Homework 1

Math 242

Due January 31, 2020 by 5pm

Topics covered:

Instructions:

- This assignment must be submitted on Canvas by the due date.
- If you collaborate with other students, please mention this near the corresponding problems.
- Some problems from this assignment come from Hatcher or Bredon, as indicated next to the problem. Note that the statements on this assignment might differ slightly from the books.
- If you are stuck please ask for help (from me or your classmates). Occasionally problems may require ingredients not discussed in the course.

Problem 1. *For a based space X, ΩX denotes the loop space, and $c \in \Omega X$ denotes the constant map.*

(a) *Show that the map $\Omega X \to \Omega X$ defined by $\gamma \mapsto \gamma * c$ is homotopic to the identity, i.e. there is a homotopy $I \times \Omega X \to \Omega X$.* [1] [2]

(b) *Observe that this argument can be used to prove that ΩX is an H-group (check the details on your own – you do not need to write it down here).*

Solution. □

Problem 2. *Show $\mathbb{R}^n$ is not a union of finitely many k-dimensional planes when $k < n$.* [3]

Solution. □

Problem 3. *Let A, B_1, B_2 be based spaces.*

(a) *Prove that $(B_1 \times B_2)^A \cong B_1^A \times B_2^A$ (homeomorphism).*

(b) *Prove that $[A, B_1 \times B_2] \cong [A, B_1] \times [A, B_2]$ (bijection of sets).*

Solution. □

Problem 4. *Identify $X = \mathbb{R}P^\infty$ with the projectivization of the space of polynomials with coefficients in $\mathbb{R}$, and use this to define a monoid structure $m : X \times X \to X$. Show that the map $X \to X$ defined by $f \mapsto m(f, f)$ is homotopic to a constant.*[4] *Conclude that $\mathbb{R}P^\infty$ is an H-group.*

Solution. □

[1] Last semester we showed that $\gamma \simeq \gamma * c$ for each fixed γ, which is weaker than what is asked for here.
[2] Hint: do not work too hard. If your solution is not short, it is not the correct proof.
[3] Note that this type of result is false for vector spaces over finite fields.
[4] Hint: use the fundamental group and covering spaces.