k(M)$ for $1\leq k\leq 3$. [5]

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[1] Warning: the Yoneda lemma is a very general result in category theory. As such, many of the precise statements you find will be abstract and potentially difficult to understand. You will use the Yoneda lemma in part(c) below, so it is recommended that you first understand part(c) and focus your answer to part(a) toward a form that is helpful for part(c).

[2] Hint: start with a short exact sequence $0\to A'\to A\to A''\to 0$. Fix M, B and apply $\operatorname{Hom}(-,\operatorname{Hom}(M,B))$ to this SES. You want to conclude that $A'\otimes M\to A\otimes M\to A''\otimes M\to 0$ is exact.

[3] Use the quaternions $\mathbb{H}$, which admits a (norm-preserving) action of the unit quaternions $S^3\subset\mathbb{H}$ by left and right multiplication. This defines $S^3\times S^3\to\mathrm{SO}(4)$. First compute the kernel, and then the image. For the latter, you may need to use Lie groups, Lie algebras, and the exponential map.

[4] This is a weak version of the Poincaré conjecture.

[5] Hint: You may find it helpful to use Poincaré duality and/or the Hurewicz theorem.

(b) Use (a) to obtain a map $f : S^3 \to M$ and prove that it is a homotopy equivalence. [6]

Solution. □

[6] Hint: You may need Whitehead's theorem and (a relative version) of Hurewicz's theorem, c.f. Hatcher Theorem 4.37.

Problem 6. *Give an outline for your final presentation. Include the definitions, theorems, examples, applications that you plan to discuss. As an example of what I'm looking for, below is an outline of the lecture in class on March 30.*

1. **Künneth theorem.**

   The Künneth theorem computes $H_k(X \times Y)$ in terms of $H_*(X)$ and $H_*(Y)$. We prove it in two steps:

   Theorem (Eilenberg–Zilber). $S_*(X \times Y)$ and $S_*(X) \otimes S_*(Y)$ have the same homology.

   Theorem (Algebraic Künneth theorem) Given (nice) chain complex C, D, there is a SES

   $$0 \to \bigoplus_{p+q=k} H_p(C) \otimes H_q(D) \to H_k(C \otimes D) \to \bigoplus_{p+q=k-1} \mathrm{Tor}(H_p(C), H_q(D)) \to 0.$$

2. **Tensor products of abelian groups.**

   Definition of $A \otimes B$. Universal property.

   Examples: (briefly, without proofs)

   - $\mathbb{Z}/m\mathbb{Z} \otimes \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/\gcd(n,m)\mathbb{Z}$
   - $\mathbb{Z}^n \otimes \mathbb{Z}^m \cong \mathbb{Z}^{nm}$
   - $A \otimes \mathbb{Z} \cong A$

   Lemma: Tensor product is right exact. (with proof)

   Example: Tensor product is not left exact, e.g. tensor $0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$ by $\mathbb{Z}/2\mathbb{Z}$.

3. **Tensor products of chain complexes.**

   Definition of $C \otimes D$. Boundary map.

   Examples:

   - If C is any chain complex and

     $$D_k = \begin{cases} M & k = 0 \\ 0 & \text{else} \end{cases}$$

     then $(C \otimes D)_k = C_k \otimes M$. In this case $H_k(C \otimes D) =: H_k(C; M)$ is called the homology of C with coefficients in M.

   - If C is any chain complex and

     $$D_k = \begin{cases} \mathbb{Z} & k = 0, 1 \\ 0 & \text{else} \end{cases}$$

     then $(C \otimes D)_k \cong C_k \oplus C_{k-1}$. Then $H_k(C \otimes D) \cong H_k(C) \oplus H_{k-1}(C)$. Note that $D = C_*(S^1)$ cellular chains with respect to the standard cell structure. It follows that $H_k(X \times S^1) \cong H_k(X) \oplus H_{k-1}(X)$. E.g. we can compute $H_k(S^1 \times \cdots \times S^1)$ quickly using this.